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Box 208281
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By Peter C. B. Phillips, Shuping Shi, and Jun Yu

Yale University, U.S.A, University of Auckland, New Zealand, University of Southampton, U.K., and Singapore Management University, Singapore; Macquarie University and CAMA, Australia; Singapore Management University, Singapore

This article provides the limit theory of real-time dating algorithms for bubble detection that were suggested in Phillips, Wu, and Yu (PWY; *International Economic Review* 52 [2011], 201–26) and in a companion paper by the present authors (Phillips, Shi, and Yu, 2015; PSY; *International Economic Review* 56 [2015a], 1099–1134. Bubbles are modeled using mildly explosive bubble episodes that are embedded within longer periods where the data evolve as a stochastic trend, thereby capturing normal market behavior as well as exuberance and collapse. Both the PWY and PSY estimates rely on recursive right-tailed unit root tests (each with a different recursive algorithm) that may be used in real time to locate the origination and collapse dates of bubbles. Under certain explicit conditions, the moving window detector of PSY is shown to be a consistent dating algorithm even in the presence of multiple bubbles. The other algorithms are consistent detectors for bubbles early in the sample and, under stronger conditions, for subsequent bubbles in some cases. These asymptotic results and accompanying simulations guide the practical implementation of the procedures. They indicate that the PSY moving window detector is more reliable than the PWY strategy, sequential application of the PWY procedure, and the CUSUM procedure.

1. INTRODUCTION

A recent article by Phillips, Wu, and Yu (2011; PWY) developed new econometric methodology for real-time bubble detection. When it was applied to NASDAQ data in the 1990s, the algorithm revealed that evidence in the data supported Greenspan's declaration of "irrational exuberance" in December 1996 and that this evidence of market exuberance had existed for some 16 months prior to that declaration. Greenspan's remark therefore amounted to an assertion that could have been evidence-based if the test had been conducted at the time.

Greenspan formulated his comment as a question: "How do we know when irrational exuberance has unduly escalated asset values?" It was this very question that the recursive test procedure in PWY was designed to address. Correspondingly, an element of the methodology that is critical for empirical applications and policy assessment is the consistency of the test. Ideally we want a test whose size goes to zero and whose power goes to unity as the sample size passes to infinity. Then in very large samples there will be no false positive declarations of exuberance and no false negative assessments where asset price bubbles are missed.

PWY gave heuristic arguments showing that their recursive methodology produced a consistent test for exuberance, and they provided a real-time dating algorithm for finding the

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bubble origination and termination dates that was used in analyzing the NASDAQ data. The present article provides a rigorous limit theory showing formal test consistency of the PWY bubble detection procedure and the consistency of its associated dating algorithm under certain conditions, notably the existence of a single bubble period in the data.[2] This limit theory is part of a much larger formal investigation undertaken here that examines the asymptotic properties of bubble detection algorithms when there may be multiple episodes of exuberance in the data, under which the PWY procedure does not perform nearly as well. As argued in the authors' companion paper (Phillips, Shi, and Yu, 2015a, hereafter PSY), data over long historical periods often include several crises involving financial exuberance and collapse. Bubble detection in this context of multiple episodes of exuberance and collapse is much more complex and is the main subject of the PSY paper, which develops a new moving window bubble detector that has some substantial advantages for long data series characterized by multiple financial crises.

The dating algorithms of PWY and PSY are now being applied to a wide range of markets that include energy, real estate, and commodities as well as financial assets.[3] This methodology and its various applications have also attracted the attention of central bank economists, fiscal regulators, and the financial press.[4] It is therefore important that the limit properties and performance characteristics of these dating algorithms be well understood to assist in guiding practitioners about the suitable choice of procedures for implementation in empirical work and policy assessment exercises.

The PWY and PSY strategies for bubble detection involve the comparison of a sequence of recursive test statistics with corresponding critical value sequences. Crossing times of these critical value lines provide the corresponding date estimates of bubble origination and termination. The PWY procedure uses recursively calculated right-sided unit root test statistics based on an expanding window of observations up to the current data point, whereas PSY use a moving window recursion of sup statistics based on a sequence of right-sided unit root tests calculated over flexible windows of varying length taken up to the current data point. Inferences from the PWY and PSY strategies about the presence of exuberance in the data, including the dating of any exuberance or collapse, are drawn from these test sequences and the corresponding critical value sequences. The goals of the present article are to explore the asymptotic and finite sample properties of these two procedures for bubble dating and to build a methodology for analyzing real-time detector asymptotics in this context.

Our findings can be summarized as follows. First, under some general conditions both the PWY and PSY detectors are consistent when there is a single bubble in the sample period. Second, when there are two bubbles in the sample period, the PWY detector for the first bubble is consistent, whereas the PWY estimates associated with the second bubble are duration-dependent. Specifically, the PWY strategy fails to detect the existence of the second bubble (and hence cannot provide consistent date estimates for the timing of that bubble) when the first bubble has longer duration than the second. But when the duration of the second bubble exceeds the first, the PWY strategy can detect the second bubble but only with some delay. Third, the PSY strategy and (under additional conditions) a sequential implementation of the PWY strategy (to each individual bubble in turn) do provide consistent detectors for both

[2] The present article therefore subsumes the results contained in the unpublished working paper of Phillips and Yu (2009) which is referenced in PWY and which first analyzed the asymptotic properties of the PWY procedure.

[3] See Phillips and Yu (2011b), Das et al. (2011), Homm and Breitung (2012), Gutierrez (2013), Bohl et al. (2013), Etienne et al. (2013), Greenaway-McGrevy and Phillips (forthcoming), and Jiang et al. (2015), among others.

[4] For example, a *Financial Times* article (Meyer, 2013) reports the work of Etienne et al. (2013), which employs the PSY dating algorithm to identify agricultural commodity bubbles. Recent working papers from the Hong Kong Monetary Authority (Yiu et al., 2013) and the Central Bank of Colombia (Ojeda-Joya et al., 2013), use PSY in studying real estate bubbles in Hong Kong and Columbia. Work for UNCTAD by Gilbert (2010) applies PWY to date bubbles in commodity prices and test congressional testimony reasoning by Masters (2008), and recent financial press articles (Phillips and Yu, 2011a, 2013) use PWY to assess current real estate and world stock market data for evidence of bubbles using these methods.

bubbles, and these results hold irrespective of bubble duration. Thus, the PSY dating algorithm and sequential application of the PWY procedure have desirable asymptotic properties in a multiple bubbles scenario. One disadvantage of sequentially applying the PWY procedure is that sufficient data are needed between bubbles to implement the procedure, and therefore some origination dates may not be consistently estimated if the origination date is excluded from the PWY sample recursion.

The article also reports simulations to evaluate the finite sample performance of these detectors and date estimators, along with an alternative procedure based on CUSUM tests, as proposed in recent work by Homm and Breitung (2012). The simulation results strongly corroborate the asymptotic theory, indicating that the PSY detector is much more reliable than PWY. On the other hand and with some exceptions that will be discussed in detail below, sequential application of the PWY procedure may perform nearly as well as the PSY algorithm. The performance characteristics of the CUSUM procedure are found to be similar to those of PWY. Overall, the results suggest that the PSY detector is a preferred procedure for practical implementation, especially with long data series involving more than one bubble/crisis episode.

The rest of the article is organized as follows. Section 2 introduces the date stamping procedures that use recursive regressions and right-tailed unit root tests of the type considered in PWY and PSY. This section also describes the models used to capture mildly explosive bubble behavior when there are single and multiple bubble episodes in the data. Section 3 derives the limit theory for the dating procedures under both single bubble and multiple bubble alternatives. Finite sample performance is studied in Section 4, and Section 5 concludes. Two appendices contain supporting lemmas and derivations for the limit theory presented in the article covering both single and multiple bubble scenarios. A technical supplement to the article (Phillips et al., 2015b) provides a complete set of additional mathematical derivations that are needed for the limit theory presented here. Computer code and Eviews software are now available for implementation of the methods in the article.[5]

2. BUBBLE DATING ALGORITHMS

This section introduces three different dating algorithms—the original PWY detector, the PSY detector, and a sequential version of the PWY detector. The approach in all of these algorithms is to use recursive right-tailed unit root tests to assess evidence for mildly explosive bubble behavior. In what follows we use the same models, tests, and notation as PSY to assist in cross referencing between the two papers.

The null hypothesis is specified as suggested in Phillips et al. (2014): a random walk (or more generally a martingale) process with an asymptotically negligible drift that we write in the form

$$X_t = kT^{-\eta} + X_{t-1} + \varepsilon_t, \text{ with constant } k \text{ and } \eta > 1/2, \tag{1}$$

where T is the sample size, $\varepsilon_t \overset{i.i.d.}{\sim} (0, \sigma^2)$, and $X_0 = O_p(1)$.[6] Under these simple conditions, partial sums of ε_t satisfy the functional law

$$T^{-1/2} \sum_{t=1}^{\lfloor T\cdot \rfloor} \varepsilon_t \Rightarrow B(\cdot) := \sigma W(\cdot), \tag{2}$$

where W is standard Brownian motion. The framework can be extended to allow for martingale difference sequence and more general weakly dependent innovations under conditions that

[5] Gauss and Matlab codes are available online at https://sites.google.com/site/shupingshi/PrgGSADF.zip?attredirects=0. An Add-In for the Eviews software package is available in Caspi (2013).

[6] See Phillips and Magdalinos (2009) for the impact of alternative initializations on the limit theory.

allow the limit theory to continue to hold under the null (1), based on the functional law (2), and under mildly explosive alternatives as in (4) below, the latter based on results in Phillips and Magdalinos (2007a, 2007b). We maintain the *i.i.d.* error assumption here to keep the exposition as simple as possible and the article to manageable length.

The fitted regression model is

$$\Delta X_t = \alpha + \beta X_{t-1} + \varepsilon_t, \varepsilon_t \overset{i.i.d.}{\sim} \left(0, \sigma^2\right), \tag{3}$$

which includes an intercept but no time trend. As in PSY, the fitted model may also be formulated in ADF regression format to allow for any short memory dependence in the innovations. The results given below continue to hold in that event but full extension to this case will substantially complicate derivations that are already extremely lengthy.

The test alternative is a mildly explosive bubble process with either a single bubble or sequence of multiple bubble episodes. The data-generating processes that are used to capture bubble effects are extended versions of the PWY bubble model. That model has a single explosive episode and collapse within the sample period $[1, T]$ and has the following form:

$$\begin{aligned} X_t = {} & (X_{t-1} + \varepsilon_t) 1\{t < \tau_e\} + (\delta_T X_{t-1} + \varepsilon_t) 1\left\{\tau_e \le t \le \tau_f\right\} \\ & + \left(\sum_{k=\tau_f+1}^{t} \varepsilon_k + X^*_{\tau_f} \right) 1\left\{t > \tau_f\right\}. \end{aligned} \tag{4}$$

As usual, it is convenient to work with fractions of the sample T, and we use the notation $t = \lfloor Tr \rfloor$ to denote the integer part of Tr for $r \in [0, 1]$. In the process (4) a mildly explosive bubble runs from $\tau_e = \lfloor Tr_e \rfloor$ to $\tau_f = \lfloor Tr_f \rfloor$ with an expansion rate determined by the mildly explosive coefficient $\delta_T = 1 + cT^{-\alpha}$ with $c > 0$ and $\alpha \in (0, 1)$. When the bubble terminates, the process collapses to a value $X^*_{\tau_f}$, which equals X_{τ_e} plus an $O_p(1)$ perturbation (i.e., $X^*_{\tau_f} = X_{\tau_e} + X^*$) at period $\tau_f + 1$, which represents a re-initialization of the process to a level that relates to the last pre-bubble observation X_{τ_e}. More general specifications of the collapse process are considered in Phillips and Shi (2014). The pre-bubble period $N_0 = [1, \tau_e)$ and post-bubble period $N_1 = (\tau_f, T]$ are assumed to follow a pure random walk process.

The model is readily extended to include multiple bubble episodes. Suppose there are K bubble episodes in the sample period, represented in terms of sample fraction intervals as $B_i = [\tau_{ie}, \tau_{if}]$ for $i = 1, 2, \ldots, K$. The shifting dynamics of X_t are then given by the model

$$\begin{aligned} X_t = {} & (X_{t-1} + \varepsilon_t) 1\{t \in N_0\} + (\delta_T X_{t-1} + \varepsilon_t) 1\{t \in B_i\} \\ & + \sum_{i=1}^{K} \left(\sum_{l=\tau_{if}+1}^{t} \varepsilon_l + X^*_{\tau_{if}} \right) 1\{t \in N_i\}, \end{aligned} \tag{5}$$

where $X^*_{\tau_{if}} = X_{\tau_{ie}} + X^*_i$ with $X^*_i = O_p(1)$ for all i and the intervening subperiods $N_0 = [1, \tau_{1e})$, $N_j = (\tau_{j-1f}, \tau_{je})$ with $j = 1, \ldots, K-1$, and $N_K = (\tau_{Kf}, T]$ are "normal" intervals of pure random walk (or more generally martingale) evolution.

The dating algorithms studied here are implemented repeatedly for observations starting from some initialization $\lfloor Tr_0 \rfloor$, where r_0 is the minimum window size required to initiate the regression. For each individual observation $t = \lfloor Tr \rfloor$, we suppose that interest centers on whether this particular observation comes from a bubble realization or an interval of normal martingale behavior. Both the PWY and PSY algorithms use data from the same information set that starts from the first observation and goes up to the observation of interest (i.e., $I_r = \{1, 2, \ldots, \lfloor Tr \rfloor\}$).

PWY conduct recursive right-tailed unit root tests with sample data running from the first observation to the current observation $t = \lfloor Tr \rfloor$. The corresponding unit root t-statistic at $t = \lfloor Tr \rfloor$ is denoted DF_r. PSY conduct recursive right-tailed unit root tests repeatedly on a sequence of (backward expanding from observation t) windows of data and perform inference based on the sup value of this t-statistic sequence. Let r_1 and r_2 denote the start and end points of the regression. The regression window width r_w then equals $r_2 - r_1$. With the end point of the regressions r_2 fixed at r (so that the test refers to the state of the process at the current observation $t = \lfloor Tr \rfloor$) and $r_1 \geq 0$, the backward expanding sample sequence extends the window size r_w from r_0 to r_2 (which is equivalent to varying r_1 from 0 to $r_2 - r_0$). The corresponding unit root test sequence is denoted by $\{DF_{r_1}^{r_2}\}_{r_1 \in [0, r_2 - r_0]}$. The sup value of the test statistic sequence is called the backward SDF statistic and is defined as

$$BSDF_r(r_0) = \sup_{r_1 \in [0, r_2 - r_0], r_2 = r} \left\{ DF_{r_1}^{r_2} \right\}.$$

The origination and termination dates of any bubbles that are detected are calculated using the first crossing principle. Specifically, in the single bubble scenario, the origination (termination) date of the bubble is the first chronological observation whose DF or BSDF statistic exceeds (goes below) its corresponding critical value. The duration of a bubble is restricted to be longer than a slowly varying (at infinity) quantity such as $\delta \log(T)$, or in sample fraction terms $L_T = \delta \log(T)/T$, where δ is a frequency dependent parameter—see PSY for further discussion. The origination and termination estimators are calculated as the crossing time fractions

$$(6) \qquad PWY : \hat{r}^e = \inf_{r \in [r_0, 1]} \left\{ r : DF_r > cv^{\beta_T} \right\} \text{ and } \hat{r}^f = \inf_{r \in [\hat{r}^e + L_T, 1]} \left\{ r : DF_r < cv^{\beta_T} \right\},$$

$$(7) \quad PSY : \hat{r}^e = \inf_{r \in [r_0, 1]} \left\{ r : BSDF_r(r_0) > scv^{\beta_T} \right\} \text{ and } \hat{r}^f = \inf_{r \in [\hat{r}^e + L_T, 1]} \left\{ r : BSDF_r(r_0) < scv^{\beta_T} \right\},$$

where cv^{β_T} and scv^{β_T} are the $100(1 - \beta_T)\%$ critical values of the DF and BSDF statistics.

In the multiple bubbles scenario, estimators associated with the first bubble are defined as in Equations (6) and (7), and denoted by $\hat{r}^{1e}$ and $\hat{r}^{1f}$. The origination (termination) of bubble i (for $i \geq 2$) is the first chronological observation after $\hat{r}^{i-1f}$ whose DF or BSDF statistic exceeds (goes below) its corresponding critical value. Structurally,

$$(8) \qquad PWY : \hat{r}^{ie} = \inf_{r \in [\hat{r}^{i-1f}, 1]} \left\{ r : DF_r > cv^{\beta_T} \right\} \text{ and } \hat{r}^{if} = \inf_{r \in [\hat{r}^{ie} + L_T, 1]} \left\{ r : DF_r < cv^{\beta_T} \right\},$$

$$(9) \qquad PSY : \hat{r}^{ie} = \inf_{r \in [\hat{r}^{i-1f}, 1]} \left\{ r : BSDF_r(r_0) > scv^{\beta_T} \right\} \text{ and}$$

$$\hat{r}^{if} = \inf_{r \in [\hat{r}^{ie} + L_T, 1]} \left\{ r : BSDF_r(r_0) < scv^{\beta_T} \right\}.$$

For the sequential PWY procedure, the dating criteria of the first bubble remain the same (i.e., Equation (6)). For all subsequent bubbles, the sequential procedure uses information starting from the termination of the previous bubble and ending at the current observation, i.e., $I_{i,r} = \left\{ \left\lfloor T\hat{r}^{i-1f} \right\rfloor + 1, \ldots, \lfloor Tr \rfloor \right\}$ for $i \geq 2$. Importantly, note that the distance between r and $\hat{r}^{i-1f}$ needs to be greater than the minimum regression window r_0, which restricts the capability of this sequential procedure to detect bubble activity in the intervening period $(\hat{r}^{i-1f}, \hat{r}^{i-1f} + r_0)$. The origination and termination dates of bubble i are then calculated as

$$(10)\, Seq\_PWY : \hat{r}^{ie} = \inf_{r \in [\hat{r}^{i-1f} + r_0, 1]} \left\{ r : {}_{\hat{r}^{i-1f}}DF_r > cv^{\beta_T} \right\} \text{ and } \hat{r}^{if} = \inf_{r \in [\hat{r}^{ie} + L_T, 1]} \left\{ r : {}_{\hat{r}^{i-1f}}DF_r < cv^{\beta_T} \right\},$$

where ${}_{\hat{r}^{i-1f}}DF_r$ is the DF statistic calculated over $(\hat{r}^{i-1f}, r]$.

