Econ. 557b
Yale University

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Time Series Econometrics II

Take Home Examination

Answer ALL Questions: Any reference material allowed.
Time Allowed: Six weeks
Due Date & Time: Friday 28 May 1999, 12:00 noon.

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Question 1

The time series Y_t is generated by the model

$$Y_t = \beta' x_t + Y_t^0, \tag{1}$$

$$Y_t^0 = aY_{t-1}^0 + u_t, \quad t = 1, ..., n \tag{2}$$

where $a = 1 + \frac{c}{n} < 1,\ Y_0 = 0,\ u_t = C(L)\varepsilon_t,$ and $C(L) = \sum_{j=0}^{\infty} c_j L^j,$ with $\sum_{j=0}^{\infty} j^{\frac{1}{2}} |c_j| < \infty$ and $(\varepsilon_t) \equiv iid(0, \sigma^2)$. The variables x_t in (1) constitute a vector of polynomial time trends for which the following conditions hold.

C1 There exist diagonal matrices D_n and F_n for which

$$D_n^{-1} x_{[nr]} \to X_p(r), \quad F_n^{-1} \Delta x_{[nr]} \to X_p^{(1)}(r),$$

where $X_p^{(1)}(r)$ is the derivative of $X_p(r)$ and $D_n = nF_n$.

C2 $\int_0^1 X_p(r) X_p(r)' dr > 0, \quad \int_0^1 X_p^{(1)}(r) X_p^{(1)}(r)' dr > 0.$

It is proposed to test the null hypothesis H$_0$: $a = 1$ using the following procedure. The data Y_t are first detrended by: (i) quasi-differencing (QD) using the operator $\Delta_g = 1 - (1 + \frac{g}{n})L = \Delta - \frac{g}{n}L$ for some given constant g; and (ii) regressing $\Delta_g Y_t$ on $\Delta_g x_t$ giving an estimate $\widehat{\beta}$ of β in (1). Detrended observations $Y_{g,t}^0$ are computed by taking the residuals $Y_{g,t}^0 = Y_t - \widehat{\beta}' x_t$. Then, the first order autoregression

$$Y_{g,t}^0 = \widehat{a}_g Y_{g,t-1}^0 + \widehat{u}_t, \tag{3}$$

is fitted and the usual Z_a test statistic is calculated with this data using the formula:

$$Z_{a,g} = n(\widehat{a}_g - 1) - \frac{\widehat{\lambda}}{\sum_{t=1}^{n} \left(Y_{g,t-1}^0\right)^2 / n^2}, \tag{4}$$

where $\widehat{\lambda}$ is an estimate of the one sided long run covariance of u_t in (2) calculated using the residuals $\widehat{u}_t$ from (3).

Part A

1. Find the limit distribution of $\rho_{a,g} = n(\widehat{a}_g - 1)$ and $Z_{a,g}$. Give results for both the null case $(a = 1)$ and the local alternative $(a = 1 + \frac{c}{n})$.

2. Take the case where $x_t = t$ and give the result in an explicit form in this case. Compare your result with that given in Elliot, Rothenberg and Stock (1996, p. 825) for their linear trend case with $x_t = (1, t)'$. (Actually, Elliot et. al. work with the $t-$ ratio statistic, and you can make corresponding adjustments in their formula to do the coefficient test comparison). What, if any, difficulties arise in the asymptotic analysis of the QD approach leading to (4) above when $x_t = (1, t)'$, rather than $x_t = t$. How, if at all, are these difficulties avoided in Elliot et al (1996). Briefly discuss the implications of your analysis.

Part B

1. Take the special case of a linear trend with $x_t = t$. Design and perform a simulation experiment for the above model with $u_t = iidN(0, 1)$ and $n = 200$. For this model, the limit distributions of $\rho_{a,g} = n(\widehat{a}_g - 1)$ and $Z_{a,g}$ are the same, so you may work with $\rho_{a,g}$ rather than $Z_{a,g}$ in your simulation analysis. Use your simulations as follows:

2. Compute kernel density estimates of the probability density of $\rho_{a,g}$ under the null hypothesis and graph your results for $g = 0$ and $g = -13.5$.

3. Compute and compare power functions for the $Z_{a,g}$ test (these will be the same as those of $\rho_{a,g}$ here since there is no serial correlation in the simulated data) for $g = 0$ and $g = -13.5$ and graph the corresponding power functions for a grid of values of the local parameter c. (Be sure to adjust the size of these tests in each case so that it is controlled to have the same value, say 5%).

Part C

1. Perform unit root tests using the statistic $Z_{a,g}$ (for both cases $g = 0$, $-13.5)$ on quarterly US real GDP data over the period 1952:1-1998:4. Discuss your findings. The data may be obtained from my web site

(Visit URL: http://korora.econ.yale.edu, and look under 'data') or my Cowles Faculty web site (Visit URL: http://cowles.econ.yale.edu/ and then look under 'faculty and staff', then 'faculty' to get to my Cowles page).

