92.2.7. *Simultaneous Equations Bias in Level VAR Estimation*—Solution, proposed by P.C.B. Phillips.

(a) Note that the first equation may be rewritten as

$$\begin{aligned} y_{1t} &= a_{12} y_{2t-1} + u_{1t} = a_{12}(y_{2t} - u_{2t}) + u_{1t} \\ &= a_{12} y_{2t} + u_{1t} - a_{12} u_{2t} \\ &= a_{12} y_{2t} + v_t, \text{ say.} \end{aligned} \tag{1}$$

Hence

$$y_{1t-1} = a_{12} y_{2t-1} + v_{t-1}$$

and we may write

$$y_{1t} = a_{11} y_{1t-1} + a_{12} y_{2t-1} + u_{1t} \tag{2}$$

$$= a_{11} v_{t-1} + a_{12}(1 + a_{11}) y_{2t-1} + u_{1t}. \tag{3}$$

Since $a_{11} = 0$, (2) is actually equivalent to

$$y_{1t} = a_{11} v_{t-1} + a_{12} y_{2t-1} + u_{1t}. \tag{4}$$

Hence, OLS on (2) is equivalent to OLS on (4). We have, in usual regression notation,

$$\begin{bmatrix} n(\hat{a}_{12} - a_{12}) \\ \sqrt{n}\hat{a}_{11} \end{bmatrix} = \begin{bmatrix} n^{-2} Y_2' Y_2 & n^{-3/2} Y_2' V \\ n^{-3/2} V' Y_2 & n^{-1} V' V \end{bmatrix}^{-1} \begin{bmatrix} n^{-1} Y_2' u_1 \\ n^{-1/2} V' u_1 \end{bmatrix}.$$

Now, using the methods in Phillips [1,2] we obtain the following limits in a straightforward way:

$$n^{-2} Y_2' Y_2 \to_d \int_0^1 B_2^2$$

$$n^{-1} Y_2' u_1 \to_d \int_0^1 B_2 \, dB_1$$

where

$$n^{-1/2} \Sigma_1^{[n \cdot]} u_t \to_d B(\cdot) = (B_1(\cdot), B_2(\cdot))' = BM(\Sigma)$$

is vector Brownian motion with covariance matrix Σ.

Moreover,

$$n^{-1}V'V \to_p \sigma_v^2 = \text{var}(v_t) = \sigma_{11} - 2a_{12}\sigma_{12} + a_{12}^2\sigma_{22}$$

$$n^{-3/2}V'Y_2 \to_p 0$$

$$n^{-1/2}V'u_1 \to_d N(0, \sigma_v^2\sigma_{11}),$$

the latter by the martingale central limit theorem. It now follows from (5) that

$$n(\hat{a}_{12} - a_{12}) \to_d \left(\int_0^1 B_2^2\right)^{-1} \int_0^1 B_2\, dB_1 \tag{6}$$

and

$$n^{1/2}\hat{a}_{11} \to_d N(0, \sigma_{11}/\sigma_v^2).$$

Further, these limit distributions are independent because $v_{t-1}u_{1t}$ is uncorrelated with u_t, ensuring that the limit variates from partial sums of these processes are independent.

(b) The limit distribution (6) suffers from "simultaneous equations bias" because the Brownian motions B_1 and B_2 are correlated if $\sigma_{12} \neq 0$. We may write, using Lemma 3.1 of Phillips [2],

$$B_1(\cdot) = (\sigma_{12}/\sigma_{22})B_2(\cdot) + B_{1\cdot2}(\cdot), \tag{7}$$

where $B_{1\cdot2}(\cdot)$ is $BM(\sigma_{11\cdot2})$, with $\sigma_{11\cdot2} = \sigma_{11} - \sigma_{12}^2/\sigma_{22}$, and is independent of $B_2(\cdot)$. Using (7) in (6) we have

$$\left(\int_0^1 B_2^2\right)^{-1} \int B_2\, dB_1 = (\sigma_{12}/\sigma_{22})\left(\int_0^1 B_2^2\right)^{-1} \int_0^1 B_2\, dB_2 + \left(\int_0^1 B_2^2\right)^{-1} \int B_2\, dB_{1\cdot2}. \tag{8}$$

The first term here, viz., $(\sigma_{12}/\sigma_{22})(\int_0^1 B_2^2)^{-1}(\int_0^2 B_2\, dB_2)$, is the "simultaneous equations bias" and represents the extent to which the limit distribution is miscentered. The second component on the right side of (8) is the distribution of an "optimal" estimate of the cointegrating coefficient a_{12} and is a symmetric mixed normal distribution. See Phillips [1] for further discussion.

(c) OLS on the restricted model $y_{1t} = a_{12}y_{2t-1} + u_{1t}$ leads to the estimate

$$\tilde{a}_{12} = (y_2'y_2)^{-1}y_2'u_1 + a_{12}$$

and

$$n(\tilde{a}_{12} - a_{12}) \to_d \left(\int_0^1 B_2^2\right)^{-1}\left(\int_0^1 B_2\, dB_1\right), \tag{9}$$

which is identical to (8). Note that although $E(y_{2t-1}u_{1t}) = 0$ (since u_t is serially independent), it is not true that the sample covariance $n^{-1}\Sigma_1^n y_{2t-1}u_{1t}$ converges to zero. In fact, as in (9) above,

$$n^{-1}\Sigma_1^n y_{2t-1}u_{1t} \to_d \int_0^1 B_2\, dB_1,$$

and the respective limit processes B_2 and B_1 that arise from $n^{-1/2}y_{2t-1}$ and $n^{-1/2}\Sigma u_{1t}$ are correlated. This is the source of the "simultaneous equations bias" that is present in OLS estimation of cointegrating relations (see Phillips and Durlauf [3] and Stock [4]). As our example shows, unrestricted VAR estimation suffers from the same bias. Again, see Phillips [1] for more discussion.

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