3. ASYMPTOTIC PROPERTIES OF THE DETECTORS

The asymptotic performance of the dating algorithms is examined in this section. Under the null hypothesis of no bubble episodes, the limit distributions of the DF and BSDF statistics follow from PSY (Theorem 1). Both the DF and BSDF statistics are special cases of the GSADF statistic introduced in PSY. For the DF statistic, the start point of the regression is $r_1 = 0$ and the end point r_2 is fixed at r. Therefore, the limit distribution of the DF statistic under the null hypothesis is

$$F_r(W) := \frac{\frac{1}{2}r[W(r)^2 - r] - \int_0^r W(s)dsW(r)}{r^{1/2}\{r\int_0^r W(s)^2ds - [\int_0^r W(s)ds]^2\}^{1/2}}, \tag{11}$$

where W is a standard Wiener process. For the BSDF statistic, the end point r_2 is fixed at r and the start point r_1 varies from 0 to $r - r_0$. The limit distribution of the BSDF statistic is

$$F_r(W, r_0) := \sup_{\substack{r_1 \in [0, r-r_0] \\ r_w = r - r_1}} \left\{ \frac{\frac{1}{2}r_w[W(r)^2 - W(r_1)^2 - r_w] - \int_{r_1}^r W(s)ds[W(r) - W(r_1)]}{r_w^{1/2}\{r_w \int_{r_1}^r W(s)^2ds - [\int_{r_1}^r W(s)ds]^2\}^{1/2}} \right\}. \tag{12}$$

Notice that $F_r(W)$ is a special case of $F_r(W, r_0)$ with $r_1 = 0$ and $r_w = r$. The asymptotic critical values cv^{β_T} and scv^{β_T} are defined as the $100(1 - \beta_T)\%$ quantiles of $F_r(W)$ and $F_r(W, r_0)$, respectively. The significance level β_T depends on the sample size T, and it is assumed that $\beta_T \to 0$ as $T \to \infty$. This control ensures that cv^{β_T} and scv^{β_T} diverge to infinity, and thereby under the null hypothesis the probabilities of (falsely) detecting a bubble using the DF and BSDF statistics, (6)–(10), tend to zero as $T \to \infty$.

An implicit restriction in the asymptotic theory is that the minimum window size r_0 is bounded below, so that it cannot pass to zero as the sample size increases. Such restrictions are commonly used in break point asymptotics and impose a practical restriction in the implementation of the test. PSY recommend an empirical rule for choosing r_0 based on a lower bound minimum window width of 1% of the sample. This rule produces stable size and good power for sample sizes typical in many applications.

We next derive the limit distributions under mildly explosive alternatives. We consider the case of a single bubble and multiple bubbles separately, as the properties of some of the detectors differ markedly in the case of multiple bubbles. The derivations require some careful calculations involving weak convergence arguments and mildly explosive limit theory, paying attention to some subtleties in the orders of magnitude of the various components of the test statistics. The details are provided in the Appendix and the technical supplement to the paper (Phillips et al., 2014).

Single Bubble Alternative.

THEOREM 1. *Under the data-generating process* (4)*, the limit forms of the DF_r and $BSDF_r(r_0)$ statistics are as follows:*

$$DF_r \sim_a \begin{cases} F_r(W) & \text{if } r \in N_0 \\ T^{1-\alpha/2} \dfrac{r^{3/2}}{\sqrt{2}(r_e - r_1)} & \text{if } r \in B \\ -T^{(1-\alpha)/2}\left(\dfrac{1}{2}cr\right)^{1/2} & \text{if } r \in N_1 \end{cases}, \tag{13}$$

$$(14)\qquad BSDF_r(r_0) \sim_a \begin{cases} F_r(W, r_0) & if\, r \in N_0 \\ T^{1-\alpha/2} \sup_{r_1 \in [0, r-r_0]} \left\{ \frac{(r-r_1)^{3/2}}{\sqrt{2(r_e-r_1)}} \right\} & if\, r \in B \\ -T^{(1-\alpha)/2} \sup_{r_1 \in [0, r-r_0]} \left\{ \left[\frac{1}{2} c (r-r_1) \right]^{1/2} \right\} & if\, r \in N_1 \end{cases},$$

where $B(r) \equiv \sigma W(r)$, $t = \lfloor Tr \rfloor$, *and* $\tau_e = \lfloor Tr_e \rfloor$.

Evidently, for all three cases the orders of magnitude of the DF and BSDF statistics are the same. Importantly, the test statistics diverge to positive infinity when the current observation falls in the explosive bubble period and to minus infinity when the observation is in a bubble collapse period. The minus infinity divergence of DF_r when $r \in N_1$ is consistent with the argument given in PWY that the DF test treats a process including both bubble expansion and collapse phases as a "pseudo stationary" process. Based on these limit forms of the recursive statistics, we obtain the following consistency results for the date detectors.

THEOREM 2 (PWY DETECTOR). *Suppose $\hat{r}_e$ and $\hat{r}_f$ are the date estimates obtained from the DF t-statistic crossing times* (6). *Under the alternative hypothesis of mildly explosive behavior in model* (4), *if*

$$(15)\qquad \frac{1}{cv^{\beta_T}} + \frac{cv^{\beta_T}}{T^{1-\alpha/2}} \to 0,$$

we have $\hat{r}_e \xrightarrow{p} r_e$ *and* $\hat{r}_f \xrightarrow{p} r_f$ *as* $T \to \infty$.

THEOREM 3 (PSY DETECTOR). *Suppose $\hat{r}_e$ and $\hat{r}_f$ are the date estimates obtained from the backward sup DF statistic crossing times* (7). *Under the alternative hypothesis of mildly explosive behavior in model* (4), *if*

$$(16)\qquad \frac{1}{scv^{\beta_T}} + \frac{scv^{\beta_T}}{T^{1-\alpha/2}} \to 0,$$

we have $\hat{r}_e \xrightarrow{p} r_e$ *and* $\hat{r}_f \xrightarrow{p} r_f$ *as* $T \to \infty$.

These results show that both strategies consistently estimate the origination and termination points when there is only a single bubble episode in the sample period. The regularity conditions in Theorems 2 and 3 imply that the orders of magnitude of the critical value expansion rates need to be smaller than $T^{1-\alpha/2}$ to deliver consistency of $\hat{r}_e$ and $\hat{r}_f$. In effect, for consistent estimation of r_e the critical value sequence needs to pass to infinity but not too fast—otherwise the signal from the mildly explosive period under the alternative is not strong enough to ensure that the critical value is exceeded. The first condition ($cv^{\beta_T}, scv^{\beta_T} \to \infty$) ensures that there are no false positives prior to the origination date r_e. The second condition ($\frac{cv^{\beta_T}}{T^{1-\alpha/2}}, \frac{scv^{\beta_T}}{T^{1-\alpha/2}} \to 0$) ensures that the signal from the data during the mildly explosive period dominates that from the earlier unit root period leading to identifying information that there is now exuberance in the data.

An implicit restriction in these two results is that the minimum window size r_0 is smaller than the origination date of the bubble r_e (i.e., $r_0 < r_e$) so that the recursive regressions provide information for some $r \in N_0$ for comparison to identify the origination point. This requirement is also implicit in what follows, in particular in later proofs of consistency of the first bubble origination date in the multiple bubbles scenario as discussed below. In the event that $r_0 \in (r_e, r_f)$, then the results given in the second panels of (13) and (14) are relevant and the origination date of the first bubble is determined to be r_0, so r_e is estimated with delay.

For consistent estimation of r_f, both conditions again come into play. The second condition ($\frac{cv^{\beta_T}}{T^{1-\alpha/2}}, \frac{scv^{\beta_T}}{T^{1-\alpha/2}} \to 0$) ensures that there is no underestimation of r_f asymptotically because for $r \leq r_f$ the signal from the data during the mildly explosive period continues to dominate. When $r > r_f$, the autoregressive estimate is calculated from data that involve the explosive episode as well as post-explosive ($r > r_f$) data, which makes the post-collapse data look mean reverting and, as shown in the proof of Theorem 1, the test statistics become negative. The expansion condition ($cv^{\beta_T}, scv^{\beta_T} \to \infty$) then ensures that there is no overestimation of r_f asymptotically.

Multiple Bubble Alternatives. The limit behavior of the recursive DF and BSDF statistics in the presence of multiple bubbles is much more complicated. The strengths and weaknesses of the various detectors are well illustrated by considering a mildly explosive process with two bubble episodes. We therefore confine much of our discussion here to the case of model (5) with $K = 2$. Even in this case, as shown below, there are several possibilities depending on the respective durations of the bubbles.

We start with the case where the duration of the first bubble exceeds that of the second bubble. Also, to obtain the BSDF asymptotics in Theorems 4 and 5, it is assumed that the distance separating the termination dates of the first and second bubbles exceeds the minimum window size (i.e., $r_{2e} - r_{1f} > r_0$). This requirement seems a natural condition to achieve identification of the second bubble. The effect of its relaxation is considered later.

THEOREM 4. *Under the data-generating process of* (5) *with* $K = 2$ *and* $r_{1f} - r_{1e} > r_{2f} - r_{2e}$, *the limit behavior of the recursive statistics* DF_r, $BSDF_r(r_0)$ *and* ${}_{\hat{r}_{1f}}DF_r$ *is given by*

$$DF_r \sim_a \begin{cases} F_r(W) & \text{if } r \in N_0 \\ T^{1-\alpha/2} \dfrac{r^{3/2}}{\sqrt{2(r_{1e} - r_1)}} & \text{if } r \in B_1 \\ -T^{(1-\alpha)/2} \left(\dfrac{1}{2} cr\right)^{1/2} & \text{if } r \in N_1 \cup B_2 \cup N_2 \end{cases}, \tag{17}$$

$$BSDF_r(r_0) \sim_a \begin{cases} F_r(W, r_0) & \text{if } r \in N_0 \\ T^{1-\alpha/2} \sup\limits_{r_1 \in [0, r-r_0]} \left\{ \dfrac{(r - r_1)^{3/2}}{\sqrt{2(r_{ie} - r_1)}} \right\} & \text{if } r \in B_i \text{ with } i = 1, 2 \\ -T^{(1-\alpha)/2} \sup\limits_{r_1 \in [0, r-r_0]} \left[\dfrac{1}{2} c(r - r_1) \right]^{1/2} & \text{if } r \in N_1 \cup N_2 \end{cases}, \tag{18}$$

$${}_{\hat{r}_{1f}}DF_r \sim_a \begin{cases} F_r(W) & \text{if } r \in N_1 \\ T^{1-\alpha/2} \dfrac{(r - r_{1f})^{3/2}}{\sqrt{2(r_{2e} - r_1)}} & \text{if } r \in B_2 \\ -T^{(1-\alpha)/2} \left[\dfrac{1}{2} c(r - r_{1f}) \right]^{1/2} & \text{if } r \in N_2 \end{cases}. \tag{19}$$

Evidently from the first panel (17), it is clear that when the duration of the first bubble exceeds that of the second bubble, the DF statistic diverges to positive infinity when $r \in B_1$, whereas for $r \in N_1 \cup B_2 \cup N_2$, the statistic is asymptotically equivalent to $-T^{(1-\alpha)/2}(\frac{1}{2}cr)^{1/2}$ and tends to negative infinity as $T \to \infty$. Importantly, therefore, the behavior of the DF statistic during the second (shorter) bubble B_2 is the same as it is for the normal martingale periods N_1 and N_2. Hence, the DF statistic does not have discriminatory power for second bubble detection when the duration of the second bubble is less than that of the first bubble.

From the second panel (18), the behavior of the BSDF statistic in both bubble periods B_1 and B_2 is the same and is distinct from that of the normal periods N_0, N_1, and N_2. Unlike the DF statistic, the BSDF statistic therefore has discriminatory power in detecting both bubbles. From the final panel (19), it is clear that the orders of magnitude characterizing the limit behavior of the sequential DF statistic ${}_{\hat{r}_{1f}}DF_r$ are the same as those of the BSDF statistic for $r \in B_2$ and $r \in N_2$. Hence, like BSDF, the sequential DF statistic has discriminatory power for the two bubble periods.

Next consider the case where the duration of the second bubble exceeds that of the first bubble.

THEOREM 5. *Under the data-generating process of* (5) *with* $K = 2$ *and* $r_{1f} - r_{1e} \leq r_{2f} - r_{2e}$, *the limit behavior of the recursive statistics* DF_r, $BSDF_r(r_0)$ *and* ${}_{\hat{r}_{1f}}DF_r$ *is as follows:*

$$DF_r \sim_a \begin{cases} F_r(W) & \text{if } r \in N_0 \\ T^{1-\alpha/2} \dfrac{r^{3/2}}{\sqrt{2(r_{ie} - r_1)}} & \text{if } r \in B_1 \\ -T^{(1-\alpha)/2}\left(\dfrac{1}{2}cr\right)^{1/2} & \text{if } r \in N_1 \cup N_2 \\ -T^{(1-\alpha)/2}\left(\dfrac{1}{2}cr\right)^{1/2} & \text{if } r \in B_2 \text{ and } r_{1f} - r_{1e} > r - r_{2e} \\ T^{1-\alpha/2}\left[\dfrac{cr^3}{2(r_{1e} + r_{2e} - r_{1f})}\right]^{1/2} & \text{if } r \in B_2 \text{ and } r_{1f} - r_{1e} \leq r - r_{2e} \end{cases}, \tag{20}$$

$$BSDF_r(r_0) \sim_a \begin{cases} F_r(W, r_0) & \text{if } r \in N_0 \\ T^{1/2}\delta_T^{t-\tau_{ie}} \sup\limits_{r_1 \in [0, r-r_0]} \left\{\frac{(r-r_1)^{3/2} B(r_{ie})}{2(r_{ie}-r_1)\int_{r_1}^{r_{ie}} B(s)ds}\right\} & \text{if } r \in B_1 \cup B_2 \\ -T^{(1-\alpha)/2} \sup\limits_{r_1 \in [0, r_2-r_0]} \left[\frac{1}{2}c(r - r_1)\right]^{1/2} & \text{if } r \in N_1 \cup N_2 \end{cases}, \tag{21}$$

$${}_{\hat{r}_{1f}}DF_r \sim_a \begin{cases} F_r(W) & \text{if } r \in N_1 \\ T^{1/2}\delta_T^{t-\tau_{2e}} \frac{(r-r_{1f})^{3/2} B(r_{ie})}{2(r_{ie}-r_1)\int_{r_{1f}}^{r_{ie}} B(s)ds} & \text{if } r \in B_2 \\ -T^{(1-\alpha)/2}\left[\frac{1}{2}c(r - r_{1f})\right]^{1/2} & \text{if } r \in N_2 \end{cases}. \tag{22}$$

As is evident in panels (21) and (22) of this theorem, the limit behaviors of the BSDF statistic and sequential DF statistic are identical to those that apply in the earlier case where $r_{1f} - r_{1e} > r_{2f} - r_{2e}$. Thus both procedures have the same discriminatory capability for identifying bubble episodes in the data. Again, results are very different for the DF statistic, where the behavior of the statistic during the second bubble ($r \in B_2$) is contingent on the timing of latest date (r) in the recursion. In particular, when $r \in B_2$, the limit behavior of the DF statistic depends on the relative length of $r_{1f} - r_{1e}$ (the duration of the first bubble) and $r - r_{2e}$ (the segment of the second bubble that is included in data used in the recursion). When $r_{1f} - r_{1e}$ exceeds $r - r_{2e}$, the statistic diverges to negative infinity, just as for the case where $r_{1f} - r_{1e} > r_{2f} - r_{2e}$. Thus, in this case there are insufficient data to identify the second bubble period. However, as is clear from the final asymptotic expression in (20), behavior changes dramatically as soon as there are more data. Specifically, when the segment of the second bubble included in the recursive regression exceeds the duration of the first bubble (i.e., when $r - r_{2e} \geq r_{1f} - r_{1e}$) the sign in the limit behavior of the DF statistic changes and the statistic now diverges to positive infinity instead of negative infinity. The order of the magnitude in the divergence also rises (from $T^{(1-\alpha)/2}$ to $T^{1-\alpha/2}$). It follows that the DF statistic has discriminatory power once there are sufficient data for this test to identify a second bubble—that is, as soon as data from the second bubble dominate those of the first bubble.

With the limit behavior of the recursive tests in hand, results on the consistency properties of the bubble date detectors now follow. It is convenient to separate the results according to each of the recursive tests and contingent conditions regarding duration of the bubbles.

THEOREM 6 (PWY DETECTOR). *Suppose $\hat{r}_{1e}$, $\hat{r}_{1f}$, $\hat{r}_{2e}$, and $\hat{r}_{2f}$ are obtained from the DF test based on the crossing times* (6) *and* (8). *Given the alternative hypothesis of mildly explosive behavior in model* (5) *with $K = 2$ and durations satisfying $r_{1f} - r_{1e} > r_{2f} - r_{2e}$, if*

$$\frac{1}{cv^{\beta_T}} + \frac{cv^{\beta_T}}{T^{1-\alpha/2}} \to 0,$$

we have $\hat{r}_{1e} \xrightarrow{p} r_{1e}$ and $\hat{r}_{1f} \xrightarrow{p} r_{1f}$ as $T \to \infty$ and $\hat{r}_{2e}$ and $\hat{r}_{2f}$ are not consistent estimators of r_{2e} and r_{2f}.

THEOREM 7 (PWY DETECTOR). *Suppose $\hat{r}_{1e}$, $\hat{r}_{1f}$, $\hat{r}_{2e}$, and $\hat{r}_{2f}$ are obtained from the DF test based on the crossing times* (6) *and* (8). *Given the alternative hypothesis of mildly explosive behavior in model* (5) *with $K = 2$ and durations satisfying $r_{1f} - r_{1e} \leq r_{2f} - r_{2e}$, if*

$$\frac{1}{cv^{\beta_T}} + \frac{cv^{\beta_T}}{T^{1-\alpha/2}} \to 0,$$

we have $\hat{r}_{1e} \xrightarrow{p} r_{1e}$, $\hat{r}_{1f} \xrightarrow{p} r_{1f}$, $\hat{r}_{2e} \xrightarrow{p} r_{2e} + r_{1f} - r_{1e}$, and $\hat{r}_{2f} \xrightarrow{p} r_{2f}$ as $T \to \infty$.

