92.1.5. *Limit Theory in Cointegrated Vector Autoregressions* – Solution, proposed by Peter C.B. Phillips and Hiro Y. Toda.

(a) We start by writing

$$y_{2t} = y_{1t-1} + y_{3t-1} + u_{2t} = y_{1t} + y_{3t} - (u_{1t} - u_{3t}) + u_{2t}$$
$$= y_{1t} + y_{3t} + v_t, \text{ say.}$$

So y_t is cointegrated with cointegrating vector $(-1,1,-1)$ and v_t is the stationary deviation from equilibrium. Next write the first equation of the VAR as

$$y_{1t} = a_{11}y_{1t-1} + a_{12}y_{2t-1} + a_{13}y_{3t-1} + u_{1t}$$
$$= (a_{11} + a_{12})y_{1t-1} + a_{12}v_{t-1} + (a_{13} + a_{12})y_{3t-1} + u_{1t} \tag{1}$$
$$= \underline{a}_{11}y_{1t-1} + \underline{a}_{12}v_{t-1} + \underline{a}_{13}y_{3t-1} + u_{1t}, \text{ say.} \tag{2}$$

Observe that (1) and (2) are equivalent and in the true model $a_{12} = a_{13} = 0$ so that $\underline{a}_{1j} = a_{1j}$. Let $\underline{a} = (\underline{a}_{12}, \underline{a}_{13})'$, $x_t = (v_{t-1}, y_{3t-1})'$ and note that the OLS estimator of $\underline{a}$ is, in usual regression notation,

$$\hat{\underline{a}} = (X'Q_1X)^{-1}(X'Q_1u_1), \tag{3}$$

since $\underline{a} = 0$ in the true model. Next observe that, if $D_n = \text{diag}(n^{1/2}, n)$,

$$D_n\hat{\underline{a}} = (D_n^{-1}X'Q_1XD_n^{-1})^{-1}D_n^{-1}X'Q_1u_1.$$

Now

$$D_n^{-1}X'XD_n^{-1} = \begin{bmatrix} n^{-1}v'v & n^{-3/2}v'y_3 \\ n^{-3/2}y_3'v & n^{-2}y_3'y_3 \end{bmatrix} \to_d \begin{bmatrix} \sigma_v^2 & 0 \\ 0 & \int_0^1 B_3^2 \end{bmatrix} \tag{4}$$

where $\sigma_v^2 = \text{var}(v_t)$ and

$$(n^{-1/2}y_{1[n\cdot]}, n^{-1/2}y_{3[n\cdot]}) \to_d (B_1(\cdot), B_3(\cdot)) \equiv BM\left(\begin{bmatrix} \sigma_{11} & \sigma_{13} \\ \sigma_{31} & \sigma_{33} \end{bmatrix}\right)$$

by the invariance principle. Also note that

$$\begin{aligned} &D_n^{-1}X'y_1(y_1'y_1)^{-1}y_1'X'D_n^{-1} \\ &\quad = (D_n^{-1}X'y_1 n^{-1})[n^{-2}y_1'y_1]^{-1}(n^{-1}y_1'XD_n^{-1}) \\ &\quad \to_d \begin{bmatrix} 0 & 0 \\ 0 & \int_0^1 B_3B_1\left(\int_0^1 B_1^2\right)^{-1}\int_0^1 B_1B_3 \end{bmatrix}, \end{aligned} \tag{5}$$

and

$$\begin{aligned} D_n^{-1}X'Q_1u_1 &= \begin{bmatrix} n^{-1/2}v'u_1 - (n^{-1/2}v'y_1n^{-1})[n^{-2}y_1'y_1]^{-1}(n^{-1}y_1'u_1) \\ n^{-1}y_3'u_1' - (n^{-1}y_3'y_1n^{-1})[n^{-2}y_1'y_1]^{-1}(n^{-1}y_1'u_1) \end{bmatrix} \\ &\to_d \begin{bmatrix} N(0, \sigma_{11}\sigma_v^2) \\ \int_0^1 B_3\,dB_1 - \int_0^1 B_3B_1\left(\int_0^1 B_1^2\right)^{-1}\int_0^1 B_1\,dB_1 \end{bmatrix} \end{aligned} \tag{6}$$

where the first and second components of the limit vector are independent since $v_{t-1}u_{1t}$ is uncorrelated with (u_{1t}, u_{3t}).

Next define the L^2 projection residual

$$\underline{B}_3(\cdot) = B_3(\cdot) - \int_0^1 B_3B_1\left(\int_0^1 B_1^2\right)^{-1} B_1(\cdot) \tag{7}$$

and, using this process, we deduce from (3), (4), (5), and (6) that

$$D_n\hat{a} \to_d \begin{bmatrix} N(0, \sigma_{11}/\sigma_v^2) \\ \left(\int_0^1 \underline{B}_3^2\right)^{-1}\int_0^1 \underline{B}_3\,dB_1 \end{bmatrix} \tag{8}$$

which has independent components and is the required limit distribution.

