90.4.7. *The Geometry of the Equivalence of OLS and GLS in the Linear Model*, proposed by Peter C.B. Phillips. Let

$$y = X\beta + u \tag{1}$$

be the general linear model with fixed regressor matrix X $(n \times k)$ and error vector u whose mean is zero and whose covariance matrix is Ω (possibly singular). Let $\mathcal{R}(\cdot)$ denote the range space of the argument matrix. Throughout, we assume that $\operatorname{rank}(\Omega) = r \leq n$ and $\mathcal{R}(X) \subset \mathcal{R}(\Omega)$. We let Ω^+ denote the Moore Penrose inverse of Ω, set $\mathcal{M} = \mathcal{R}(X)$ and use $\mathcal{M}_\Omega^\perp$ to signify that part of the orthogonal complement of $\mathcal{M}$ in $\mathcal{R}(\Omega)$.

Define

$$\mathcal{M}_\Omega^c = \{\eta \in \mathcal{R}(\Omega) \mid \eta'\Omega^+\epsilon = 0,\ \forall \epsilon \in \mathcal{M}\}$$

and call this the subspace of $\mathcal{R}(\Omega)$ that is Ω-conjugate to $\mathcal{M}$, following the related terminology in Malinvaud ([2], p. 168).

Prove the following two results, using (i) to establish (ii).

(i) Lemma
   (a) $\mathcal{M}_\Omega^c = \Omega(\mathcal{M}_\Omega^\perp)$,
   (b) $[\Omega(\mathcal{M}_\Omega^\perp)]_\Omega^\perp = \Omega^+\mathcal{M}$.

(ii) Theorem (Kruskal [1])

OLS = GLS on (1)

iff

$\Omega^+\mathcal{M} = \mathcal{M}$

iff

$$\Omega\mathfrak{M} = \mathfrak{M}.$$

Comment on these results.

REFERENCES

1. Kruskal, W. When are Gauss Markov and least squares estimators identical: A coordinate free approach. *Annals of Mathematical Statistics*, 39 (1968): 70–75.
2. Malinvaud, E. *Statistical methods of econometrics*. Amsterdam: North Holland, 1980.