Question 1.

Let $W(r)$ be standard Brownian motion on $C[0,1]$ and define the sequence of processes $W_n(r) = W(\frac{[2^n r]}{2^n})$ for $n = 1, 2, \dots$. Let $0 < t \leq 1$, $l = [2^n t]$, and $\Delta_k = W(\frac{k}{2^n}) - W(\frac{k-1}{2^n})$. Prove the following hold as $n \to \infty$:

(a) $\sum_{k \leq l} \Delta_k^3 \overset{a.s}{\to} 0$;

(b) $\sum_{k \leq l} W(\frac{k-1}{2^n}) \Delta_k^2 \overset{a.s}{\to} \int_0^t W(r)dr$;

(c) $\int_0^t W_n(r)^2 dW \overset{a.s}{\to} \frac{1}{3} \left\{ W(t)^3 - 3 \int_0^t W(r)dr \right\}$;

(d) $\int_0^t \{W_n(r)^2 - W(r)^2\}^2 \, dr \overset{a.s.}{\to} 0$.

Question 2.

Consider the model

$$y_t = \beta_x' x_t + y_t^s, \quad x_t' = (t, t^2, ..., t^p) \tag{1}$$

$$y_t^s = \alpha y_{t-1}^s + u_t, \quad \alpha = 1$$

$$u_t = c(L)\varepsilon_t, \quad \varepsilon_t \equiv iid(0, \sigma^2), \quad c(L) = \sum_{i=0}^{\infty} c_i L^i, \quad \sum_{i=0}^{\infty} i^{1/2} |c_i| < \infty,$$

where n is the sample size. The initial value $y_0^s = O_{a.s.}(1)$. Residuals $\widehat{v}_t$ are calculated from the following least squares regression of Δy_t on Δx_t

$$\Delta y_t = \widehat{\beta}' \Delta x_t + \widehat{v}_t$$

and from these residuals the fitted partial sum process $\widehat{y}_t^s = \sum_{i=1}^t \widehat{v}_i$ is calculated.

(a) Find a functional central limit law for the standardised process $n^{-1/2} \widehat{y}_{[nr]}^s$.

(b) Find the limit distribution of the statistic nV_n, where

$$V_n = \frac{\sum_{i=1}^n (\Delta \widehat{y}_t^s)^2}{\sum_{i=1}^n (\widehat{y}_t^s)^2}$$

Question 3.

Consider the model

$$y_t = \beta_0 + \beta_x' x_t + u_t, \quad x_t' = (t, t^2, ..., t^p) \tag{2}$$

$$u_t = \alpha u_{t-1} + v_t, \quad \alpha = 1 + \frac{c}{n}$$

$$v_t = d(L)\varepsilon_t, \quad \varepsilon_t \equiv iid(0,\sigma^2), \quad d(L) = \sum_{i=0}^{\infty} d_i L^i, \quad \sum_{i=0}^{\infty} i^{1/2} |d_i| < \infty,$$

where n is the sample size and c is assumed to be a fixed constant. The initial value $u_0 = O_{a.s.}(1)$.

Our interest is in the estimation of the parameter vector β_x. Two methods of estimation are considered. The first is OLS regression on (2) which yields $\widehat{\beta}_x$. The second is OLS regression of $\widetilde{y}_t$ on $\widetilde{x}_t$, where

$$\widetilde{y}_t = \begin{cases} y_1 & t = 1 \\ y_t - \alpha y_{t-1} & t \geq 2 \end{cases}$$

and

$$\widetilde{x}_t = \begin{cases} x_1 & t = 1 \\ x_t - \alpha x_{t-1} & t \geq 2 \end{cases}$$

and α is taken to be known. The second regression yields the estimator $\widetilde{\beta}_x$.

(a) Find the limit distributions of $\widehat{\beta}_x$ and $\widetilde{\beta}_x$. Let V_{ols} and V_{qd} be the variance matrices of these limit distributions.

(b) Let $R = \det V_{ols}/\det V_{qd}$. Find the limit of R as $c \to -\infty$.