91.4.3. *Testing for Stationarity in the Components Representation of a Time Series*—Solution, proposed by D. Kwiatkowski, P.C.B. Phillips, and P. Schmidt. (a) Note that $r_t = \Sigma_1^t v_j$ and set $w_t = r_t + u_t$ so that the model can be written as

$$y_t = x_t\gamma + w_t; \qquad \gamma' = (\gamma_0, \gamma_1),\ x_t' = (1, t)$$

or in observation format as

$y = X\gamma + w.$

Now $E(w) = 0$ and

$$\begin{aligned}\text{var}(w) &= \text{var}(u) + \text{var}(r)\\ &= \sigma_u^2 I_n + \sigma_v^2 LL' = \sigma_u^2 I_n + \sigma_v^2 A\end{aligned}$$

where

$$L = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ 1 & 1 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & 1 & \cdots & 1 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 2 & \cdots & 2 \\ 1 & 2 & 3 & \cdots & 3 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 2 & 3 & \cdots & n \end{bmatrix}$$

$= \Omega(\sigma_u^2, \sigma_v^2)$, say.

The log likelihood is then

$$L(\gamma, \sigma_u^2, \sigma_v^2; y) = -\frac{n}{2}\ln 2\pi - \frac{1}{2}\ln|\Omega| - \frac{1}{2}(y - X\gamma)'\Omega^{-1}(y - X\gamma)$$

(b) $\partial L/\partial\sigma_v^2 = -\frac{1}{2}\,\text{tr}(\Omega^{-1}A) + \frac{1}{2}(y - X\gamma)'\Omega^{-1}A\Omega^{-1}(y - X\gamma)$ and then

$\tilde{\lambda} = \partial L(\tilde{\gamma}, \tilde{\sigma}_u^2, \tilde{\sigma}_v^2 = 0)/\partial\sigma_v^2 = (1/2\tilde{\sigma}_u^2)\,\text{tr}(A) + (1/2\tilde{\sigma}_u^4)(y - X\tilde{\gamma})'A(y - X\tilde{\gamma})$

where $\tilde{\gamma}, \tilde{\sigma}_u^2$ are the restricted ML estimates, that is, the OLS estimates of $y = X\gamma + w$. Write $\tilde{u} = y - X\tilde{\gamma}$ and then we have

$$\tilde{\lambda} = -\frac{1}{2\tilde{\sigma}_u^2}\,\text{tr}(A) + \frac{1}{2\tilde{\sigma}_u^4}\,\tilde{u}'A\tilde{u}.$$

The LM test of

$H_0: \sigma_v^2 = 0$

is based on $\tilde{\lambda}$. We can construct a "studentized test" based on $\tilde{\lambda}$ and an estimate of its standard error. Note that

$$\text{var}\left(\frac{1}{2\sigma_u^4}\,u'Au\right) = \left(\frac{1}{4\sigma_u^8}\right)2\sigma_u^4\,\text{tr}(A^2), \qquad \text{under normality.} \tag{1}$$

We set

$$\begin{aligned} LM_1 &= \frac{\tilde{\lambda}}{[(1/2\sigma_u^4)\text{tr}(A^2)]^{1/2}} \\ &= \frac{\tilde{u}'A\tilde{u}}{2^{1/2}\tilde{\sigma}_u^2(\text{tr}(A^2))^{1/2}} - \frac{\text{tr}(A)}{2^{1/2}(\text{tr}(A^2))^{1/2}}. \end{aligned} \tag{2}$$

Equivalently, we may work with

$$LM_2 = \frac{\tilde{u}'A\tilde{u}}{\tilde{\sigma}_u^2}$$

(removing the fixed term and scale coefficient of (2)).

Next note that

$$\tilde{u}'A\tilde{u} = \tilde{u}'LL'\tilde{u} = \sum_1^n \tilde{S}_{t-1}^2$$

where $\tilde{S}_t = \sum_1^t \tilde{u}_j$. Hence we have the representation

$$LM_2 = \frac{\sum_1^n \tilde{S}_{t-1}^2}{\tilde{\sigma}_u^2}. \tag{3}$$

(c) Under the null

$$n^{-1/2}S_{[nr]} = n^{-1/2}\sum_1^{[nr]} u_j \Rightarrow B(r) = BM(\sigma_u^2)$$

whereas

$$\begin{aligned} n^{-1/2}\tilde{S}_{[nr]} &= n^{-1/2}\sum_1^{[nr]} \tilde{u}_j \\ &= n^{-1/2}\sum_1^{[nr]} u_j - \left(n^{-1/2}\sum_1^{[nr]} x_t'\right)(X'X)^{-1}X'u. \end{aligned}$$

Now $x_t' = (1, t)$ and setting

$$D_n = \begin{pmatrix} n^{1/2} & 0 \\ 0 & n^{3/2} \end{pmatrix}$$

we have

$$D_n^{-1}X'XD_n^{-1} = \begin{bmatrix} 1 & \sum_1^n t/n^2 \\ \sum_1^n t/n^2 & \sum t^2/n^3 \end{bmatrix} \to \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3} \end{bmatrix}.$$

$$D_n^{-1}X'u = \begin{pmatrix} \frac{1}{\sqrt{n}}\sum u_t \\ \frac{1}{n^{3/2}}\sum tu_t \end{pmatrix} \Rightarrow \begin{pmatrix} \int_0^1 dB \\ \int_0^1 r\,dB \end{pmatrix}.$$