THEOREM 8 (PSY DETECTOR). *Suppose $\hat{r}_{1e}$, $\hat{r}_{1f}$, $\hat{r}_{2e}$, and $\hat{r}_{2f}$ are obtained from the backward sup DF test based on the crossing times* (7) *and* (9). *Given the alternative hypothesis of mildly explosive behavior in model* (5) *with $K = 2$, if*

$$\frac{1}{scv^{\beta_T}} + \frac{scv^{\beta_T}}{T^{1-\alpha/2}} \to 0 \quad \text{with } i = 1, 2,$$

we have $\hat{r}_{1e} \xrightarrow{p} r_{1e}$, $\hat{r}_{1f} \xrightarrow{p} r_{1f}$, $\hat{r}_{2e} \xrightarrow{p} r_{2e}$, and $\hat{r}_{2f} \xrightarrow{p} r_{2f}$ as $T \to \infty$.

THEOREM 9 (SEQUENTIAL PWY DETECTOR). *Suppose $\hat{r}_{1e}$, $\hat{r}_{1f}$, $\hat{r}_{2e}$, and $\hat{r}_{2f}$ are obtained from sequential application of the DF test based on the crossing times* (6) *and* (10). *Given the alternative hypothesis of mildly explosive behavior in model* (5) *with $K = 2$, if*

$$\frac{1}{cv^{\beta_T}} + \frac{cv^{\beta_T}}{T^{1-\alpha/2}} \to 0,$$

we have $\hat{r}_{1e} \xrightarrow{p} r_{1e}$, $\hat{r}_{1f} \xrightarrow{p} r_{1f}$, $\hat{r}_{2e} \xrightarrow{p} r_{2e}$, and $\hat{r}_{2f} \xrightarrow{p} r_{2f}$ as $T \to \infty$.

Theorems 6–9 characterize the consistency properties of the detectors when there are two bubble episodes in the observed data. The results depend on the detector and certain side conditions regarding the duration of the bubbles. Importantly, the PWY strategy consistently estimates the origination and termination of the first bubble but not the second bubble. When the duration of the first bubble exceeds that of the second bubble, the PWY strategy fails to detect the second bubble. When the duration of the second bubble exceeds the first, the PWY recursion detects the presence of a second bubble but with a delay measured by the duration of the first bubble ($r_{1f} - r_{1e}$). The PWY detector is therefore inconsistent in date stamping the second bubble even when the conditions favor its detection. In contrast, the PSY and sequential PWY recursions are both consistent date detectors for the origination and termination of the two bubbles irrespective of their relative durations. These procedures are therefore robust to bubble duration.

Theorems 6–9 can be extended to scenarios with multiple bubbles ($K > 2$). In this case, if the duration of bubble $i + 1$ is less than that of bubble i for some $i \in \{1, 2, \cdots, K - 1\}$, then the PWY recursion may, under certain conditions such as increasing duration up to bubble i, detect the presence of bubble i, but it will not detect bubble $i + 1$. In contrast, the PSY and sequential PWY strategies detect each of the K bubbles, with fully consistent date detection by the PSY recursion.

We now consider the extreme scenario, mentioned earlier, where the minimum window length r_0 exceeds the distance between the termination dates of the two bubbles. Suppose $K = 2$. For the sequential PWY procedure, the first regression after re-initialization from the end point of the first bubble now runs from period N_1 directly to N_2, so this procedure completely passes over the second bubble and is unable to detect it. Somewhat remarkably however, the PSY strategy still has some detective capability for the second bubble depending on the relative length of $r_{1f} - r_{1e}$ and $r - r_{2e}$. Specifically, for observations in the second bubble episode (i.e., $r \in B_2$), their backward expanding regression sample sequences does not include the case of $\tau_1 \in N_1$ and $\tau_2 \in B_2$ when $r_0 > r_{2f} - r_{1f}$. Hence, the limit behavior of $BSDF_r(r_0)$ under the two-bubble data-generating process is

$$(23)\; BSDF_r(r_0) \sim_a \begin{cases} -T^{(1-\alpha)/2} \sup\limits_{r_1 \in [0, r_2 - r_0]} \left[\frac{1}{2} c\,(r - r_1)\right]^{1/2} & \text{if } r \in B_2 \text{ and } r_{1f} - r_{1e} > r - r_{2e} \\ T^{1-\alpha/2} \sup\limits_{r_1 \in [0, r_2 - r_0]} \left[\frac{c(r-r_1)^3}{2(r_{1e} - r_1 + r_{2e} - r_{1f})}\right]^{1/2} & \text{if } r \in B_2 \text{ and } r_{1f} - r_{1e} \le r - r_{2e} \end{cases}.$$

Then, if $r_{1f} - r_{1e} > r - r_{2e}$, the limit behavior of $BSDF_r(r_0)$ at $r \in B_2$ is the same as when $r \in N_1 \cup N_2$, so in that event the PSY strategy also cannot detect the second bubble. But when $r_{1f} - r_{1e} \le r - r_{2e}$, the limit behavior of $BSDF_r(r_0)$ at $r \in B_2$ is divergent with an order magnitude of $T^{1-\alpha/2}$. Hence, even though $r_0 > r_{2f} - r_{1f}$, the PSY strategy is still able to detect the second bubble (with a delay of $r_{1f} - r_{1e}$ in the estimated origination date) as long as the duration of the second bubble exceeds the first bubble.

A less extreme scenario is the case where $r_{2e} - r_{1f} < r_0 \le r_{2f} - r_{1f}$. That is, the minimum window size exceeds the distance separating the two bubbles but does not exceed the distance between the termination dates of these two bubbles. In this circumstance, the limit behaviors of $BSDF_r(r_0)$ and ${}_{\hat{r}_{1f}}DF_r$ remain the same as in (21) and (22) for $r_{1f} + r_0 \le r \le r_{2f}$ (the later segment of B_2). However, for observations prior to that in B_2, the ${}_{\hat{r}_{1f}}DF_r$ statistic does not exist by construction and the BSDF statistic follows the limit behavior of (23). Therefore, there will be delay in estimates of the second bubble origination date using both the PSY and sequential PWY strategies. However, the delay is potentially smaller using the PSY strategy due to the last panel of (23).

The advantage of the PSY strategy over the sequential PWY procedure is revealed in the simulations reported below that consider some less extreme cases. For instance, when $r_{3e} - r_{2f} < r_0 < r_{3f} - r_{2f}$ (i.e., $0.05 < 0.12 < 0.15$) as in the first panel of Table 10, the detection rate of the sequential PWY strategy is zero compared with 62% for the PSY strategy.

TABLE 1

DETECTION RATE AND ESTIMATION OF THE ORIGINATION AND TERMINATION DATES UNDER SINGLE BUBBLE DGP AND DIFFERENT BUBBLE EXPANSION RATES

	PWY	PSY	Seq	CUSUM
$\alpha = 0.60, \delta_T = 1.06$				
Detection Rate	0.77	0.85	0.79	0.85
$r_e = 0.40$	0.46 (0.03)	0.45 (0.03)	0.46 (0.03)	0.46 (0.03)
$r_f = 0.55$	0.55 (0.01)	0.55 (0.01)	0.55 (0.01)	0.55 (0.01)
$\alpha = 0.55, \delta_T = 1.08$				
Detection Rate	0.84	0.90	0.85	0.90
$r_e = 0.40$	0.45 (0.03)	0.44 (0.03)	0.45 (0.03)	0.45 (0.03)
$r_f = 0.55$	0.55 (0.00)	0.55 (0.01)	0.55 (0.00)	0.55 (0.01)
$\alpha = 0.50, \delta_T = 1.10$				
Detection Rate	0.89	0.93	0.90	0.93
$r_e = 0.40$	0.45 (0.03)	0.43 (0.03)	0.44 (0.03)	0.44 (0.03)
$r_f = 0.55$	0.55 (0.00)	0.55 (0.01)	0.55 (0.00)	0.55 (0.01)

NOTES: Calculations are based on 5,000 replications. The minimum window has 12 observations. Parameters are set to: $y_0 = 100$, $c = 1$, $\sigma = 6.79$, $\tau_e = \lfloor 0.4T \rfloor$, $\tau_f - \tau_e = \lfloor 0.15T \rfloor$, $T = 100$. Figures in parentheses are standard deviations.

4. SIMULATION EVIDENCE

This section reports simulations to explore the finite sample performance of the PSY, PWY, sequential PWY, and CUSUM procedures for bubble detection. These simulations focus on detection rates and estimation accuracy of the dating algorithms of these procedures. They complement the findings reported in PSY and examine performance characteristics in systems with many bubbles.

Experiments are conducted with generating models that involve up to three separate bubbles. The generating system for single, dual, and three bubbles are as in (4) and (5). The parameter settings follow those used in PSY, so that $y_0 = 100$, $\sigma = 6.79$, $c = 1$, and $T = 100$.[7] In the single bubble setting, we explore the sensitivities of the dating strategies to the parameters determining the magnitude of the bubbles (the bubble expansion rate α and the bubble duration $d_T = \tau_f - \tau_e$), the bubble location parameter τ_e, and the sample size T. We focus our attention on the impact of bubble durations in the two bubble and three bubble settings. For each parameter constellation, 5,000 replications were used. Bubbles were identified using respective finite sample 95% quantiles, obtained from simulations with 5,000 replications. The minimum window size has 12 observations.

We report the proportion of samples in which a bubble was successfully detected along with the empirical mean and standard deviation (in parentheses) of the estimated origination and termination dates. Successful detection of a bubble is defined as an outcome where the estimated origination date is greater than or equal to the true origination date and smaller than the true termination date of that particular bubble (i.e., $\tau_{ie} \leq \hat{\tau}_{ie} < \tau_{if}$).

4.1. *A Single Bubble.* In Tables 1 and 2, the bubble expansion rate α and bubble duration d_T can each take three values: specifically, the expansion rate $\alpha \in \{0.60, 0.55, 0.50\}$ with corresponding autoregressive coefficient $\delta_T \in \{1.06, 1.08, 1.10\}$ when $T = 100$, and duration is $d_T \in \{\lfloor 0.10T \rfloor, \lfloor 0.15T \rfloor, \lfloor 0.20T \rfloor\}$. Evidently for all algorithms the bubble detection rate increases with the value of the autoregressive coefficient δ_T and the bubble duration d_T. Moreover, a higher autoregressive coefficient results in more timely detection of the bubble, whereas longer bubble duration is associated with longer delay (i.e., $\hat{r}_e - r_e$). For instance, the delay in the PSY estimate reduces from 0.05 to 0.03 when δ_T increases from 1.06 to 1.10 and the delay increases from 0.04 to 0.06 when the bubble duration extends from $\lfloor 0.10T \rfloor$ to $\lfloor 0.20T \rfloor$.

[7] The parameters y_0 and σ match the initial value and the sample standard deviation of the differenced series of the normalized S&P 500 price–dividend ratio examined in PSY.

TABLE 2

DETECTION RATE AND ESTIMATION OF THE ORIGINATION AND TERMINATION DATES UNDER SINGLE BUBBLE DGP AND DIFFERENT BUBBLE DURATIONS

	PWY	PSY	Seq	CUSUM
$\tau_f - \tau_e = \lfloor 0.10T \rfloor$				
Detection Rate	0.59	0.70	0.59	0.67
$r_e = 0.40$	0.44 (0.02)	0.44 (0.02)	0.44 (0.02)	0.44 (0.02)
$r_f = 0.50$	0.50 (0.00)	0.50 (0.01)	0.50 (0.01)	0.50 (0.01)
$\tau_f - \tau_e = \lfloor 0.15T \rfloor$				
Detection Rate	0.77	0.85	0.79	0.85
$r_e = 0.40$	0.46 (0.03)	0.45 (0.03)	0.46 (0.03)	0.46 (0.03)
$r_f = 0.55$	0.55 (0.01)	0.55 (0.01)	0.55 (0.01)	0.55 (0.01)
$\tau_f - \tau_e = \lfloor 0.20T \rfloor$				
Detection Rate	0.86	0.90	0.88	0.91
$r_e = 0.40$	0.47 (0.04)	0.46 (0.04)	0.47 (0.04)	0.48 (0.04)
$r_f = 0.60$	0.60 (0.01)	0.60 (0.01)	0.60 (0.01)	0.60 (0.01)

NOTES: Calculations are based on 5,000 replications. The minimum window has 12 observations. Parameters are set to: $y_0 = 100$, $c = 1$, $\sigma = 6.79$, $\alpha = 0.6$, $\tau_e = \lfloor 0.4T \rfloor$, $T = 100$. Figures in parentheses are standard deviations.

TABLE 3

DETECTION RATE AND ESTIMATION OF THE ORIGINATION AND TERMINATION DATES UNDER SINGLE BUBBLE DGP AND DIFFERENT BUBBLE LOCATIONS

	PWY	PSY	Seq	CUSUM
$\tau_e = \lfloor 0.2T \rfloor$				
Detection Rate	0.87	0.90	0.87	0.87
$r_e = 0.20$	0.25 (0.03)	0.25 (0.03)	0.25 (0.03)	0.26 (0.03)
$r_f = 0.35$	0.35 (0.00)	0.35 (0.01)	0.35 (0.01)	0.35 (0.01)
$\tau_e = \lfloor 0.4T \rfloor$				
Detection Rate	0.77	0.85	0.79	0.85
$r_e = 0.40$	0.46 (0.03)	0.45 (0.03)	0.46 (0.03)	0.46 (0.03)
$r_f = 0.55$	0.55 (0.01)	0.55 (0.01)	0.55 (0.01)	0.55 (0.01)
$\tau_e = \lfloor 0.6T \rfloor$				
Detection Rate	0.72	0.82	0.74	0.82
$r_e = 0.60$	0.66 (0.03)	0.65 (0.03)	0.66 (0.03)	0.65 (0.03)
$r_f = 0.75$	0.75 (0.01)	0.75 (0.01)	0.75 (0.01)	0.75 (0.01)

NOTES: Calculations are based on 5,000 replications. The minimum window has 12 observations. Parameters are set to: $y_0 = 100$, $c = 1$, $\sigma = 6.79$, $\alpha = 0.6$, $\tau_f - \tau_e = \lfloor 0.15T \rfloor$, $T = 100$. Figures in parentheses are standard deviations.

In Table 3, the location parameter τ_e varies from $\lfloor 0.2T \rfloor$ to $\lfloor 0.6T \rfloor$. When the bubble originates at a later stage of the sample, the bubble detection rates of all strategies are lower. Table 4 monitors the effects of increasing the sample size from 100 to 400. Evidently, whereas the detection rate of the CUSUM strategy decreases with the sample size, the detection rates of the PWY, PSY, and sequential PWY strategies increase with the sample size. The time needed to detect bubbles in all algorithms is largely unaffected by the location of the bubble and the sample size.

The most striking finding in Tables 1–3 is the superiority of the PSY strategy relative to the other algorithms in the single bubble case. The PSY strategy has a higher rate of bubble detection and provides a more accurate estimate of the origination date. All strategies deliver a good detection rate of the termination date of the bubble, which is no doubt associated with the sharp collapse specification in the model formulation.

4.2. *Two Bubbles.* Two duration scenarios feature in the dual bubble simulations. In one of these, the first bubble has longer duration (Table 5), whereas in the other the second bubble has longer duration (Table 6). The bubbles originate 20% and 60% into the sample, and the expansion rate of the two bubbles is 1.04 (i.e., $\alpha = 0.6$).

TABLE 4

DETECTION RATE AND ESTIMATION OF THE ORIGINATION AND TERMINATION DATES UNDER SINGLE BUBBLE DGP AND DIFFERENT SAMPLE SIZES

	PWY	PSY	Seq	CUSUM
$T = 100$				
Detection Rate	0.77	0.85	0.79	0.85
$r_e = 0.40$	0.46 (0.03)	0.45 (0.03)	0.46 (0.03)	0.46 (0.03)
$r_f = 0.55$	0.55 (0.01)	0.55 (0.01)	0.55 (0.01)	0.55 (0.01)
$T = 200$				
Detection Rate	0.79	0.85	0.81	0.83
$r_e = 0.40$	0.46 (0.04)	0.45 (0.03)	0.46 (0.04)	0.45 (0.03)
$r_f = 0.55$	0.55 (0.01)	0.54 (0.02)	0.55 (0.01)	0.55 (0.02)
$T = 400$				
Detection Rate	0.83	0.89	0.85	0.78
$r_e = 0.40$	0.46 (0.04)	0.44 (0.03)	0.45 (0.04)	0.45 (0.03)
$r_f = 0.55$	0.55 (0.02)	0.53 (0.04)	0.55 (0.02)	0.54 (0.03)

NOTES: Calculations are based on 5,000 replications. The minimum window has 12 observations. Parameters are set to: $y_0 = 100$, $c = 1$, $\sigma = 6.79$, $\alpha = 0.60$, $\tau_e = \lfloor 0.4T \rfloor$, $\tau_f - \tau_e = \lfloor 0.15T \rfloor$. Figures in parentheses are standard deviations.

TABLE 5

DETECTION RATE AND ESTIMATION OF THE ORIGINATION AND TERMINATION DATES UNDER TWO BUBBLE DGP WITH SHORTER SECOND BUBBLE DURATIONS

	PWY	PSY	Seq	CUSUM
$\tau_{2f} - \tau_{2e} = \lfloor 0.10T \rfloor$				
Detection Rate (1)	0.94	0.95	0.94	0.93
$r_{1e} = 0.20$	0.26 (0.04)	0.26 (0.04)	0.26 (0.04)	0.27 (0.04)
$r_{1f} = 0.40$	0.40 (0.01)	0.40 (0.01)	0.40 (0.01)	0.40 (0.01)
Detection Rate (2)	0.01	0.73	0.68	0.03
$r_{2e} = 0.60$	0.66 (0.02)	0.64 (0.02)	0.64 (0.02)	0.66 (0.02)
$r_{2f} = 0.70$	0.70 (0.00)	0.70 (0.00)	0.70 (0.00)	0.70 (0.01)
$\tau_{2f} - \tau_{2e} = \lfloor 0.15T \rfloor$				
Detection Rate (1)	0.94	0.95	0.94	0.93
$r_{1e} = 0.20$	0.26 (0.04)	0.26 (0.04)	0.26 (0.04)	0.27 (0.04)
$r_{1f} = 0.40$	0.40 (0.01)	0.40 (0.01)	0.40 (0.01)	0.39 (0.01)
Detection Rate (2)	0.04	0.89	0.84	0.11
$r_{2e} = 0.60$	0.71 (0.03)	0.65 (0.03)	0.65 (0.03)	0.70 (0.03)
$r_{2f} = 0.75$	0.75 (0.00)	0.75 (0.01)	0.75 (0.01)	0.75 (0.01)

NOTES: Calculations are based on 5,000 replications. The minimum window has 12 observations. Parameters are set to: $y_0 = 100$, $c = 1$, $\sigma = 6.79$, $\alpha = 0.6$, $\tau_{1e} = \lfloor 0.20T \rfloor$, $\tau_{2e} = \lfloor 0.60T \rfloor$, $\tau_{1f} - \tau_{1e} = \lfloor 0.20T \rfloor$, $T = 100$. Figures in parentheses are standard deviations.