2. Perform a conventional Z_a unit root test on the same data, allowing for a linear trend in the regression and using OLS instead of QD detrending. Compare the two sets of empirical results.

Question 2

An observed continuous time process $X(t)$ is generated by the linear system

$$X(t) = \beta' Z(t) + W(t), \quad t \in [0,1] \tag{5}$$

where $W(t)$ is an unobserved standard Brownian motion, $Z(t) = (t, ..., t^p)'$ is a time polynomial vector and β is a parameter vector to be estimated.

The following two estimators of β are proposed:

$$\widehat{\beta} = \left(\int_0^1 Z(t)Z(t)'dt \right)^{-1} \left(\int_0^1 Z(t)X(t)dt \right),$$

and

$$\widetilde{\beta} = \left(\int_0^1 Z^{(1)}(t)Z^{(1)}(t)'dt \right)^{-1} \left(\int_0^1 Z^{(1)}(t)dX(t) \right),$$

where $Z^{(1)}$ is the vector of first derivatives of Z.

Part A

1. Show that both $\widehat{\beta}' Z(t)$ and $\widetilde{\beta}' Z(t)$ are Hilbert projections in $L_2[0,1]$. How do these projections differ?

2. Find the distributions of $\widehat{\beta}$ and $\widetilde{\beta}$ and and compare them in the case where $p = 1$. What do you conclude?

Part B

Suppose the system generating $X(t)$ is

$$X(t) = \beta' Z(t) + J_c(t), \quad t \in [0,1], \tag{6}$$

where $J_c(t) = \int_0^t e^{(t-s)c} dW(s)$ is a linear diffusion.

1. Are $\widehat{\beta}' Z(t)$ and $\widetilde{\beta}' Z(t)$ still Hilbert projections?
2. Calculate the distributions of $\widehat{\beta}$ and $\widetilde{\beta}$ and compare them in the case where $p = 1$? What do you conclude?
3. Can you suggest an unbiased linear estimator of β which has smaller variance than $\widehat{\beta}$ and $\widetilde{\beta}$? Does it correspond to another Hilbert projection?

Question 3

The time series X_t is generated by the model

$$X_t = aX_{t-1} + u_t, \quad t = 1, ..., n \tag{7}$$

where $a = 1$ and $X_0 = 0$. The errors u_t form a strictly stationary and ergodic martingale difference sequence with respect to the natural filtration $\mathcal{F}_t$ and satisfy the following conditions (so that u_t is an ARCH (1) error process):

C1 $E(u_t|\mathcal{F}_{t-1}) = 0$, $E(u_t^2|\mathcal{F}_{t-1}) = \sigma_t^2$, $E(u_t^2) = \sigma^2$.

C2 $\sigma_t^2 = \alpha_0 + \alpha_1 u_{t-1}^2, \qquad \alpha_0, \alpha_1 > 0$ and $\alpha_1 < 1$.

C3 $E\left((u_t^2 - \sigma_t^2)^2 |\mathcal{F}_{t-1}\right) = \mu_t^c$, $E\left((u_t^2 - \sigma_t^2)^2\right) = \mu_4$.

C4 $E(u_t^3|\mathcal{F}_{t-1}) = 0$.

It is proposed to estimate (7) by least squares, giving

$$\widehat{a} = \sum_{t=1}^{n} X_{t-1}X_t / \sum_{t=1}^{n} X_{t-1}^2,$$

and (conditional) generalized least squares, giving

$$\widetilde{a} = \sum_{t=1}^{n} \frac{X_{t-1}X_t}{\sigma_t^2} / \sum_{t=1}^{n} \frac{X_{t-1}^2}{\sigma_t^2}.$$

1. Show that

$$n^{-\frac{1}{2}} \sum_{t=1}^{[nr]} \begin{bmatrix} u_t \\ u_t/\sigma_t^2 \end{bmatrix}$$

   satisfies a functional central limit theorem and obtain the limit process.

2. Find the limit distributions of $n(\widehat{a}-1)$ and $n(\widetilde{a}-1)$.

3. Perform a simulation experiment to compare the sampling distributions of $n(\widehat{a}-1)$ and $n(\widetilde{a}-1)$ when $n = 100$, $\alpha_1 = 0.8$, $\alpha_0 = 0.4$. Graph kernel estimates of the probability densities of these distributions. Comment on your results.

References

1. Elliot, G., T. J. Rothenberg and J. H. Stock (1996). "Efficient Tests for an Autoregressive Unit Root", Econometrica, 64, 813-836.