We now use the decomposition

$$B_3(\cdot) = (\sigma_{31}/\sigma_{11})B_1(\cdot) + B_{3\cdot 1}(\cdot) \tag{9}$$

where $B_{3\cdot 1}(\cdot) \equiv BM(\sigma_{33} - \sigma_{31}^2/\sigma_{11}) \equiv BM(\sigma_{33\cdot 1})$ is independent of B_1. Employing (9) in (7) we have

$$\begin{aligned} \underline{B}_3(\cdot) &= B_{3\cdot 1}(\cdot) - \int_0^1 B_{3\cdot 1}B_1\left(\int_0^1 B_1^2\right)B_1(\cdot) \\ &= \underline{B}_{3\cdot 1}(\cdot), \text{ say.} \end{aligned} \tag{10}$$

It follows that

$$\left(\int_0^1 \underline{B}_3^2\right)^{-1}\int_0^1 \underline{B}_3\,dB_1 = \left(\int_0^1 \underline{B}_{3\cdot1}^2\right)^{-1}\int_0^1 \underline{B}_{3\cdot1}\,dB_1.$$

(b) The Wald test of the noncausality hypothesis $\mathcal{H}_0: a_{12} = a_{13} = 0$ is given by

$$\begin{aligned}
W &= \hat{a}'(X'Q_1X)\hat{a}/\hat{\sigma}_{11}\\
&= u_1'Q_1X(X'Q_1X)^{-1}X'Q_1u_1/\hat{\sigma}_{11}\\
&= u_1'Q_1XD_n^{-1}[D_n^{-1}X'Q_1XD_n^{-1}]^{-1}D_n^{-1}X'Q_1u_1/\hat{\sigma}_{11}\\
&\to_d \left[N(0,\sigma_{11}\sigma_v^2), \int_0^1 \underline{B}_3\,dB_1\right]\begin{bmatrix}\sigma_v^2 & 0\\ 0 & \int_0^1 \underline{B}_3^2\end{bmatrix}^{-1}\begin{bmatrix}N(0,\sigma_{11}\sigma_v^2)\\ \int_0^1 \underline{B}_3\,dB_1\end{bmatrix}\Bigg/\sigma_{11}\\
&= \chi_1^2 + \int_0^1 dB_1\underline{B}_3\left(\int_0^1 \underline{B}_3^2\right)^{-1}\int_0^1 \underline{B}_3\,dB_1/\sigma_{11},
\end{aligned} \tag{11}$$

which is the required limit distribution.

Now write $B_1(\cdot) = \sigma_{11}^{1/2}W_1(\cdot)$ and $B_3(\cdot) = \sigma_{33}^{1/2}W_3(\cdot)$. Then (9) becomes

$$\begin{aligned}
W_3(\cdot) &= \frac{\sigma_{31}}{\sigma_{33}^{1/2}\sigma_{11}^{1/2}}W_1(\cdot) + (\sigma_{33\cdot1}/\sigma_{33})^{1/2}W_{3\cdot1}(\cdot)\\
&= \rho_{31}W_1(\cdot) + (1-\rho_{31}^2)^{1/2}W_{3\cdot1}(\cdot)
\end{aligned}$$

with $W_{3\cdot1}$ independent of W_1. Define

$$\begin{aligned}
\underline{W}_3(\cdot) &= W_3(\cdot) - \int_0^1 W_3W_1\left(\int_0^1 W_1^2\right)^{-1}W_1(\cdot)\\
&= (1-\rho_{31}^2)^{1/2}\left\{W_{3\cdot1}(\cdot) - \int W_{3\cdot1}W_1\left(\int_0^1 W_1^2\right)^{-1}W_1(\cdot)\right\}\\
&= (1-\rho_{31}^2)^{1/2}\underline{W}_{3\cdot1}(\cdot)
\end{aligned} \tag{12}$$

and then the limit variate (11) can also be written in the form

$$\chi_1^2 + \int_0^1 dW_1\underline{W}_{3\cdot1}\left(\int_0^1 W_{3\cdot1}^2\right)\int_0^1 \underline{W}_{3\cdot1}\,dW_1. \tag{13}$$

(c) From the form of (13) it is apparent that the limit distribution of the Wald test is a mixture of a chi-squared one variate and a nonstandard distribution. Note that the limit distribution is free of nuisance parameters since $W_1(\cdot)$ and $\underline{W}_{3\cdot1}$ depend only on standard Brownian motions. Note, however, that there is dependence between W_1 and $\underline{W}_{3\cdot1}$

since the latter depends on W_1 from its definition in (12). Finally, observe that this nonstandard limit distribution applies, even though both excluded variables y_2 and y_3 in equation one are cointegrated. In effect, there is "insufficient" cointegration to ensure a chi-squared limit theory. The reader is referred to Toda and Phillips (1991) for a full development of the theory of causality tests.

REFERENCE

Toda, H.Y. & P.C.B. Phillips (1991). Vector Autoregressions and Causality. Cowles Foundation Discussion Paper No. 977.