In Table 5, the duration of the first bubble is 20% of the total sample. The duration of the second bubble is shorter than the first one, taking values $d_T = \tau_{2f} - \tau_{2e} = \lfloor 0.10T \rfloor, \lfloor 0.15T \rfloor$. As anticipated from asymptotic theory, PWY fails to detect the second bubble in this duration scenario. For instance, when $d_T = \lfloor 0.10T \rfloor$, the proportion of samples where the second bubble is detected using PWY is negligible (around 0.01). Noticeably, all algorithms perform well in identifying the first bubble. The average delay in detecting this bubble is four to seven observations.

The opposite setting is considered in the simulations reported in Table 6. Here the duration of the first bubble is fixed at $\lfloor 0.10T \rfloor$ and the duration of the second bubble varies from $\lfloor 0.10T \rfloor$ to $\lfloor 0.20T \rfloor$. Several results emerge from the table. First, there is no dramatic performance difference in identifying the first bubble among the dating algorithms. It is interesting to note that, due to its shorter bubble duration, the detection rates for the first bubble are lower than those in Table 5. Second, we observe a significant boost in the second bubble detection rate for the PWY strategy. In particular, when the duration of the second bubble is twice as long as the first, the detection rates of the PWY strategy is 77%. This outcome contrasts sharply with the

TABLE 6

DETECTION RATE AND ESTIMATION OF THE ORIGINATION AND TERMINATION DATES UNDER TWO BUBBLE DGP WITH LONGER SECOND BUBBLE DURATIONS

	PWY	PSY	Seq	CUSUM
$\tau_{2f} - \tau_{2e} = \lfloor 0.10T \rfloor$				
Detection Rate (1)	0.71	0.76	0.69	0.64
$r_{1e} = 0.20$	0.24 (0.02)	0.24 (0.02)	0.24 (0.02)	0.25 (0.02)
$r_{1f} = 0.30$	0.30 (0.00)	0.30 (0.00)	0.30 (0.01)	0.30 (0.01)
Detection Rate (2)	0.20	0.72	0.60	0.44
$r_{2e} = 0.60$	0.66 (0.02)	0.64 (0.02)	0.64 (0.02)	0.66 (0.02)
$r_{2f} = 0.70$	0.70 (0.00)	0.70 (0.00)	0.70 (0.01)	0.70 (0.00)
$\tau_{2f} - \tau_{2e} = \lfloor 0.15T \rfloor$				
Detection Rate (1)	0.71	0.76	0.69	0.64
$r_{1e} = 0.20$	0.24 (0.02)	0.24 (0.02)	0.24 (0.02)	0.25 (0.02)
$r_{1f} = 0.30$	0.30 (0.00)	0.30 (0.00)	0.30 (0.02)	0.30 (0.01)
Detection Rate (2)	0.54	0.86	0.78	0.76
$r_{2e} = 0.60$	0.69 (0.03)	0.65 (0.03)	0.66 (0.03)	0.68 (0.03)
$r_{2f} = 0.75$	0.75 (0.00)	0.75 (0.01)	0.75 (0.01)	0.75 (0.00)
$\tau_{2f} - \tau_{2e} = \lfloor 0.20T \rfloor$				
Detection Rate (1)	0.71	0.76	0.69	0.64
$r_{1e} = 0.20$	0.24 (0.02)	0.24 (0.02)	0.24 (0.02)	0.25 (0.02)
$r_{1f} = 0.30$	0.30 (0.00)	0.30 (0.00)	0.30 (0.00)	0.30 (0.01)
Detection Rate (2)	0.77	0.91	0.87	0.90
$r_{2e} = 0.60$	0.71 (0.04)	0.66 (0.04)	0.67 (0.04)	0.69 (0.04)
$r_{2f} = 0.80$	0.80 (0.00)	0.80 (0.01)	0.80 (0.01)	0.80 (0.01)

NOTES: Calculations are based on 5,000 replications. The minimum window has 12 observations. Parameters are set to: $y_0 = 100$, $c = 1$, $\sigma = 6.79$, $\alpha = 0.6$, $\tau_{1e} = \lfloor 0.20T \rfloor$, $\tau_{2e} = \lfloor 0.60T \rfloor$, $\tau_{1f} - \tau_{1e} = \lfloor 0.10T \rfloor$, $T = 100$. Figures in parentheses are standard deviations.

PWY detection rates for the second bubble displayed in Table 5. Third, there are relatively long delays in PWY detection of the second bubble. As a case in the point, when the duration of the second bubble is $\lfloor 0.20T \rfloor$, the PWY estimate of the origination date of the second bubble is 0.71 with a delay of 11 observations (nearly twice as long as the delay in detection of 6 observations when using PSY). Those findings corroborate closely the asymptotic theory, which shows how the PWY detector consistently estimates the first bubble but only identifies the second bubble with some delay when $r_{2f} - r_{2e} > r_{1f} - r_{1e}$.

In both experiments (Tables 5 and 6), the performance of the CUSUM procedure follows closely that of the PWY procedure. The PSY and the sequential PWY detectors are much more reliable in all cases, as shown in their higher detection rates and more timely detection of both bubbles. Overall, the findings indicate that the PSY strategy provides the best performance when there are two bubbles in the time series.

4.3 *Three Bubbles.* Table 7–10 report findings for the three bubble case. In Tables 7–9, we adjust the duration of one bubble to $d_T \in \{\lfloor 0.10T \rfloor, \lfloor 0.20T \rfloor\}$ and fix the durations of the other two bubbles. The bubbles originate 15%, 45%, and 75% into the sample and the bubble expansion rate is 1.04 in each case.

Results are similar to the two bubble case and are consistent with asymptotic theory in the more complex scenarios of multiple bubbles. First, when the duration of bubble i (for $i = 1, 2$) is longer than bubble $i + 1$, theory indicates that the PWY strategy is not capable of detecting the presence of bubble $i + 1$. The simulation findings in Table 7 show that, due to the longer duration of the second bubble, where $d_T = \lfloor 0.20T \rfloor$, the PWY detection rate is zero for the third bubble, whose duration is $d_T = \lfloor 0.10T \rfloor$. Similar results are found in Table 9 where the duration of the first bubble is longer than the second. An interesting feature of the PWY outcomes is that the presence of a long duration bubble causes weak identification of all subsequent bubbles. In

TABLE 7

DETECTION RATE AND ESTIMATES OF THE ORIGINATION AND TERMINATION DATES UNDER THREE BUBBLE DGP WITH DIFFERENT FIRST BUBBLE DURATIONS

	PWY	PSY	Seq	CUSUM
$\tau_{1f}-\tau_{1e}=\lfloor 0.1T\rfloor$, $\tau_{2f}-\tau_{2e}=\lfloor 0.2T\rfloor$, $\tau_{3f}-\tau_{3e}=\lfloor 0.1T\rfloor$				
Detection Rate (1)	0.72	0.75	0.69	0.64
$r_{1e}=0.15$	0.19 (0.02)	0.19 (0.02)	0.20 (0.02)	0.17 (0.02)
$r_{1f}=0.25$	0.25 (0.00)	0.25 (0.00)	0.25 (0.00)	0.25 (0.02)
Detection Rate (2)	0.79	0.93	0.90	0.90
$r_{2e}=0.45$	0.57 (0.04)	0.51 (0.04)	0.51 (0.04)	0.55 (0.04)
$r_{2f}=0.65$	0.65 (0.00)	0.65 (0.01)	0.65 (0.01)	0.65 (0.01)
Detection Rate (3)	0.00	0.72	0.82	0.01
$r_{3e}=0.75$	0.82 (0.01)	0.79 (0.02)	0.79 (0.02)	0.80 (0.02)
$r_{3f}=0.85$	0.85 (0.00)	0.85 (0.01)	0.85 (0.00)	0.85 (0.01)
$\tau_{1f}-\tau_{1e}=\lfloor 0.2T\rfloor$, $\tau_{2f}-\tau_{2e}=\lfloor 0.2T\rfloor$, $\tau_{3f}-\tau_{3e}=\lfloor 0.1T\rfloor$				
Detection Rate (1)	0.92	0.93	0.89	0.92
$r_{1e}=0.15$	0.21 (0.04)	0.21 (0.04)	0.21 (0.04)	0.22 (0.05)
$r_{1f}=0.35$	0.35 (0.01)	0.35 (0.01)	0.35 (0.01)	0.35 (0.02)
Detection Rate (2)	0.14	0.97	0.96	0.25
$r_{2e}=0.45$	0.60 (0.03)	0.51 (0.04)	0.50 (0.04)	0.60 (0.03)
$r_{2f}=0.65$	0.65 (0.00)	0.65 (0.01)	0.65 (0.01)	0.65 (0.01)
Detection Rate (3)	0.00	0.74	0.85	0.00
$r_{3e}=0.75$	—	0.79 (0.02)	0.79 (0.02)	0.80 (0.03)
$r_{3f}=0.85$	—	0.85 (0.01)	0.85 (0.00)	0.85 (0.00)

NOTES: Calculations are based on 5,000 replications. The minimum window has 12 observations. Parameters are set to: $y_0=100$, $c=1$, $\sigma=6.79$, $\alpha=0.6$, $T=100$, $\tau_{1e}=\lfloor 0.15T\rfloor$, $\tau_{2e}=\lfloor 0.45T\rfloor$, $\tau_{3e}=\lfloor 0.75T\rfloor$. Figures in parentheses are standard deviations.

TABLE 8

DETECTION RATE AND ESTIMATES OF THE ORIGINATION AND TERMINATION DATES UNDER THREE BUBBLE DGP WITH DIFFERENT SECOND BUBBLE DURATIONS

	PWY	PSY	Seq	CUSUM
$\tau_{1f}-\tau_{1e}=\lfloor 0.1T\rfloor$, $\tau_{2f}-\tau_{2e}=\lfloor 0.1T\rfloor$, $\tau_{3f}-\tau_{3e}=\lfloor 0.2T\rfloor$				
Detection Rate (1)	0.72	0.75	0.69	0.64
$r_{1e}=0.15$	0.19 (0.02)	0.19 (0.02)	0.20 (0.02)	0.19 (0.02)
$r_{1f}=0.25$	0.25 (0.00)	0.25 (0.00)	0.25 (0.00)	0.24 (0.01)
Detection Rate (2)	0.17	0.73	0.64	0.33
$r_{2e}=0.45$	0.51 (0.02)	0.49 (0.02)	0.49 (0.02)	0.51 (0.02)
$r_{2f}=0.55$	0.55 (0.00)	0.55 (0.00)	0.55 (0.01)	0.55 (0.00)
Detection Rate (3)	0.67	0.92	0.86	0.87
$r_{3e}=0.75$	0.88 (0.03)	0.81 (0.04)	0.81 (0.05)	0.86 (0.04)
$r_{3f}=0.95$	0.95 (0.00)	0.95 (0.01)	0.95 (0.01)	0.95 (0.01)
$\tau_{1f}-\tau_{1e}=\lfloor 0.1T\rfloor$, $\tau_{2f}-\tau_{2e}=\lfloor 0.2T\rfloor$, $\tau_{3f}-\tau_{3e}=\lfloor 0.2T\rfloor$				
Detection Rate (1)	0.72	0.75	0.69	0.64
$r_{1e}=0.15$	0.19 (0.02)	0.19 (0.02)	0.20 (0.02)	0.19 (0.02)
$r_{1f}=0.25$	0.25 (0.00)	0.25 (0.00)	0.25 (0.00)	0.25 (0.02)
Detection Rate (2)	0.79	0.93	0.90	0.90
$r_{2e}=0.45$	0.57 (0.04)	0.51 (0.04)	0.51 (0.04)	0.55 (0.04)
$r_{2f}=0.65$	0.65 (0.00)	0.65 (0.01)	0.65 (0.01)	0.65 (0.01)
Detection Rate (3)	0.12	0.94	0.92	0.23
$r_{3e}=0.75$	0.91 (0.02)	0.81 (0.04)	0.80 (0.04)	0.90 (0.04)
$r_{3f}=0.95$	0.95 (0.00)	0.95 (0.02)	0.95 (0.01)	0.95 (0.01)

NOTES: Calculations are based on 5,000 replications. The minimum window has 12 observations. Parameters are set to: $y_0=100$, $c=1$, $\sigma=6.79$, $\alpha=0.6$, $T=100$, $\tau_{1e}=\lfloor 0.15T\rfloor$, $\tau_{2e}=\lfloor 0.45T\rfloor$, $\tau_{3e}=\lfloor 0.75T\rfloor$. Figures in parentheses are standard deviations.

TABLE 9

DETECTION RATE AND ESTIMATION OF THE ORIGINATION AND TERMINATION DATES UNDER THREE BUBBLE DGP WITH DIFFERENT THIRD BUBBLE DURATIONS

	PWY	PSY	Seq	CUSUM
$\tau_{1f} - \tau_{1e} = \lfloor 0.2T \rfloor$, $\tau_{2f} - \tau_{2e} = \lfloor 0.1T \rfloor$, $\tau_{3f} - \tau_{3e} = \lfloor 0.1T \rfloor$				
Detection Rate (1)	0.92	0.93	0.89	0.92
$r_{1e} = 0.15$	0.21 (0.04)	0.21 (0.04)	0.21 (0.04)	0.22 (0.04)
$r_{1f} = 0.35$	0.35 (0.01)	0.35 (0.01)	0.35 (0.01)	0.35 (0.01)
Detection Rate (2)	0.00	0.75	0.84	0.01
$r_{2e} = 0.45$	0.50 (0.02)	0.49 (0.02)	0.49 (0.02)	0.51 (0.02)
$r_{2f} = 0.55$	0.55 (0.00)	0.55 (0.00)	0.55 (0.03)	0.55 (0.01)
Detection Rate (3)	0.01	0.74	0.67	0.04
$r_{3e} = 0.75$	0.81 (0.02)	0.79 (0.02)	0.79 (0.02)	0.81 (0.02)
$r_{3f} = 0.85$	0.85 (0.00)	0.85 (0.00)	0.85 (0.00)	0.85 (0.00)
$\tau_{1f} - \tau_{1e} = \lfloor 0.2T \rfloor$, $\tau_{2f} - \tau_{2e} = \lfloor 0.1T \rfloor$, $\tau_{3f} - \tau_{3e} = \lfloor 0.2T \rfloor$				
Detection Rate (1)	0.92	0.93	0.89	0.92
$r_{1e} = 0.15$	0.21 (0.04)	0.21 (0.04)	0.21 (0.04)	0.22 (0.04)
$r_{1f} = 0.35$	0.35 (0.01)	0.35 (0.01)	0.35 (0.01)	0.33 (0.01)
Detection Rate (2)	0.00	0.75	0.84	0.01
$r_{2e} = 0.45$	0.50 (0.02)	0.49 (0.02)	0.49 (0.02)	0.51 (0.02)
$r_{2f} = 0.55$	0.55 (0.00)	0.55 (0.00)	0.55 (0.03)	0.55 (0.01)
Detection Rate (3)	0.20	0.93	0.89	0.69
$r_{3e} = 0.75$	0.90 (0.03)	0.80 (0.04)	0.81 (0.04)	0.89 (0.03)
$r_{3f} = 0.95$	0.95 (0.00)	0.95 (0.01)	0.95 (0.01)	0.95 (0.00)

NOTES: Calculations are based on 5,000 replications. The minimum window has 12 observations. Parameters are set to: $y_0 = 100$, $c = 1$, $\sigma = 6.79$, $\alpha = 0.6$, $T = 100$, $\tau_{1e} = \lfloor 0.15T \rfloor$, $\tau_{2e} = \lfloor 0.45T \rfloor$, $\tau_{3e} = \lfloor 0.75T \rfloor$. Figures in parentheses are standard deviations.

TABLE 10

DETECTION RATE AND ESTIMATES OF THE ORIGINATION AND TERMINATION DATES UNDER THREE BUBBLE DGP AND SPECIAL EXAMPLES

	PWY	PSY	Seq	CUSUM
$\tau_{1f} - \tau_{1e} = \lfloor 0.1T \rfloor$, $\tau_{2f} - \tau_{2e} = \lfloor 0.2T \rfloor$, $\tau_{3f} - \tau_{3e} = \lfloor 0.10T \rfloor$, $\tau_{2e} = \lfloor 0.45T \rfloor$, $\tau_{3e} = \lfloor 0.70T \rfloor$				
Detection Rate (1)	0.72	0.75	0.69	0.64
$r_{1e} = 0.15$	0.19 (0.02)	0.19 (0.02)	0.20 (0.02)	0.19 (0.02)
$r_{1f} = 0.25$	0.25 (0.00)	0.25 (0.00)	0.25 (0.00)	0.25 (0.02)
Detection Rate (2)	0.79	0.93	0.90	0.90
$r_{2e} = 0.45$	0.57 (0.04)	0.51 (0.04)	0.51 (0.04)	0.55 (0.04)
$r_{2f} = 0.65$	0.65 (0.00)	0.65 (0.01)	0.65 (0.01)	0.65 (0.01)
Detection Rate (3)	0.00	0.61	0.01	0.01
$r_{3e} = 0.70$	0.77 (0.02)	0.77 (0.00)	0.77 (0.02)	0.75 (0.02)
$r_{3f} = 0.80$	0.79 (0.02)	0.80 (0.00)	0.84 (0.06)	0.80 (0.01)
$\tau_{1f} - \tau_{1e} = \lfloor 0.2T \rfloor$, $\tau_{2f} - \tau_{2e} = \lfloor 0.1T \rfloor$, $\tau_{3f} - \tau_{3e} = \lfloor 0.10T \rfloor$, $\tau_{2e} = \lfloor 0.40T \rfloor$, $\tau_{3e} = \lfloor 0.75T \rfloor$				
Detection Rate (1)	0.92	0.93	0.89	0.92
$r_{1e} = 0.15$	0.21 (0.04)	0.20 (0.04)	0.21 (0.04)	0.22 (0.04)
$r_{1f} = 0.35$	0.35 (0.01)	0.35 (0.01)	0.35 (0.01)	0.35 (0.01)
Detection Rate (2)	0.00	0.60	0.06	0.01
$r_{2e} = 0.40$	0.46 (0.02)	0.47 (0.00)	0.47 (0.01)	0.45 (0.03)
$r_{2f} = 0.50$	0.50 (0.00)	0.50 (0.00)	0.50 (0.01)	0.50 (0.01)
Detection Rate (3)	0.01	0.74	0.18	0.04
$r_{3e} = 0.75$	0.82 (0.02)	0.79 (0.02)	0.81 (0.02)	0.81 (0.02)
$r_{3f} = 0.85$	0.85 (0.00)	0.85 (0.00)	0.85 (0.00)	0.85 (0.00)

NOTES: Calculations are based on 5,000 replications. The minimum window has 12 observations. Parameters are set to: $y_0 = 100$, $c = 1$, $\sigma = 6.79$, $\alpha = 0.6$, $T = 100$, $\tau_{1e} = \lfloor 0.15T \rfloor$. Figures in parentheses are standard deviations.

particular, when the first bubble lasts longer than the second and third bubbles (the first panel of Table 9), the PWY detection rates of these two bubbles are 0.00 and 0.01.