Hence,

$$n^{-1/2}\tilde{S}_{[nr]} = n^{-1/2}S_{[nr]} - \left(\frac{[nr]}{n}, \frac{\sum_{1}^{[nr]} t}{n^2} \right) (D_n^{-1}X'XD_n^{-1})^{-1}D_n^{-1}X'u$$

$$\Rightarrow B(r) - \left[r, \frac{r^2}{2} \right] \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3} \end{bmatrix}^{-1} \begin{pmatrix} \int_0^1 dB \\ \int_0^1 r\,dB \end{pmatrix}$$

$$\Rightarrow \tilde{B}_2(r), \text{ say.} \tag{4}$$

We also obtain

$$\frac{1}{n^2}\sum_{1}^{n} \tilde{S}_{t-1}^2 = \frac{1}{n}\sum_{1}^{n}\left(\frac{\tilde{S}_{t-1}}{\sqrt{n}}\right)^2 \Rightarrow \int_0^1 \tilde{B}_2(r)^2\,dr.$$

Hence

$$n^{-2}LM_2 = n^{-2}\sum_{1}^{n} \tilde{S}_{t-1}^2/\tilde{\sigma}_u^2$$

$$\Rightarrow \int_0^1 \tilde{B}_2(r)^2\,dr/\sigma_u^2$$

$$= \int_0^1 \tilde{W}_2(r)^2\,dr$$

since $\tilde{B}_2(r) = \sigma_u \tilde{W}_2(r)$. Note that $\tilde{W}_2(r)$, which is defined in the same way as (4), is free of nuisance parameters.

Remark. Observe that in the LM_1 form we have $(2\,\mathrm{tr}\,A^2)^{1/2}$ in the denominator. Now

$$\mathrm{tr}(A^2) = \mathrm{tr}\left\{ \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 2 & \cdots & 2 \\ 1 & 2 & 3 & \cdots & 3 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 2 & 3 & \cdots & n \end{bmatrix}^2 \right\}$$

$$= n1^2 + (n-1)2^2 + (n-2)3^2 + \cdots + n^2 + (n-1)1^2$$
$$+ (n-2)2^2 + \cdots + 1(n-1)^2$$

(continued)

$$= (2n-1)1^2 + (2n-3)2^2 + (2n-5)3^2 + \cdots + 2(n-1)^2 + n^2$$

$$= \sum_{k-1}^{n} [2n - (2k-1)]k^2$$

$$= 2n\sum_1^n k^2 - 2\sum_1^n k^3 + \sum_1^n k^2$$

$$= 2nn(n+1)(2n+1)/6 - 2[n(n+1)/2)]^2 + n(n+1)(2n+1)/6$$

$$= \frac{n(n+1)}{6}[2n(2n+1) - 3n(n+1) + 2n + 1]$$

$$= \frac{n(n+1)}{6}[n^2 + n + 1].$$

That is, $\operatorname{tr}(A^2) \sim n^4/6$ and $2\operatorname{tr}(A^2) \sim n^4/3$ as $n \to \infty$. So

$$LM_1 \sim \frac{\tilde{u}'A\tilde{u}}{\tilde{\sigma}_u^2(2\operatorname{tr}A^2)^{1/2}} - \frac{\operatorname{tr}(A)}{(2\operatorname{tr}A^2)^{1/2}}$$

$$\sim \frac{3^{1/2}\tilde{u}'A\tilde{u}}{n^2\tilde{\sigma}_u^2} - \frac{3^{1/2}n(n+1)/2}{n^2}$$

$$\sim 3^{1/2}\left[\frac{1}{\sigma_u^2 n^2}\sum_1^n \tilde{S}_{t-1}^2 - \frac{1}{2}\right]$$

$$\Rightarrow 3^{1/2}\left[\int_0^1 \widetilde{W}_2^2(r)\,dr - \frac{1}{2}\right].$$

Thus, the factor $1/(2\operatorname{tr}(A^2))^{1/2}$ gives the right normalization, as a power of n, in standardizing $\tilde{u}'A\tilde{u}$.

(d) The normality assumption affects the likelihood and the LM statistic in consequence. Note also that the variance formula (1) relies on normality, otherwise 4'th moments would be involved.

However, the limit theory for

$$\frac{1}{n^2}LM_2 = \frac{1}{n^2}\sum_1^n \frac{\tilde{S}_{t-1}^2}{\tilde{\sigma}_u^2}$$

$$\Rightarrow \int_0^1 \widetilde{W}_2^2(r)\,dr$$

is invariant to the normality assumption so that the statistic $n^{-2}LM_2$ based on the calculation (1) is in this sense robust.

Remark. The reader is referred to Kwiatkowski, Phillips, and Schmidt [1] for a theoretical development and empirical application of the LM test derived herein.

REFERENCES

1. Kwiatkowski, D., P.C.B. Phillips & P. Schmidt. "Testing the null hypothesis of stationarity against the alternative of a unit root: How sure are we that economic time series have a unit root," mimeographed, Michigan State University, 1990.