94.2.6. *Convergence of a Nonlinear Time Series Model*—Solution, proposed by Peter C.B. Phillips. Write $X_t = (\frac{1}{2})(1 + \eta_t)X_{t-1}$, where $\eta_t \equiv$ iid(0,1), and let Z_t be defined by

$$Z_t = (\tfrac{1}{2})^{1/2}(1 + \eta_t)Z_{t-1}, \qquad t = 1,2,\ldots,$$

with $Z_0 = X_0$. Then,

$$X_t = \prod_{j=1}^{t} \left\{ \left(\frac{1}{2}\right)(1 + \eta_j) \right\} X_0 = \left(\frac{1}{2}\right)^{t/2} \prod_{j=1}^{t} \left\{ \left(\frac{1}{2}\right)^{1/2} (1 + \eta_j) \right\} X_0 = \left(\frac{1}{2}\right)^{t/2} Z_t.$$

Because X_0 is independent of the η_j and has zero mean, we have $E(Z_t) = 0$. Next, note that

$$E(Z_t^2 \mid \mathcal{F}_{t-1}) = E\{(\tfrac{1}{2})^{1/2}(1 + \eta_t)\}^2 Z_{t-1}^2 = Z_{t-1}^2,$$

so that Z_t^2 is a martingale. Furthermore,

$$E(Z_t^2) = E\{E(Z_t^2 \mid \mathcal{F}_0)\} = E(Z_0^2) = \sigma^2,$$

and hence $\sup_t E(Z_t^2) = \sigma^2 < \infty$. Thus, Z_t^2 is a martingale with uniformly bounded first moment. By the martingale convergence theorem, Z_t^2 converges almost surely as $t \to \infty$. It follows that Z_t converges almost surely; i.e., the infinite product

$$Z_\infty = \prod_{j=1}^{\infty} \{(\tfrac{1}{2})^{1/2}(1 + \eta_j)\} Z_0$$

is convergent almost surely. Hence,

$$X_t = (\tfrac{1}{2})^{t/2} Z_t \to_{\text{a.s.}} 0,$$

as required.