Second, the simulations confirm that when the duration of bubble i is shorter than that of bubble $i+1$, the PWY strategy detects the existence of both bubbles but with a delay in the identification of bubble $i+1$. A case in point occurs in the first panel of Table 8, where the duration of the second bubble, is shorter than that of the third bubble. The detection rate of the third bubble using the PWY strategy is 0.67 and the length of the delay in the detection of this bubble is $\lfloor 0.13T \rfloor$, more than twice the delay incurred by the PSY detector. Third, just as for the two bubble case, the behavior of the CUSUM detector resembles that of PWY.

Fourth, the performances of PSY and sequential PWY are invariant to the relative durations among the bubbles. In other words, the frequency of detecting bubble i and the time needed to detect this bubble depend on the duration of this particular bubble, not on the duration of bubble j (for $j \neq i$).

Overall best performance is delivered by the PSY algorithm, followed by the sequential PWY strategy. Notice that when the duration of bubble i is twice as long as the duration of bubble $i+1$, the sequential PWY detection rate of bubble $i+1$ rises to a higher level than PSY. For example, in the first panel of Table 7 where $\tau_{2f} - \tau_{2e} = \lfloor 0.20T \rfloor$ and $\tau_{3f} - \tau_{3e} = \lfloor 0.10T \rfloor$, the third bubble detection rate of sequential PWY is 0.82, exceeding that of PSY at 0.72. This is due to the fact that the sequential procedure re-initializes after the collapse of the second bubble, and the first regression following re-initialization already covers several observations of the third bubble episode. This situation resembles the case of bubbles occurring at the beginning of the sample, which increases the bubble detection rate as shown in Table 3.

In extreme cases when the first regression after re-initialization covers most observations of the particular bubble episode, the sequential PWY procedure may fail to detect this bubble. Table 10 gives examples that forcefully illustrate this point. In the first panel of the table, the sequential PWY procedure re-initiates at $\lfloor 0.65T \rfloor$, and the undetectable period (due to the minimum regression window requirement of 12 observations) following this re-initialization is over the period $\lfloor 0.65T \rfloor$ to $\lfloor 0.77T \rfloor$ and covers most of the third bubble episode. As a result, the detection rate of the third bubble episode using the sequential PWY procedure is 0.01, whereas the detection rate of the third bubble using PSY is 61%. A further example occurs in the bottom panel of the same table. For the same reason, the sequential procedure fails to detect the second bubble episode in 94% of the cases—the detection rate reported in the table is 0.06. Noticeably, the unsuccessful detection of the second bubble also leads to a low detection rate for the third bubble, which may be partly explained by the fact that the remaining sample period includes two bubble episodes. In all of these cases the PSY detector works well with a high average detection rate (93%, 60%, and 74% for bubbles 1, 2, and 3, respectively) and an average delay of 4–7 observations in detection.

5. CONCLUSIONS

We develop limit theory for real-time dating of the origination and termination of mildly explosive periods using detectors based on the PWY, PSY, and sequential PWY algorithms. All three strategies rely on recursive right-tailed unit root tests but involve different types of recursion. The asymptotic performance of the detectors are evaluated using the extended PWY bubble model where mildly explosive bubble episodes are embedded within a longer period of normal stochastic trend behavior.

The PWY date estimates are shown to depend on the number of bubble episodes within the sample period and the relative durations of the bubbles when there are multiple bubble episodes. Specifically, in the single bubble case, the PWY estimators are consistent under some mild regularity conditions. When the sample period includes two bubble episodes, the PWY approach can consistently estimate the first bubble but not the second. The dating accuracy of the second bubble is related to the relative duration of the two bubbles. If the first bubble lasts longer than the second, the PWY strategy cannot detect occurrence of the second bubble. Alternatively, if the duration of the second bubble exceeds the first, the PWY detector finds the second bubble but with some delay even asymptotically. In contrast, the PSY approach and a sequential implementation of the PWY strategy both provide consistent estimators of all bubbles regardless of the number of bubble episodes occurring in the sample period and their relative duration.

Finite sample simulation are strongly confirmative of the asymptotics, indicating that the PSY algorithm is more reliable as a detector than the PWY strategy. The second best procedure is the sequential PWY strategy. The performance of the CUSUM procedure resembles that of the PWY strategy and has similar disadvantages in multiple bubble cases.

The results obtained here require some detailed and complex calculations to obtain the limit theory of the various recursive detection algorithms. Although these results are specific to the bubble model context under study, the methods should be useful in other recursive regression contexts. Also, with some modifications, the results continue to hold under more general conditions on the innovations than those used here and may be extended to deal with more general crisis and collapse processes (Phillips and Shi, 2014). The main requirements for these extensions are that the weak convergence (2) applies under normal periods and the limit theory for mildly explosive and mildly integrated periods applies, as it is known to do, under general forms of weak dependence (Phillips and Magdalinos, 2007b).

APPENDIX

A. *The Dating Algorithms (A Single Bubble)*

Section A.1 provides some useful preliminary results that characterize the limit behavior of the regression components over the various subperiods of the data. Section A2 provides test asymptotics and gives proofs of Theorems 1–3 which describe the consistency properties of the PWY and PSY dating strategies.

A.1. *Notation and useful preliminary lemmas.* We define the following notation:

- The bubble period $B = [\tau_e, \tau_f]$, where $\tau_e = \lfloor Tr_e \rfloor$ and $\tau_f = \lfloor Tr_f \rfloor$.
- The normal market periods $N_0 = [1, \tau_e)$ and $N_1 = [\tau_f + 1, T]$.
- The starting point of the regression $\tau_1 = \lfloor Tr_1 \rfloor$, the ending point of the regression $\tau_2 = \lfloor Tr_2 \rfloor$, the regression sample size $\tau_w = \lfloor Tr_w \rfloor$ with $r_w = r_2 - r_1$ and observation $t = \lfloor Tr \rfloor$.
- $B(.) \equiv \sigma W(.)$, where W is standard Brownian motion.

We use the data-generating process

$$X_t = \begin{cases} X_{t-1} + \varepsilon_t & \text{for } t \in N_0 \\ \delta_T X_{t-1} + \varepsilon_t & \text{for } t \in B \\ X^*_{\tau_f} + \sum_{k=\tau_f+1}^{t} \varepsilon_k & \text{for } t \in N_1 \end{cases}, \tag{A.1}$$

where $\delta_T = 1 + cT^{-\alpha}$ with $c > 0$ and $\alpha \in (0, 1)$, $\varepsilon_t \overset{i.i.d.}{\sim} (0, \sigma^2)$ and $X^*_{\tau_f} = X_{\tau_e} + X^*$ with $X^* = O_p(1)$. Under (A.1) we have the following lemmas.

LEMMA A1. *Under the data-generating process,*

(1) *For* $t \in N_0$, $X_{t=\lfloor Tr \rfloor} \sim_a T^{1/2} B(r)$.
(2) *For* $t \in B$, $X_{t=\lfloor Tr \rfloor} = \delta_T^{t-\tau_e} X_{\tau_e} \left\{1 + o_p(1)\right\} \sim_a T^{1/2} \delta_T^{t-\tau_e} B(r_e)$.
(3) *For* $t \in N_1$, $X_{t=\lfloor Tr \rfloor} \sim_a T^{1/2} [B(r) - B(r_f) + B(r_e)]$.

PROOF. (1) For $t \in N_0$, X_t is a unit root process. We know that $T^{-1/2}X_{t=\lfloor Tr \rfloor} \Rightarrow B(r)$ as $T \to \infty$. (2) For $t \in B$, the data-generating process

$$X_t = \delta_T X_{t-1} + \varepsilon_t = \delta_T^{t-\tau_e+1} X_{\tau_e-1} + \sum_{j=0}^{t-\tau_e} \delta_T^j \varepsilon_{t-j}.$$

Based on Phillips and Magdalinos (2007a, Lemma 4.2), we know that for $\alpha < 1$,

$$T^{-\alpha/2} \sum_{j=0}^{t-\tau_e} \delta_T^{-(t-\tau_e)+j} \varepsilon_{t-j} \xrightarrow{L} X_c \equiv N\left(0, \sigma^2/2c\right),$$

as $t - \tau_e \to \infty$. Furthermore, we know that $T^{-1/2}X_{\tau_e-1} \xrightarrow{L} B(r_e)$ and $\delta_T \to 1$ as $T \to \infty$. Therefore,

$$\begin{aligned}\delta_T^{-(t-\tau_e)} T^{-1/2} X_t &= \delta_T T^{-1/2} X_{\tau_e-1} + T^{-1/2} \sum_{j=0}^{t-\tau_e} \delta_T^{-(t-\tau_e)+j} \varepsilon_{t-j} \\ &= \delta_T T^{-1/2} X_{\tau_e-1} + T^{-(1-\alpha)/2} T^{-\alpha/2} \sum_{j=0}^{t-\tau_e} \delta_T^{-(t-\tau_e)+j} \varepsilon_{t-j} \xrightarrow{L} B(r_e).\end{aligned}$$

This implies that the first term has a higher order than the second term. Hence,

$$X_t = \delta_T^{t-\tau_e} X_{\tau_e} \left\{ 1 + \frac{\sum_{j=0}^{t-\tau_e-1} \delta_T^j \varepsilon_{t-j}}{\delta_T^{t-\tau_e} X_{\tau_e}} \right\} = \delta_T^{t-\tau_e} X_{\tau_e} \left\{1 + o_p(1)\right\} \sim_a T^{1/2} \delta_T^{t-\tau_e} B(r_e).$$

(3) For $t \in N_1$,

$$X_t = \sum_{k=\tau_f+1}^{t} \varepsilon_k + X_{\tau_f}^* = \sum_{k=\tau_f+1}^{t} \varepsilon_k + X_{\tau_e} + X^* \sim_a T^{1/2}\left[B(r) - B(r_f) + B(r_e)\right]$$

due to the fact that $X_{\tau_e} \sim_a T^{1/2}B(r_e)$, $\sum_{k=\tau_f+1}^{t} \varepsilon_k \sim_a T^{1/2}[B(r) - B(r_f)]$, and $X^* = O_p(1)$. ■

LEMMA A2. *Under the data-generating process,*

(1) *For $\tau_1 \in N_0$ and $\tau_2 \in B$,*

$$\frac{1}{\tau_w} \sum_{j=\tau_1}^{\tau_2} X_j = \frac{T^\alpha \delta_T^{\tau_2-\tau_e}}{\tau_w c} X_{\tau_e} \left\{1 + o_p(1)\right\} \sim_a T^{\alpha-1/2} \delta_T^{\tau_2-\tau_e} \frac{1}{r_w c} B(r_e).$$

(2) *For $\tau_1 \in B$ and $\tau_2 \in N_1$,*

$$\frac{1}{\tau_w} \sum_{j=\tau_1}^{\tau_2} X_j = \frac{T^\alpha \delta_T^{\tau_f-\tau_1}}{\tau_w c} X_{\tau_e} \left\{1 + o_p(1)\right\} \sim_a T^{\alpha-1/2} \delta_T^{\tau_f-\tau_1} \frac{1}{r_w c} B(r_e).$$

(3) *For $\tau_1 \in N_0$ and $\tau_2 \in N_1$,*

$$\frac{1}{\tau_w} \sum_{j=\tau_1}^{\tau_2} X_j = X_{\tau_e} \frac{T^\alpha \delta_T^{\tau_f-\tau_e}}{\tau_w c} \left\{1 + o_p(1)\right\} \sim_a T^{\alpha-1/2} \delta_T^{\tau_f-\tau_e} \frac{1}{r_w c} B(r_e).$$

PROOF. (1) For $\tau_1 \in N_0$ and $\tau_2 \in B$, we have

$$\frac{1}{\tau_w}\sum_{j=\tau_1}^{\tau_2} X_j = \frac{1}{\tau_w}\sum_{j=\tau_1}^{\tau_e-1} X_j + \frac{1}{\tau_w}\sum_{j=\tau_e}^{\tau_2} X_j.$$

The first term is

$$\begin{aligned}\frac{1}{\tau_w}\sum_{j=\tau_1}^{\tau_e-1} X_j &= T^{1/2}\frac{\tau_e-\tau_1}{\tau_w}\left(\frac{1}{\tau_e-\tau_1}\sum_{j=\tau_1}^{\tau_e-1}\frac{X_j}{\sqrt{T}}\right)\\ &\sim_a T^{1/2}\frac{r_e-r_1}{r_w}\int_{r_1}^{r_e} B(s)\,ds. \qquad \text{(A.2)}\end{aligned}$$

The second term is

$$\begin{aligned}\frac{1}{\tau_w}\sum_{j=\tau_e}^{\tau_2} X_j &= \frac{X_{\tau_e}}{\tau_w}\sum_{j=\tau_e}^{\tau_2}\delta_T^{j-\tau_e}\left\{1+o_p(1)\right\} \text{ from Lemma A1}\\ &= \frac{X_{\tau_e}}{\tau_w}\frac{\delta_T^{\tau_2-\tau_e+1}-1}{\delta_T-1}\left\{1+o_p(1)\right\}\\ &= X_{\tau_e}\frac{T^{\alpha}\delta_T^{\tau_2-\tau_e}+c\delta_T^{\tau_2-\tau_e}-T^{\alpha}}{\tau_w c}\left\{1+o_p(1)\right\}\\ &= X_{\tau_e}\frac{T^{\alpha}\delta_T^{\tau_2-\tau_e}}{\tau_w c}\left\{1+o_p(1)\right\} \sim_a T^{\alpha-1/2}\delta_T^{\tau_2-\tau_e}\frac{1}{r_w c}B(r_e). \qquad \text{(A.3)}\end{aligned}$$

Furthermore, we have

$$\frac{T^{\alpha-1/2}\delta_T^{\tau_2-\tau_e}}{T^{1/2}} = \frac{\delta_T^{\tau_2-\tau_e}}{T^{1-\alpha}} = \frac{e^{c(r_2-r_e)T^{1-\alpha}}}{T^{1-\alpha}} > 1.$$

This implies that $\tau_w^{-1}\sum_{j=\tau_e}^{\tau_2} X_j$ has a higher order than $\tau_w^{-1}\sum_{j=\tau_1}^{\tau_e-1} X_j$. Hence,

$$\begin{aligned}\frac{1}{\tau_w}\sum_{j=\tau_1}^{\tau_2} X_j &= \frac{1}{\tau_w}\sum_{j=\tau_e}^{\tau_2} X_j\left\{1+o_p(1)\right\}\\ &= \frac{T^{\alpha}\delta_T^{\tau_2-\tau_e}}{\tau_w c}X_{\tau_e}\left\{1+o_p(1)\right\} \text{ from Equation (A.3)}\\ &\sim_a T^{\alpha-1/2}\delta_T^{\tau_2-\tau_e}\frac{1}{r_w c}B(r_e).\end{aligned}$$

(2) For $\tau_1 \in B$ and $\tau_2 \in N_1$, we have

$$\frac{1}{\tau_w}\sum_{j=\tau_1}^{\tau_2} X_j = \frac{1}{\tau_w}\sum_{j=\tau_1}^{\tau_f} X_j + \frac{1}{\tau_w}\sum_{j=\tau_f+1}^{\tau_2} X_j.$$

The first term is

$$\begin{aligned}
\frac{1}{\tau_w}\sum_{j=\tau_1}^{\tau_f} X_j &= \frac{X_{\tau_e}}{\tau_w}\sum_{j=\tau_1}^{\tau_f}\delta_T^{j-\tau_e}\left\{1+o_p(1)\right\} \text{ from Lemma A1} \\
&= \frac{X_{\tau_e}}{\tau_w}\frac{\delta_T^{\tau_f-\tau_1+1}-1}{\delta_T-1}\left\{1+o_p(1)\right\} \\
&= \frac{X_{\tau_e}}{\tau_w}\frac{T^\alpha\delta_T^{\tau_f-\tau_1}+c\delta_T^{\tau_f-\tau_1}-T^\alpha}{c}\left\{1+o_p(1)\right\} \\
&= \frac{T^\alpha\delta_T^{\tau_f-\tau_1}}{\tau_w c}X_{\tau_e}\left\{1+o_p(1)\right\} \\
&\sim_a T^{\alpha-1/2}\delta_T^{\tau_f-\tau_1}\frac{1}{r_w c}B(r_e).
\end{aligned}$$

The second term is

(A.4)
$$\begin{aligned}
&\frac{1}{\tau_w}\sum_{j=\tau_f+1}^{\tau_2} X_j \\
&= \frac{1}{\tau_w}\sum_{j=\tau_f+1}^{\tau_2}\left[\sum_{k=\tau_f+1}^{j}\varepsilon_k+X_{\tau_e}\right] \\
&= T^{1/2}\frac{\tau_2-\tau_f}{\tau_w}\left[\frac{1}{\tau_2-\tau_f}\sum_{j=\tau_f+1}^{\tau_2}\left(T^{-1/2}\sum_{k=\tau_f+1}^{j}\varepsilon_k\right)\right]+T^{1/2}\frac{\tau_2-\tau_f}{\tau_w}\left(T^{-1/2}X_{\tau_e}\right) \\
&\sim_a T^{1/2}\frac{r_2-r_f}{r_w}\int_{r_f}^{r_2}\left[B(s)-B(r_f)\right]ds+T^{1/2}\frac{r_2-r_f}{r_w}B(r_e)
\end{aligned}$$

(A.5)
$$= T^{1/2}\frac{r_2-r_f}{r_w}\left\{\int_{r_f}^{r_2}\left[B(s)-B(r_f)\right]ds-B(r_e)\right\}.$$

