Econ. 553a
Yale University

Peter C. B. Phillips
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Econometrics IV: Time Series Econometrics

Take Home Examination

Answer ALL[1] *Questions*: Any reference material allowed.
Time Allowed: Six weeks
Due Date & Time: Friday 15 January 1999, 12:00 noon.

[1]In the event that you are unable to complete all your answers in the allotted time, please be sure to attempt Questions A and C.

Question A.

The time series X_t is generated by the model

$$X_t = \theta X_{t-1} + u_t, \quad t = 1, ..., n \tag{1}$$

where $|\theta| < 1$ and $X_0 = 0$. The errors u_t form a strictly stationary and ergodic martingale difference sequence with respect to the natural filtration $\mathcal{F}_t$ and satisfy the following conditions (so that u_t is an ARCH(1) error process):

C1 $E(u_t|\mathcal{F}_{t-1}) = 0$, $E(u_t^2|\mathcal{F}_{t-1}) = \sigma_t^2$, $E(u_t^2) = \sigma^2$.

C2 $\sigma_t^2 = \alpha_0 + \alpha_1 u_{t-1}^2, \qquad \alpha_0, \alpha_1 > 0$ and $\alpha_1 < 1$.

C3 $E\left((u_t^2 - \sigma_t^2)^2 |\mathcal{F}_{t-1}\right) = \mu_t^c$, $E\left((u_t^2 - \sigma_t^2)^2\right) = \mu_4$.

Let

$$\widehat{\theta} = \left(\sum_{t=2}^{n} X_{t-1}^2\right)^{-1} \left(\sum_{t=2}^{n} X_{t-1} X_t\right) \tag{2}$$

be the least squares regression estimator of θ in (1). Define the following alternative estimator of θ

$$\widehat{\theta}_g = \left(\sum_{t=2}^{n} X_{t-1}^2/\sigma_t^2\right)^{-1} \left(\sum_{t=2}^{n} X_{t-1} X_t/\sigma_t^2\right).$$

1. Show that $\widehat{\theta}$ is strongly consistent for θ as $n \to \infty$.
2. Find the limit distribution of $\widehat{\theta}$ as $n \to \infty$.
3. Compare this limit distribution with the case where u_t is $iid\,(0, \sigma^2)$.
4. Show that $\widehat{\theta}_g$ is strongly consistent for θ.
5. Find the limit distribution of $\widehat{\theta}_g$.
6. Compare the limit distributions of $\widehat{\theta}_g$ and $\widehat{\theta}$. Is $\widehat{\theta}_g$ more efficient than $\widehat{\theta}$? Is $\widehat{\theta}_g$ asymptotically efficient in some sense?

Question B.

Let the time series $(X_t)_1^n$ be generated by the following linear process

$$X_t = \sum_{i=0}^{\infty} d_i u_{t-i}, \quad \text{with } \sum_{j=0}^{\infty} j^{\frac{1}{2}} |d_j| < \infty, d_0 = 1 \tag{3}$$

and where the innovations u_t constitute a strictly stationary and ergodic sequence of martingale differences satisfying the following conditions:

C1 $E(u_t|\mathcal{F}_{t-1}) = 0$, $E(u_t^2|\mathcal{F}_{t-1}) = \sigma_t^2$, $E(u_t^2) = \sigma^2$.

C2′ $E(u_t^2 u_{t-k}^2) = \mu_{kk}$, for all integer k

C3′ $E(u_t^2 u_{t-k} u_{t-l}) = \mu_{kl}$. for all integers $k \neq l$, and $\sup_l \sum_{k=1}^{\infty} |\mu_{kl}| < \infty$.

An econometrician proposes to use a simple first order autoregression to model X_t and fits the following regression equation by ordinary least squares

$$X_t = \widehat{\theta} X_{t-1} + \widehat{v}_t,$$

where $\widehat{\theta}$ is defined in (2) above.

1. Find the almost sure limit of $\widehat{\theta}$ as $n \to \infty$.

2. Find the limit distribution of $\widehat{\theta}$ as $n \to \infty$.

3. Show that when $d_i = \theta^i$ (i.e. the econometrician's AR (1) model for the conditional mean is satisfied) and u_t satisfies **C2,** this limit distribution reduces to your result in Question A.

4. Show that when $X_t = u_t$ and u_t satisfies **C2** (i.e. the data form a martingale difference and follow an ARCH(1) process) then

$$\sqrt{n}\widehat{\theta} \Rightarrow N(0, \mu_{11}/\sigma^4)$$

   where

$$\mu_{11} = \alpha_1 E(u_t^2 - \sigma^2)^2 + \sigma^4 > \sigma^4,$$

   so that the limit distribution is not standard normal in general.

5. Use the results obtained above to suggest a procedure for testing that X_t is a martingale difference (i.e. test the null hypothesis that $X_t = u_t$ is serially uncorrelated).

Question C.

Set up a dynamic trend regression model for the logarithm of real GDP in the form

$$X_t = \sum_{j=1}^{p} a_j X_{t-j} + \sum_{k=-1}^{q} b_k t^k + u_t,$$

where u_t is taken to be *iid* $(0, \sigma^2)$ and $b_{-1} = 0$, so that $q = -1$ corresponds to the case of no fitted intercept.

Collect quarterly real GDP data for New Zealand from 1983:1 - 1998:1. The data may be downloaded from my web site at the following URL: http://korora.econ.yale.edu (check under the heading 'data' in the left panel).

1. Using the data over the period 1983:1-1995:4 and the BIC and PIC model selection criteria (see Phillips, 1996, and Phillips & Ploberger, 1994) choose the order parameters (p, q) in this model. You may set the maximum values of the order parameters at the values $p_{\max} = 4$ and $q_{\max} = 2$. Give a graphical representation of the BIC and PIC surfaces to show how well determined these parameters are by the data.

2. Using the same subsample of data, fit the selected models as well as the model with $(p_{\max}, q_{\max})$ by least squares regression. Report the results and provide some analysis of your findings.

3. Use these three models to make *ex ante* forecasts of real GDP from 1996:1-1998:1. You may compute forecasts $h-$periods ahead with $h \leq 6$. You may also reestimate your model (and even the model orders, if you wish) with the latest data in making your forecasts. (That is, in forecasting 1-period ahead, for example, you may use data to period n in forecasting period $n+1$ and you may, if you wish, reestimate the model orders as you proceed recursively through the data from 1996:1-1998:1.) Be clear in your discussion about how you have proceeded.

4. Evaluate the forecasts from the three models in terms of their root mean squared forecast errors.

5. Discuss your findings.

Write your answers to the above as a scientific paper, paying attention to the quality of your presentation. Be sure to provide a full discussion of the methods being used and indicate limitations of the approach you are using wherever you think it is appropriate.

References

1. Phillips, P.C.B (1996): Econometric model determination, *Econometrica,* **64**: 763–812.

2. Phillips, P. C. B. and W. Ploberger (1994). "Posterior odds testing for a unit root with data–based model selection," *Econometric Theory* 10, 774–808.