Furthermore, we have

$$\frac{T^{\alpha-1/2}\delta_T^{\tau_f-\tau_1}}{T^{1/2}}=\frac{\delta_T^{\tau_f-\tau_1}}{T^{1-\alpha}}=\frac{e^{c(r_f-r_1)T^{1-\alpha}}}{T^{1-\alpha}}>1.$$

This implies that $\tau_w^{-1}\sum_{j=\tau_1}^{\tau_f}X_j$ has a higher order than $\tau_w^{-1}\sum_{j=\tau_f+1}^{\tau_2}X_j$. Hence,

$$\frac{1}{\tau_w}\sum_{j=\tau_1}^{\tau_2}X_j=\frac{T^\alpha\delta_T^{\tau_f-\tau_1}}{\tau_w c}X_{\tau_e}\left\{1+o_p(1)\right\}\sim_a T^{\alpha-1/2}\delta_T^{\tau_f-\tau_1}\frac{1}{r_w c}B(r_e).$$

(3) For $\tau_1\in N_0$ and $\tau_2\in N_1$,

$$\frac{1}{\tau_w}\sum_{j=\tau_1}^{\tau_2}X_j=\frac{1}{\tau_w}\sum_{j=\tau_1}^{\tau_e-1}X_j+\frac{1}{\tau_w}\sum_{j=\tau_e}^{\tau_f}X_j+\frac{1}{\tau_w}\sum_{j=\tau_f+1}^{\tau_2}X_j.$$

The first term is

$$\frac{1}{\tau_w}\sum_{j=\tau_1}^{\tau_e-1}X_j\sim_a T^{1/2}\frac{r_e-r_1}{r_w}\int_{r_1}^{r_e}B(s)\,ds \text{ from Equation (A.2).}$$

The second term is

$$\begin{aligned}
\frac{1}{\tau_w}\sum_{j=\tau_e}^{\tau_f} X_j &= \frac{X_{\tau_e}}{\tau_w}\sum_{j=\tau_e}^{\tau_f}\delta_T^{j-\tau_e}\left\{1+o_p\left(1\right)\right\} \text{ from Lemma A1} \\
&= \frac{X_{\tau_e}}{\tau_w}\frac{\delta_T^{\tau_f-\tau_e+1}-1}{\delta_T-1}\left\{1+o_p\left(1\right)\right\} \\
&= \frac{X_{\tau_e}}{\tau_w c}\left(T^{\alpha}\delta_T^{\tau_f-\tau_e}+c\delta_T^{\tau_f-\tau_e}-T^{\alpha}\right)\left\{1+o_p\left(1\right)\right\} \\
&= \frac{T^{\alpha}\delta_T^{\tau_f-\tau_e}}{\tau_w c}X_{\tau_e}\left\{1+o_p\left(1\right)\right\} \qquad \text{(A.6)} \\
&\sim_a T^{\alpha-1/2}\delta_T^{\tau_f-\tau_e}\frac{1}{r_w c}B\left(r_e\right).
\end{aligned}$$

The third term is

$$\frac{1}{\tau_w}\sum_{j=\tau_f+1}^{\tau_2} X_j \sim_a T^{1/2}\frac{r_2-r_f}{r_w}\left\{\int_{r_f}^{r_2}\left[B\left(s\right)-B\left(r_f\right)\right]ds-B\left(r_e\right)\right\} \text{ from Equation (A.5).}$$

Furthermore, we know

$$\frac{T^{\alpha-1/2}\delta_T^{\tau_f-\tau_e}}{T^{1/2}}=\frac{e^{c(r_f-r_e)T^{1-\alpha}}}{T^{1-\alpha}}>1.$$

This implies that $\tau_w^{-1}\sum_{j=\tau_e}^{\tau_f}X_j$ dominates $\tau_w^{-1}\sum_{j=\tau_1}^{\tau_e-1}X_j$ and $\tau_w^{-1}\sum_{j=\tau_f+1}^{\tau_2}X_j$. Therefore,

$$\frac{1}{\tau_w}\sum_{j=\tau_1}^{\tau_2}X_j=\frac{T^{\alpha}\delta_T^{\tau_f-\tau_e}}{\tau_w c}X_{\tau_e}\left\{1+o_p\left(1\right)\right\}\sim_a T^{\alpha-1/2}\delta_T^{\tau_f-\tau_e}\frac{1}{cr_w}B\left(r_e\right). \qquad \blacksquare$$

LEMMA A3. *Define the centered quantity* $\tilde{X}_t=X_t-\tau_w^{-1}\sum_{j=\tau_1}^{\tau_2}X_j$.

(1) *For* $\tau_1\in N_0$ *and* $\tau_2\in B$,

$$\tilde{X}_t=\begin{cases} -\frac{T^{\alpha}\delta_T^{\tau_2-\tau_e}}{\tau_w c}X_{\tau_e}\left\{1+o_p\left(1\right)\right\} & \text{if } t\in N_0 \\ \left[\delta_T^{t-\tau_e}-\frac{T^{\alpha}\delta_T^{\tau_2-\tau_e}}{\tau_w c}\right]X_{\tau_e}\left\{1+o_p\left(1\right)\right\} & \text{if } t\in B \end{cases}.$$

(2) *For* $\tau_1\in B$ *and* $\tau_2\in N_1$,

$$\tilde{X}_t=\begin{cases} \left[\delta_T^{t-\tau_e}-\frac{T^{\alpha}\delta_T^{\tau_f-\tau_1}}{\tau_w c}\right]X_{\tau_e}\left\{1+o_p\left(1\right)\right\} & \text{if } t\in B \\ -\frac{T^{\alpha}\delta_T^{\tau_f-\tau_1}}{\tau_w c}X_{\tau_e}\left\{1+o_p\left(1\right)\right\} & \text{if } t\in N_1 \end{cases}.$$

(3) *For* $\tau_1\in N_0$ *and* $\tau_2\in N_1$,

$$\tilde{X}_t=\begin{cases} -\frac{T^{\alpha}\delta_T^{\tau_f-\tau_e}}{\tau_w c}X_{\tau_e}\left\{1+o_p\left(1\right)\right\} & \text{if } t\in N_0\cup N_1 \\ \left[\delta_T^{t-\tau_e}-\frac{T^{\alpha}\delta_T^{\tau_f-\tau_e}}{\tau_w c}\right]X_{\tau_e}\left\{1+o_p\left(1\right)\right\} & \text{if } t\in B \end{cases}.$$

PROOF. (1) Suppose $\tau_1 \in N_0$ and $\tau_2 \in B$. If $t \in N_0$,

$$\tilde{X}_t = X_t - \tau_w^{-1} \sum_{j=\tau_1}^{\tau_2} X_j = -\frac{T^{\alpha} \delta_T^{\tau_2 - \tau_e}}{\tau_w c} X_{\tau_e} \left\{1 + o_p(1)\right\}, \tag{A.7}$$

where the second term dominates the first term due to the fact that

$$T^{-1/2} X_t \sim_a B(r) \text{ from Lemma A1}$$

$$\frac{1}{\tau_w} \sum_{j=\tau_1}^{\tau_2} X_j \sim_a T^{\alpha - 1/2} \delta_T^{\tau_2 - \tau_e} \frac{1}{r_w c} B(r_e) \text{ from Lemma A2}$$

and

$$\frac{T^{\alpha - 1/2} \delta_T^{\tau_2 - \tau_e}}{T^{1/2}} = \frac{e^{c(r_2 - r_e) T^{1-\alpha}}}{T^{1-\alpha}} > 1.$$

If $t \in B$,

$$\tilde{X}_t = X_t - \tau_w^{-1} \sum_{j=\tau_1}^{\tau_2} X_j = \left[\delta_T^{t - \tau_e} - \frac{T^{\alpha} \delta_T^{\tau_2 - \tau_e}}{\tau_w c}\right] X_{\tau_e} \left\{1 + o_p(1)\right\}.$$

(2) Suppose $\tau_1 \in B$ and $\tau_2 \in N_1$. If $t \in B$,

$$\tilde{X}_t = X_t - \tau_w^{-1} \sum_{j=\tau_1}^{\tau_2} X_j = \left[\delta_T^{t - \tau_e} - \frac{T^{\alpha} \delta_T^{\tau_f - \tau_1}}{\tau_w c}\right] X_{\tau_e} \left\{1 + o_p(1)\right\}.$$

If $t \in N_1$,

$$\tilde{X}_t = X_t - \tau_w^{-1} \sum_{j=\tau_1}^{\tau_2} X_j = -\frac{T^{\alpha} \delta_T^{\tau_f - \tau_1}}{\tau_w c} X_{\tau_e} \left\{1 + o_p(1)\right\},$$

where the second term dominates the first term due to the fact that

$$X_{t = \lfloor Tr \rfloor} \sim_a T^{1/2} \left[B(r) - B(r_f) + B(r_e)\right] \text{ from Lemma A1}$$

$$\frac{1}{\tau_w} \sum_{j=\tau_1}^{\tau_2} X_j \sim_a T^{\alpha - 1/2} \delta_T^{\tau_f - \tau_1} \frac{1}{r_w c} B(r_e) \text{ from Lemma A2}$$

and

$$\frac{T^{\alpha - 1/2} \delta_T^{\tau_f - \tau_1}}{T^{1/2}} = \frac{\delta_T^{\tau_f - \tau_1}}{T^{1-\alpha}} = \frac{e^{c(r_f - r_1) T^{1-\alpha}}}{T^{1-\alpha}} > 1.$$

(3) Suppose $\tau_1 \in N_0$ and $\tau_2 \in N_1$. If $t \in N_0$,

$$\tilde{X}_t = X_t - \tau_w^{-1} \sum_{j=\tau_1}^{\tau_2} X_j = -\frac{T^{\alpha} \delta_T^{\tau_f - \tau_e}}{\tau_w c} X_{\tau_e} \left\{1 + o_p(1)\right\},$$

where the second term dominates the first term due to the fact that

$$X_{t=\lfloor Tr \rfloor} \sim_a T^{1/2} B(r) \text{ from Lemma A1}$$

$$\frac{1}{\tau_w} \sum_{j=\tau_1}^{\tau_2} X_j \sim_a T^{\alpha-1/2} \delta_T^{\tau_f - \tau_e} \frac{1}{r_w c} B(r_e) \text{ from Lemma A2}$$

and

$$\frac{T^{\alpha-1/2} \delta_T^{\tau_f-\tau_e}}{T^{1/2}} > 1.$$

If $t \in B$,

$$\tilde{X}_t = X_t - \tau_w^{-1} \sum_{j=\tau_1}^{\tau_2} X_j = \left[\delta_T^{t-\tau_e} - \frac{T^{\alpha} \delta_T^{\tau_f-\tau_e}}{\tau_w c} \right] X_{\tau_e} \left\{ 1 + o_p(1) \right\}.$$

If $t \in N_1$,

$$\tilde{X}_t = X_t - \tau_w^{-1} \sum_{j=\tau_1}^{\tau_2} X_j = -\frac{T^{\alpha} \delta_T^{\tau_f-\tau_e}}{\tau_w c} X_{\tau_e} \left\{ 1 + o_p(1) \right\},$$

since $X_{t=\lfloor Tr \rfloor} \sim_a T^{1/2} [B(r) - B(r_f) + B(r_e)]$ (from Lemma A1). ∎

LEMMA A4. *The sample variance terms involving $\tilde{X}_t$ behave as follows.*

(1) *For $\tau_1 \in N_0$ and $\tau_2 \in B$,*

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}^2 = \frac{T^{\alpha} \delta_T^{2(\tau_2-\tau_e)}}{2c} X_{\tau_e}^2 \left\{ 1 + o_p(1) \right\} \sim_a \frac{T^{1+\alpha} \delta_T^{2(\tau_2-\tau_e)}}{2c} B(r_e)^2.$$

(2) *For $\tau_1 \in B$ and $\tau_2 \in N_1$,*

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}^2 = \frac{T^{\alpha} \delta_T^{2(\tau_f-\tau_e)}}{2c} X_{\tau_e}^2 \left\{ 1 + o_p(1) \right\} \sim_a \frac{T^{\alpha+1} \delta_T^{2(\tau_f-\tau_e)}}{2c} B(r_e)^2.$$

(3) *For $\tau_1 \in N_0$ and $\tau_2 \in N_1$,*

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}^2 = \frac{T^{\alpha} \delta_T^{2(\tau_f-\tau_e)}}{2c} X_{\tau_e}^2 \left\{ 1 + o_p(1) \right\} \sim_a \frac{T^{\alpha+1} \delta_T^{2(\tau_f-\tau_e)}}{2c} B(r_e)^2.$$

PROOF. (1) For $\tau_1 \in N_0$ and $\tau_2 \in B$,

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}^2 = \sum_{j=\tau_1}^{\tau_e} \tilde{X}_{j-1}^2 + \sum_{j=\tau_e}^{\tau_2} \tilde{X}_{j-1}^2.$$

The first term is

$$\begin{aligned}
\sum_{j=\tau_1}^{\tau_e-1} \tilde{X}_{j-1}^2 &= \sum_{j=\tau_1}^{\tau_e-1} \frac{T^{2\alpha}\delta_T^{2(\tau_2-\tau_e)}}{\tau_w^2 c^2} X_{\tau_e}^2 \left\{1+o_p(1)\right\} \text{ from Lemma A3} \\
&= \frac{\tau_e-\tau_1}{\tau_w^2 c^2} T^{2\alpha}\delta_T^{2(\tau_2-\tau_e)} X_{\tau_e}^2 \left\{1+o_p(1)\right\} \\
&\sim_a \frac{r_e-r_1}{r_w^2 c} T^{2\alpha}\delta_T^{2(\tau_2-\tau_e)} B(r_e).
\end{aligned}$$

Given that

$$\sum_{j=\tau_e}^{\tau_2} \delta_T^{2(j-1-\tau_e)} = \frac{\delta_T^{2(\tau_2-\tau_e)}-\delta_T^{-2}}{\delta_T^2-1} = \frac{T^{\alpha}\delta_T^{2(\tau_2-\tau_e)}}{2c}\left\{1+o_p(1)\right\}$$

$$\sum_{j=\tau_e}^{\tau_2} \delta_T^{j-1-\tau_e} = \frac{\delta_T^{\tau_2-\tau_e}-\delta_T^{-1}}{\delta_T-1} = \frac{T^{\alpha}\delta_T^{\tau_2-\tau_e}}{c}\left\{1+o_p(1)\right\},$$

the second term

$$\begin{aligned}
\sum_{j=\tau_e}^{\tau_2} \tilde{X}_{j-1}^2 &= \sum_{j=\tau_e}^{\tau_2}\left[\delta_T^{j-1-\tau_e} - \frac{T^{\alpha}\delta_T^{\tau_2-\tau_e}}{\tau_w c}\right]^2 X_{\tau_e}^2\left\{1+o_p(1)\right\} \\
&= \sum_{j=\tau e}^{\tau_2}\left[\delta_T^{2(j-1-\tau_e)} - 2\delta_T^{j-1-\tau_e}\frac{T^{\alpha}\delta_T^{\tau_2-\tau_e}}{\tau_w c} + \frac{T^{2\alpha}\delta_T^{2(\tau_2-\tau_e)}}{\tau_w^2 c^2}\right] X_{\tau_e}^2\left\{1+o_p(1)\right\} \\
&= \left[\frac{T^{\alpha}\delta_T^{2(\tau_2-\tau_e)}}{2c} - 2\frac{T^{2\alpha-1}\delta_T^{2(\tau_2-\tau_e)}}{r_w c^2} + \frac{r_2-r_e+\frac{1}{T}}{r_w^2 c^2} T^{2\alpha-1}\delta_T^{2(\tau_2-\tau_e)}\right] X_{\tau_e}^2\left\{1+o_p(1)\right\} \\
&= \frac{T^{\alpha}\delta_T^{2(\tau_2-\tau_e)}}{2c} X_{\tau_e}^2\left\{1+o_p(1)\right\} \text{ (since } \alpha > 2\alpha-1) \sim_a \frac{T^{1+\alpha}\delta_T^{2(\tau_2-\tau_e)}}{2c} B(r_e)^2.
\end{aligned}$$

Since $1+\alpha > 2\alpha$, $\sum_{j=\tau_e}^{\tau_2} \tilde{X}_{j-1}^2$ dominates $\sum_{j=\tau_1}^{\tau_e} \tilde{X}_{j-1}^2$. Therefore,

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}^2 = \sum_{j=\tau_e}^{\tau_2} \tilde{X}_{j-1}^2\left\{1+o_p(1)\right\} = \frac{T^{\alpha}\delta_T^{2(\tau_2-\tau_e)}}{2c} X_{\tau_e}^2\left\{1+o_p(1)\right\} \sim_a \frac{T^{1+\alpha}\delta_T^{2(\tau_2-\tau_e)}}{2c} B(r_e)^2.$$

(2) For $\tau_1 \in B$ and $\tau_2 \in N_1$,

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}^2 = \sum_{j=\tau_1}^{\tau_f} \tilde{X}_{j-1}^2 + \sum_{j=\tau_f+1}^{\tau_2} \tilde{X}_{j-1}^2.$$

Given that

$$\sum_{j=\tau_1}^{\tau_f} \delta_T^{2(j-1-\tau_e)} = \frac{T^{\alpha}\left[\delta_T^{2(\tau_f-\tau_e)} - \delta_T^{2(\tau_1-\tau_e-1)}\right]}{2c+c^2T^{-\alpha}} = \frac{T^{\alpha}\delta_T^{2(\tau_f-\tau_e)}}{2c}\left\{1+o_p(1)\right\}$$

$$\sum_{j=\tau_1}^{\tau_f} \delta_T^{j-1-\tau_e} = \frac{T^{\alpha}\left[\delta_T^{\tau_f-\tau_e} - \delta_T^{\tau_1-\tau_e-1}\right]}{c} = \frac{T^{\alpha}\delta_T^{\tau_f-\tau_e}}{c}\left\{1+o_p(1)\right\},$$

the first term is

$$
\begin{aligned}
\sum_{j=\tau_1}^{\tau_f} \tilde{X}_{j-1}^2 &= \sum_{j=\tau_1}^{\tau_f} \left[\delta_T^{j-1-\tau_e} - \frac{T^\alpha \delta_T^{\tau_f-\tau_1}}{\tau_w c} \right]^2 X_{\tau_e}^2 \left\{1 + o_p(1)\right\} \\
&= \left[\frac{T^\alpha \delta_T^{2(\tau_f-\tau_e)}}{2c} - 2\frac{T^\alpha \delta_T^{\tau_f-\tau_1}}{\tau_w c} \frac{T^\alpha \delta_T^{\tau_f-\tau_e}}{c} + \frac{\tau_f - \tau_1 + 1}{\tau_w^2 c^2} T^{2\alpha} \delta_T^{2(\tau_f-\tau_1)} \right] X_{\tau_e}^2 \left\{1 + o_p(1)\right\} \\
&= \left[\frac{T^\alpha \delta_T^{2(\tau_f-\tau_e)}}{2c} - 2\frac{\delta_T^{(\tau_f-\tau_1)+(\tau_f-\tau_e)}}{T^{1-2\alpha} r_w c^2} + \frac{r_f - r_1 + \frac{1}{T}}{T^{1-2\alpha} r_w^2 c^2} \delta_T^{2(\tau_f-\tau_1)} \right] X_{\tau_e}^2 \left\{1 + o_p(1)\right\} \\
&= \frac{T^\alpha \delta_T^{2(\tau_f-\tau_e)}}{2c} X_{\tau_e}^2 \left\{1 + o_p(1)\right\} \text{ (since } \alpha > 2\alpha - 1 \text{ and } \tau_f - \tau_e > \tau_f - \tau_1\text{)} \\
&\sim_a \frac{T^{\alpha+1} \delta_T^{2(\tau_f-\tau_e)}}{2c} B(r_e)^2 .
\end{aligned}
$$

The second term is

$$
\begin{aligned}
\sum_{j=\tau_f+1}^{\tau_2} \tilde{X}_{j-1}^2 &= \sum_{j=\tau_f+1}^{\tau_2} \frac{T^{2\alpha} \delta_T^{2(\tau_f-\tau_1)}}{\tau_w^2 c^2} X_{\tau_e}^2 \left\{1 + o_p(1)\right\} \\
&= \frac{\tau_2 - \tau_f}{\tau_w^2 c^2} T^{2\alpha} \delta_T^{2(\tau_f-\tau_1)} X_{\tau_e}^2 \left\{1 + o_p(1)\right\} \\
&\sim_a \frac{r_2 - r_f}{r_w^2 c^2} T^{2\alpha} \delta_T^{2(\tau_f-\tau_1)} B(r_e)^2 .
\end{aligned}
$$

Since $1 + \alpha > 2\alpha$, $\sum_{j=\tau_1}^{\tau_f} \tilde{X}_{j-1}^2$ dominates $\sum_{j=\tau_f+1}^{\tau_2} \tilde{X}_{j-1}^2$. Therefore,

$$
\begin{aligned}
\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}^2 &= \sum_{j=\tau_1}^{\tau_f} \tilde{X}_{j-1}^2 \left\{1 + o_p(1)\right\} = \frac{T^\alpha \delta_T^{2(\tau_f-\tau_e)}}{2c} X_{\tau_e}^2 \left\{1 + o_p(1)\right\} \\
&\sim_a \frac{T^{\alpha+1} \delta_T^{2(\tau_f-\tau_e)}}{2c} B(r_e)^2 .
\end{aligned}
$$

(3) For $\tau_1 \in N_0$ and $\tau_2 \in N_1$,

$$
\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}^2 = \sum_{j=\tau_1}^{\tau_e-1} \tilde{X}_{j-1}^2 + \sum_{j=\tau_e}^{\tau_f} \tilde{X}_{j-1}^2 + \sum_{j=\tau_f+1}^{\tau_2} \tilde{X}_{j-1}^2 .
$$

The first term is

$$
\begin{aligned}
\sum_{j=\tau_1}^{\tau_e-1} \tilde{X}_{j-1}^2 &= \sum_{j=\tau_1}^{\tau_e-1} \frac{T^{2\alpha} \delta_T^{2(\tau_f-\tau_e)}}{\tau_w^2 c^2} X_{\tau_e}^2 \left\{1 + o_p(1)\right\} \\
&= \frac{\tau_e - \tau_1}{\tau_w^2 c^2} T^{2\alpha} \delta_T^{2(\tau_f-\tau_e)} X_{\tau_e}^2 \left\{1 + o_p(1)\right\} \\
&\sim_a \frac{r_e - r_1}{r_w^2 c^2} T^{2\alpha} \delta_T^{2(\tau_f-\tau_e)} B(r_e)^2 .
\end{aligned}
$$

Given that

$$\sum_{j=\tau_e}^{\tau_f} \delta_T^{2(j-1-\tau_e)} = \frac{\delta_T^{2(\tau_f-\tau_e)} - \delta_T^{-2}}{\delta_T^2 - 1} = \frac{T^\alpha \delta_T^{2(\tau_f-\tau_e)}}{2c}\left\{1 + o_p(1)\right\}$$

$$\sum_{j=\tau_e}^{\tau_f} \delta_T^{j-1-\tau_e} = \frac{\delta_T^{\tau_f-\tau_e} - \delta_T^{-1}}{\delta_T - 1} = \frac{T^\alpha \delta_T^{\tau_f-\tau_e}}{c}\left\{1 + o_p(1)\right\},$$

the second term

$$\begin{aligned}
\sum_{j=\tau_e}^{\tau_f} \tilde{X}_{j-1}^2 &= \sum_{j=\tau_e}^{\tau_f}\left[\delta_T^{j-1-\tau_e} - \frac{T^\alpha \delta_T^{\tau_f-\tau_e}}{\tau_w c}\right]^2 X_{\tau_e}^2\left\{1 + o_p(1)\right\} \\
&= \left[\frac{T^\alpha \delta_T^{2(\tau_f-\tau_e)}}{2c} - 2\frac{\delta_T^{2(\tau_f-\tau_e)}}{T^{1-2\alpha} r_w c^2} + \frac{r_f - r_e + \frac{1}{T}}{T^{1-2\alpha} r_w^2 c^2}\delta_T^{2(\tau_f-\tau_e)}\right] X_{\tau_e}^2\left\{1 + o_p(1)\right\} \\
&= \frac{T^\alpha \delta_T^{2(\tau_f-\tau_e)}}{2c} X_{\tau_e}^2\left\{1 + o_p(1)\right\} \ (\text{since } \alpha > 2\alpha - 1) \\
&\sim_a \frac{T^{\alpha+1} \delta_T^{2(\tau_f-\tau_e)}}{2c} B(r_e)^2 .
\end{aligned}$$

The third term is

$$\begin{aligned}
\sum_{j=\tau_f+1}^{\tau_2} \tilde{X}_{j-1}^2 &= \sum_{j=\tau_f+1}^{\tau_2} \frac{T^{2\alpha} \delta_T^{2(\tau_f-\tau_e)}}{\tau_w^2 c^2} X_{\tau_e}^2\left\{1 + o_p(1)\right\} \\
&= \frac{\tau_2 - \tau_f}{\tau_w^2 c^2} T^{2\alpha} \delta_T^{2(\tau_f-\tau_e)} X_{\tau_e}^2\left\{1 + o_p(1)\right\} \\
&\sim_a \frac{r_2 - r_f}{r_w^2 c^2} T^{2\alpha} \delta_T^{2(\tau_f-\tau_e)} B(r_e)^2 .
\end{aligned}$$

Since $1 + \alpha > 2\alpha$, $\sum_{j=\tau_e}^{\tau_f} \tilde{X}_{j-1}^2$ dominates the other two terms. Therefore,

$$\begin{aligned}
\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}^2 &= \sum_{j=\tau_e}^{\tau_f} \tilde{X}_{j-1}^2\left\{1 + o_p(1)\right\} = \frac{T^\alpha \delta_T^{2(\tau_f-\tau_e)}}{2c} X_{\tau_e}^2\left\{1 + o_p(1)\right\} \\
&\sim_a \frac{T^{\alpha+1} \delta_T^{2(\tau_f-\tau_e)}}{2c} B(r_e)^2 .
\end{aligned}$$

■

LEMMA A5. *The sample covariance of $\tilde{X}_t$ and ε_t behaves as follows.*

(1) *For $\tau_1 \in N_0$ and $\tau_2 \in B$,*

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}\varepsilon_j = \sum_{j=\tau_e}^{\tau_2} \tilde{X}_{j-1}\varepsilon_j\left\{1 + o_p(1)\right\} \sim_a T^{(\alpha+1)/2} \delta_T^{\tau_2-\tau_e} X_c B(r_e) .$$

(2) *For* $\tau_1 \in B$ *and* $\tau_2 \in N_1$,

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}\varepsilon_j = \sum_{j=\tau_1}^{\tau_f} \tilde{X}_{j-1}\varepsilon_j \left\{1 + o_p(1)\right\} \sim_a T^{(\alpha+1)/2}\delta_T^{\tau_f-\tau_e} X_c B(r_e).$$

(3) *For* $\tau_1 \in N_0$ *and* $\tau_2 \in N_1$,

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}\varepsilon_j = \sum_{j=\tau_e}^{\tau_f} \tilde{X}_{j-1}\varepsilon_j \left\{1 + o_p(1)\right\} \sim_a T^{(\alpha+1)/2}\delta_T^{\tau_f-\tau_e} X_c B(r_e).$$

PROOF. (1) For $\tau_1 \in N_0$ and $\tau_2 \in B$,

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}\varepsilon_j = \sum_{j=\tau_1}^{\tau_e-1} \tilde{X}_{j-1}\varepsilon_j + \sum_{j=\tau_e}^{\tau_2} \tilde{X}_{j-1}\varepsilon_j.$$

The first term is

$$\begin{aligned}
\sum_{j=\tau_1}^{\tau_e-1} \tilde{X}_{j-1}\varepsilon_j &= \sum_{j=\tau_1}^{\tau_e-1} -\frac{T^\alpha \delta_T^{\tau_2-\tau_e}}{\tau_w c} X_{\tau_e}\varepsilon_j \left\{1 + o_p(1)\right\} \\
&= -\frac{T^\alpha \delta_T^{\tau_2-\tau_e}}{\tau_w c} X_{\tau_e} \sum_{j=\tau_1}^{\tau_e-1} \varepsilon_j \left\{1 + o_p(1)\right\} \\
&= -\frac{T^\alpha \delta_T^{\tau_2-\tau_e}}{r_w c} \left(T^{-1/2} X_{\tau_e}\right) \left(T^{-1/2} \sum_{j=\tau_1}^{\tau_e-1} \varepsilon_j\right) \left\{1 + o_p(1)\right\} \\
&\sim_a -\frac{T^\alpha \delta_T^{\tau_2-\tau_e}}{r_w c} B(r_e)\left[B(r_e) - B(r_1)\right].
\end{aligned}$$

The second term is

$$\begin{aligned}
\sum_{j=\tau_e}^{\tau_2} \tilde{X}_{j-1}\varepsilon_j &= \sum_{j=\tau_e}^{\tau_2} \left[\delta_T^{j-1-\tau_e} - \frac{T^\alpha \delta_T^{\tau_2-\tau_e}}{\tau_w c}\right] X_{\tau_e}\varepsilon_j \left\{1 + o_p(1)\right\} \\
&= \left[\sum_{j=\tau_e}^{\tau_2} \delta_T^{j-1-\tau_e}\varepsilon_j - \frac{T^\alpha \delta_T^{\tau_2-\tau_e}}{\tau_w c} \sum_{j=\tau_e}^{\tau_2} \varepsilon_j\right] X_{\tau_e} \left\{1 + o_p(1)\right\} \\
&= \left[T^{\alpha/2}\delta_T^{\tau_2-\tau_e} \left(\frac{1}{T^{\alpha/2}} \sum_{j=\tau_e}^{\tau_2} \delta_T^{-(\tau_2-j+1)}\varepsilon_j\right) - \frac{\delta_T^{\tau_2-\tau_e}}{T^{1/2-\alpha} r_w c} \left(\frac{1}{\sqrt{T}} \sum_{j=\tau_e}^{\tau_2} \varepsilon_j\right)\right] X_{\tau_e} \left\{1 + o_p(1)\right\} \\
&= T^{\alpha/2}\delta_T^{\tau_2-\tau_e} \left(T^{-\alpha/2} \sum_{j=\tau_e}^{\tau_2} \delta_T^{-(\tau_2-j+1)}\varepsilon_j\right) X_{\tau_e} \left\{1 + o_p(1)\right\} \ (\text{since } \alpha/2 > \alpha - 1/2) \\
&\sim_a T^{(\alpha+1)/2}\delta_T^{\tau_2-\tau_e} X_c B(r_e).
\end{aligned}$$

Since $(\alpha+1)/2 > \alpha$, $\sum_{j=\tau_e}^{\tau_2} \tilde{X}_{j-1}\varepsilon_j$ dominates $\sum_{j=\tau_1}^{\tau_e-1} \tilde{X}_{j-1}\varepsilon_j$. Therefore,

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}\varepsilon_j = \sum_{j=\tau_e}^{\tau_2} \tilde{X}_{j-1}\varepsilon_j \left\{1+o_p(1)\right\} \sim_a T^{(\alpha+1)/2}\delta_T^{\tau_2-\tau_e} X_c B(r_e).$$

(2) For $\tau_1 \in B$ and $\tau_2 \in N_1$,

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}\varepsilon_j = \sum_{j=\tau_1}^{\tau_f} \tilde{X}_{j-1}\varepsilon_j + \sum_{j=\tau_f+1}^{\tau_2} \tilde{X}_{j-1}\varepsilon_j.$$

The first term is

$$\begin{aligned}
\sum_{j=\tau_1}^{\tau_f} \tilde{X}_{j-1}\varepsilon_j &= \sum_{j=\tau_1}^{\tau_f} \left[\delta_T^{j-1-\tau_e} - \frac{T^{\alpha}\delta_T^{\tau_f-\tau_1}}{\tau_w c}\right] X_{\tau_e}\varepsilon_j \left\{1+o_p(1)\right\} \\
&= \left[\sum_{j=\tau_1}^{\tau_f} \delta_T^{j-1-\tau_e}\varepsilon_j - \frac{T^{\alpha}\delta_T^{\tau_f-\tau_1}}{\tau_w c}\sum_{j=\tau_1}^{\tau_f}\varepsilon_j\right] X_{\tau_e}\left\{1+o_p(1)\right\} \\
&= \left[T^{\alpha/2}\delta_T^{\tau_f-\tau_e}\left(\frac{1}{T^{\alpha/2}}\sum_{j=\tau_1}^{\tau_f}\delta_T^{-(\tau_f-j+1)}\varepsilon_j\right) - \frac{T^{\alpha+1/2}\delta_T^{\tau_f-\tau_1}}{\tau_w c}\left(\frac{1}{\sqrt{T}}\sum_{j=\tau_1}^{\tau_f}\varepsilon_j\right)\right] X_{\tau_e}\left\{1+o_p(1)\right\} \\
&= T^{\alpha/2}\delta_T^{\tau_f-\tau_e}\left(T^{-\alpha/2}\sum_{j=\tau_1}^{\tau_f}\delta_T^{-(\tau_f-j+1)}\varepsilon_j\right) X_{\tau_e}\left\{1+o_p(1)\right\} \\
&\sim_a T^{(\alpha+1)/2}\delta_T^{\tau_f-\tau_e} X_c B(r_e).
\end{aligned}$$

The second term is

$$\begin{aligned}
\sum_{j=\tau_f+1}^{\tau_2} \tilde{X}_{j-1}\varepsilon_j &= \sum_{j=\tau_f+1}^{\tau_2} -\frac{T^{\alpha}\delta_T^{\tau_f-\tau_1}}{\tau_w c} X_{\tau_e}\varepsilon_j \left\{1+o_p(1)\right\} \\
&= -\frac{T^{\alpha}\delta_T^{\tau_f-\tau_1}}{r_w c}\left(T^{-1/2}X_{\tau_e}\right)\left(T^{-1/2}\sum_{j=\tau_f+1}^{\tau_2}\varepsilon_j\right)\left\{1+o_p(1)\right\} \\
&\sim_a -\frac{T^{\alpha}\delta_T^{\tau_f-\tau_1}}{r_w c} B(r_e)\left[B(r_2)-B(r_f)\right].
\end{aligned}$$

Since $(\alpha+1)/2 > \alpha$, $\sum_{j=\tau_1}^{\tau_f} \tilde{X}_{j-1}\varepsilon_j$ dominates $\sum_{j=\tau_f+1}^{\tau_2} \tilde{X}_{j-1}\varepsilon_j$. Therefore,

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}\varepsilon_j = \sum_{j=\tau_1}^{\tau_f} \tilde{X}_{j-1}\varepsilon_j \left\{1+o_p(1)\right\} \sim_a T^{(\alpha+1)/2}\delta_T^{\tau_f-\tau_e} X_c B(r_e).$$

(3) For $\tau_1 \in N_0$ and $\tau_2 \in N_1$,

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}\varepsilon_j = \sum_{j=\tau_1}^{\tau_e-1} \tilde{X}_{j-1}\varepsilon_j + \sum_{j=\tau_e}^{\tau_f} \tilde{X}_{j-1}\varepsilon_j + \sum_{j=\tau_f+1}^{\tau_2} \tilde{X}_{j-1}\varepsilon_j.$$

The first term is

$$\begin{aligned}\sum_{j=\tau_1}^{\tau_e-1}\tilde{X}_{j-1}\varepsilon_j &= \sum_{j=\tau_1}^{\tau_e-1}-\frac{T^{\alpha}\delta_T^{\tau_f-\tau_e}}{\tau_w c}X_{\tau_e}\varepsilon_j\left\{1+o_p(1)\right\}\\ &= -\frac{T^{\alpha}\delta_T^{\tau_f-\tau_e}}{r_w c}\left(T^{-1/2}X_{\tau_e}\right)\left(T^{-1/2}\sum_{j=\tau_1}^{\tau_e-1}\varepsilon_j\right)\left\{1+o_p(1)\right\}\\ &\sim_a -\frac{T^{\alpha}\delta_T^{\tau_f-\tau_e}}{r_w c}B(r_e)\left[B(r_e)-B(r_1)\right].\end{aligned}$$

The second term is

$$\begin{aligned}\sum_{j=\tau_e}^{\tau_f}\tilde{X}_{j-1}\varepsilon_j &= \sum_{j=\tau_e}^{\tau_f}\left[\delta_T^{j-1-\tau_e}-\frac{T^{\alpha}\delta_T^{\tau_f-\tau_e}}{\tau_w c}\right]X_{\tau_e}\varepsilon_j\left\{1+o_p(1)\right\}\\ &= \left[\sum_{j=\tau_e}^{\tau_f}\delta_T^{j-1-\tau_e}\varepsilon_j-\frac{T^{\alpha}\delta_T^{\tau_f-\tau_e}}{\tau_w c}\sum_{j=\tau_e}^{\tau_f}\varepsilon_j\right]X_{\tau_e}\left\{1+o_p(1)\right\}\\ &= \left[T^{\alpha/2}\delta_T^{\tau_f-\tau_e}\left(\frac{1}{T^{\alpha/2}}\sum_{j=\tau_e}^{\tau_f}\delta_T^{-(\tau_f-j+1)}\varepsilon_j\right)-\frac{T^{\alpha-1/2}\delta_T^{\tau_f-\tau_e}}{r_w c}\left(\frac{1}{\sqrt{T}}\sum_{j=\tau_e}^{\tau_f}\varepsilon_j\right)\right]X_{\tau_e}\left\{1+o_p(1)\right\}\\ &= T^{\alpha/2+1/2}\delta_T^{\tau_f-\tau_e}\left(T^{-\alpha/2}\sum_{j=\tau_e}^{\tau_f}\delta_T^{-(\tau_f-j+1)}\varepsilon_j\right)\left(T^{-1/2}X_{\tau_e}\right)\left\{1+o_p(1)\right\}\\ &\sim_a T^{(\alpha+1)/2}\delta_T^{\tau_f-\tau_e}X_cB(r_e).\end{aligned}$$

The third term is

$$\begin{aligned}\sum_{j=\tau_f+1}^{\tau_2}\tilde{X}_{j-1}\varepsilon_j &= \sum_{j=\tau_f+1}^{\tau_2}-\frac{T^{\alpha}\delta_T^{\tau_f-\tau_e}}{\tau_w c}X_{\tau_e}\varepsilon_j\left\{1+o_p(1)\right\}\\ &= -\frac{T^{\alpha}\delta_T^{\tau_f-\tau_e}}{r_w c}\left(T^{-1/2}X_{\tau_e}\right)\left(T^{-1/2}\sum_{j=\tau_f+1}^{\tau_2}\varepsilon_j\right)\left\{1+o_p(1)\right\}\\ &\sim_a -\frac{T^{\alpha}\delta_T^{\tau_f-\tau_e}}{r_w c}B(r_e)\left[B(r_2)-B(r_f)\right].\end{aligned}$$

Since $(\alpha+1)/2>\alpha$, $\sum_{j=\tau_e}^{\tau_f}\tilde{X}_{j-1}\varepsilon_j$ dominates the other two terms. Therefore,

$$\sum_{j=\tau_1}^{\tau_2}\tilde{X}_{j-1}\varepsilon_j=\sum_{j=\tau_e}^{\tau_f}\tilde{X}_{j-1}\varepsilon_j\left\{1+o_p(1)\right\}\sim_a T^{(\alpha+1)/2}\delta_T^{\tau_f-\tau_e}X_cB(r_e).$$

■

LEMMA A6. *The sample covariance of $\tilde{X}_{j-1}$ and $X_j-\delta_T X_{j-1}$ behaves as follows.*

(1) *For* $\tau_1 \in N_0$ *and* $\tau_2 \in B$,

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}\left(X_j - \delta_T X_{j-1}\right) \sim_a \frac{r_e - r_1}{r_w} T\delta_T^{\tau_2-\tau_e} B\left(r_e\right) \int_{r_1}^{r_e} B\left(s\right) ds.$$

(2) *For* $\tau_1 \in B$ *and* $\tau_2 \in N_1$,

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}\left(X_j - \delta_T X_{j-1}\right) \sim_a -T\delta_T^{2(\tau_f-\tau_e)} B\left(r_e\right)^2.$$

(3) *For* $\tau_1 \in N_0$ *and* $\tau_2 \in N_1$,

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}\left(X_j - \delta_T X_{j-1}\right) \sim_a -T\delta_T^{2(\tau_f-\tau_e)} B\left(r_e\right)^2.$$

PROOF. (1) When $\tau_1 \in N_0$ and $\tau_2 \in B$,

$$\begin{aligned}
\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}\left(X_j - \delta_T X_{j-1}\right) &= \sum_{j=\tau_e}^{\tau_2} \tilde{X}_{j-1}\varepsilon_j + \sum_{j=\tau_1}^{\tau_e-1} \tilde{X}_{j-1}\left(X_j - X_{j-1} + X_{j-1} - \delta_T X_{j-1}\right) \\
&= \sum_{j=\tau_e}^{\tau_2} \tilde{X}_{j-1}\varepsilon_j + \sum_{j=\tau_1}^{\tau_e-1} \tilde{X}_{j-1}\left(\varepsilon_j - \frac{c}{T^\alpha} X_{j-1}\right) \\
&= \sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}\varepsilon_j - \frac{c}{T^\alpha} \sum_{j=\tau_1}^{\tau_e-1} \tilde{X}_{j-1} X_{j-1}. \qquad \text{(A.8)}
\end{aligned}$$

The first term is

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}\varepsilon_j \sim_a T^{(\alpha+1)/2} \delta_T^{\tau_2-\tau_e} X_c B\left(r_e\right) \text{ (from Lemma A5).}$$

The second term is

$$\begin{aligned}
\frac{c}{T^\alpha} \sum_{j=\tau_1}^{\tau_e-1} \tilde{X}_{j-1} X_{j-1} &= \frac{c}{T^\alpha} \sum_{j=\tau_1}^{\tau_e-1} -\frac{T^\alpha \delta_T^{\tau_2-\tau_e}}{\tau_w c} X_{\tau_e} X_{j-1} \left\{1 + o_p\left(1\right)\right\} \\
&= -\frac{\delta_T^{\tau_2-\tau_e}}{\tau_w} X_{\tau_e} \sum_{j=\tau_1}^{\tau_e-1} X_{j-1} \left\{1 + o_p\left(1\right)\right\} \\
&= -\frac{\tau_e - \tau_1}{\tau_w} T\delta_T^{\tau_2-\tau_e} \left(T^{-1/2} X_{\tau_e}\right) \left[\frac{1}{\tau_e - \tau_1} \sum_{j=\tau_1}^{\tau_e-1} \left(T^{-1/2} X_{j-1}\right)\right] \left\{1 + o_p\left(1\right)\right\} \\
&\sim_a -\frac{r_e - r_1}{r_w} T\delta_T^{\tau_2-\tau_e} B\left(r_e\right) \int_{r_1}^{r_e} B\left(s\right) ds.
\end{aligned}$$

Since $(\alpha+1)/2 < 1$, $\frac{c}{T^\alpha} \sum_{j=\tau_1}^{\tau_e-1} \tilde{X}_{j-1} X_{j-1}$ dominates $\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}\varepsilon_j$. Therefore,

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}(X_j - \delta_T X_{j-1}) = -\frac{c}{T^\alpha} \sum_{j=\tau_1}^{\tau_e - 1} \tilde{X}_{j-1} X_{j-1} \{1 + o_p(1)\}$$

$$\sim_a \frac{r_e - r_1}{r_w} T \delta_T^{\tau_2 - \tau_e} B(r_e) \int_{r_1}^{r_e} B(s)\, ds.$$

(2) When $\tau_1 \in B$ and $\tau_2 \in N_1$,

$$\begin{aligned}
\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}(X_j - \delta_T X_{j-1}) &= \sum_{j=\tau_1}^{\tau_f} \tilde{X}_{j-1}\varepsilon_j + \tilde{X}_{\tau_f}\left(X_{\tau_f+1} - \delta_T X_{\tau_f}\right) \\
&\quad + \sum_{j=\tau_f+2}^{\tau_2} \tilde{X}_{j-1}(X_j - X_{j-1} + X_{j-1} - \delta_T X_{j-1}) \\
&= \sum_{j=\tau_1}^{\tau_f} \tilde{X}_{j-1}\varepsilon_j + \tilde{X}_{\tau_f}\left(X_{\tau_e} + X^* + \varepsilon_{\tau_f+1} - \delta_T X_{\tau_f}\right) \\
&\quad + \sum_{j=\tau_f+2}^{\tau_2} \tilde{X}_{j-1}\left(\varepsilon_j - \frac{c}{T^\alpha} X_{j-1}\right) \\
&= \sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}\varepsilon_j - \delta_T \tilde{X}_{\tau_f} X_{\tau_f} - \frac{c}{T^\alpha} \sum_{j=\tau_f+2}^{\tau_2} \tilde{X}_{j-1} X_{j-1}.
\end{aligned}$$

The first term is

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}\varepsilon_j \sim_a T^{(\alpha+1)/2} \delta_T^{\tau_f - \tau_e} X_c B(r_e) \text{ (from Lemma A5).}$$

The second term is

$$\begin{aligned}
\delta_T \tilde{X}_{\tau_f} X_{\tau_f} &= \delta_T \left[\delta_T^{\tau_f - \tau_e} - \frac{T^\alpha \delta_T^{\tau_f - \tau_1}}{\tau_w c}\right] X_{\tau_e} X_{\tau_f} \{1 + o_p(1)\} \\
&= \delta_T^{\tau_f - \tau_e + 1} X_{\tau_e} X_{\tau_f} \{1 + o_p(1)\} \sim_a T \delta_T^{2(\tau_f - \tau_e)} B(r_e)^2
\end{aligned}$$

due to the fact that

$$\frac{\delta_T^{\tau_f - \tau_e}}{T^{\alpha-1} \delta_T^{\tau_f - \tau_1}} = T^{1-\alpha} \delta_T^{\tau_1 - \tau_e} > 1.$$

The third term is

$$\begin{aligned}
\frac{c}{T^\alpha} \sum_{j=\tau_f+2}^{\tau_2} \tilde{X}_{j-1} X_{j-1} &= \frac{c}{T^\alpha} \sum_{j=\tau_f+2}^{\tau_2} -\frac{T^\alpha \delta_T^{\tau_f - \tau_1}}{\tau_w c} X_{\tau_e} X_{j-1} \{1 + o_p(1)\} \\
&= -\frac{\delta_T^{\tau_f - \tau_1}}{\tau_w} X_{\tau_e} \sum_{j=\tau_f+2}^{\tau_2} X_{j-1} \{1 + o_p(1)\} \\
&= -\frac{\tau_2 - \tau_f - 1}{\tau_w} T \delta_T^{\tau_f - \tau_1} \left(T^{-1/2} X_{\tau_e}\right) \left(\frac{1}{\tau_2 - \tau_f - 1} \sum_{j=\tau_f+2}^{\tau_2} T^{-1/2} X_{j-1}\right) \\
&\quad \times \{1 + o_p(1)\}
\end{aligned}$$

$$\sim_a -\frac{r_2 - r_f}{r_w} T\delta_T^{\tau_f - \tau_1} B(r_e) \int_{r_f}^{r_2} B(s)\, ds.$$

The quantity $\delta_T \tilde{X}_{\tau_f} X_{\tau_f}$ dominates the other two terms and hence

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}(X_j - \delta_T X_{j-1}) = -\delta_T \tilde{X}_{\tau_f} X_{\tau_f} \left\{1 + o_p(1)\right\} \sim_a -T\delta_T^{2(\tau_f - \tau_e)} B(r_e)^2.$$

(3) When $\tau_1 \in N_0$ and $\tau_2 \in N_1$,

$$\begin{aligned}
\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}(X_j - \delta_T X_{j-1}) &= \sum_{j=\tau_e}^{\tau_f} \tilde{X}_{j-1}\varepsilon_j + \sum_{j=\tau_1}^{\tau_e - 1} \tilde{X}_{j-1}(X_j - X_{j-1} + X_{j-1} - \delta_T X_{j-1}) \\
&\quad + \tilde{X}_{\tau_f}\left(X_{\tau_f+1} - \delta_T X_{\tau_f}\right) + \sum_{j=\tau_f+2}^{\tau_2} \tilde{X}_{j-1}(X_j - X_{j-1} + X_{j-1} - \delta_T X_{j-1}) \\
&= \sum_{j=\tau_e}^{\tau_f} \tilde{X}_{j-1}\varepsilon_j + \sum_{j=\tau_1}^{\tau_e - 1} \tilde{X}_{j-1}\left(\varepsilon_j - \frac{c}{T^\alpha} X_{j-1}\right) \\
&\quad + \tilde{X}_{\tau_f}\left(X_{\tau_f+1} - \delta_T X_{\tau_f}\right) + \sum_{j=\tau_f+2}^{\tau_2} \tilde{X}_{j-1}\left(\varepsilon_j - \frac{c}{T^\alpha} X_{j-1}\right) \\
&= \sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}\varepsilon_j - \frac{c}{T^\alpha}\sum_{j=\tau_1}^{\tau_e - 1} \tilde{X}_{j-1}X_{j-1} - \delta_T \tilde{X}_{\tau_f} X_{\tau_f} - \frac{c}{T^\alpha}\sum_{j=\tau_f+2}^{\tau_2} \tilde{X}_{j-1}X_{j-1}.
\end{aligned}$$

The first term is

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}\varepsilon_j \sim_a T^{(\alpha+1)/2}\delta_T^{\tau_f - \tau_e} X_c B(r_e) \text{ (from Lemma A5).}$$

The second term is

$$\begin{aligned}
\frac{c}{T^\alpha}\sum_{j=\tau_1}^{\tau_e - 1} \tilde{X}_{j-1}X_{j-1} &= \frac{c}{T^\alpha}\sum_{j=\tau_1}^{\tau_e - 1} -\frac{T^\alpha \delta_T^{\tau_f - \tau_e}}{\tau_w c} X_{\tau_e} X_{j-1}\left\{1 + o_p(1)\right\} \\
&= -\frac{\delta_T^{\tau_f - \tau_e}}{\tau_w} X_{\tau_e} \sum_{j=\tau_1}^{\tau_e - 1} X_{j-1}\left\{1 + o_p(1)\right\} \\
&= -\frac{\tau_e - \tau_1}{\tau_w} T\delta_T^{\tau_f - \tau_e}\left(T^{-1/2}X_{\tau_e}\right)\left(\frac{1}{\tau_e - \tau_1}\sum_{j=\tau_1}^{\tau_e - 1} T^{-1/2}X_{j-1}\right)\left\{1 + o_p(1)\right\} \\
&\sim_a -\frac{r_e - r_1}{r_w} T\delta_T^{\tau_f - \tau_e} B(r_e) \int_{r_1}^{r_e} B(s)\, ds.
\end{aligned}$$

The third term is

$$\delta_T \tilde{X}_{\tau_f} X_{\tau_f} = \delta_T \left[\delta_T^{\tau_f - \tau_e} - \frac{T^\alpha \delta_T^{\tau_f - \tau_e}}{\tau_w c}\right] X_{\tau_e} X_{\tau_f}\left\{1 + o_p(1)\right\}$$

$$= \delta_T^{\tau_f - \tau_e + 1} X_{\tau_e} X_{\tau_f} \left\{ 1 + o_p(1) \right\} \sim_a T \delta_T^{2(\tau_f - \tau_e)} B(r_e)^2 ,$$

due to the fact that

$$\frac{\delta_T^{\tau_f - \tau_e}}{T^{\alpha - 1} \delta_T^{\tau_f - \tau_e}} = T^{1-\alpha} > 1.$$

The fourth term is

$$\begin{aligned} \frac{c}{T^\alpha} \sum_{j=\tau_f+2}^{\tau_2} \tilde{X}_{j-1} X_{j-1} &= \frac{c}{T^\alpha} \sum_{j=\tau_f+2}^{\tau_2} -\frac{T^\alpha \delta_T^{\tau_f - \tau_e}}{\tau_w c} X_{\tau_e} X_{j-1} \left\{ 1 + o_p(1) \right\} \\ &= -\frac{\delta_T^{\tau_f - \tau_e}}{\tau_w} X_{\tau_e} \sum_{j=\tau_f+2}^{\tau_2} X_{j-1} \left\{ 1 + o_p(1) \right\} \\ &= -\frac{\tau_2 - \tau_f - 1}{\tau_w} T \delta_T^{\tau_f - \tau_e} \left(T^{-1/2} X_{\tau_e} \right) \left(\frac{1}{\tau_2 - \tau_f - 1} \sum_{j=\tau_f+2}^{\tau_2} T^{-1/2} X_{j-1} \right) \\ &\quad \times \left\{ 1 + o_p(1) \right\} \\ &\sim_a -\frac{r_2 - r_f}{r_w} T \delta_T^{\tau_f - \tau_e} B(r_e) \int_{r_f}^{r_2} B(s)\, ds. \end{aligned}$$

The quantity $\delta_T \tilde{X}_{\tau_f} X_{\tau_f}$ dominates the other three terms and hence

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1} \left(X_j - \delta_T X_{j-1} \right) = -\delta_T \tilde{X}_{\tau_f} X_{\tau_f} \left\{ 1 + o_p(1) \right\} \sim_a -T \delta_T^{2(\tau_f - \tau_e)} B(r_e)^2 .$$

■

A.2. *Test asymptotics and proofs of Theorems 1–3.* The fitted regression model for the subperiod unit root test is

$$X_t = \hat{\alpha}_{r_1, r_2} + \hat{\rho}_{r_1, r_2} X_{t-1} + \hat{\varepsilon}_t. \tag{A.9}$$

The intercept $\hat{\alpha}_{r_1,r_2}$ and slope coefficient $\hat{\rho}_{r_1,r_2}$ are obtained using data over the subperiod $[r_1, r_2]$.

REMARK 1. We calculate the asymptotic distribution of the unit root statistic under the alternative hypothesis. Based on Lemma A4 and Lemma A6, we can obtain the limit distribution of $\hat{\rho}_{r_1,r_2} - \delta_T$ using

$$\hat{\rho}_{r_1,r_2} - \delta_T = \frac{\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1} \left(X_j - \delta_T X_{j-1} \right)}{\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}^2}.$$

(1) When $\tau_1 \in N_0$ and $\tau_2 \in B$,

$$\hat{\rho}_{r_1,r_2} - \delta_T \sim_a T^{-\alpha} \delta_T^{-(\tau_2 - \tau_{ie})} \frac{\frac{r_e - r_1}{r_w} \int_{r_1}^{r_e} B(s)\, ds}{B(r_e)};$$

(2) when $\tau_1 \in B$ and $\tau_2 \in N_1$,

$$\hat{\rho}_{r_1,r_2} - \delta_T \sim_a -2T^{-\alpha} c;$$

(3) when $\tau_1 \in N_0$ and $\tau_2 \in N_1$,

$$\hat{\rho}_{r_1,r_2} - \delta_T \sim_a -2T^{-\alpha}c.$$

REMARK 2. The asymptotic distributions of the unit root coefficient Z-statistics can be obtained using

$$DF^z_{r_1,r_2} = \tau_w\left(\hat{\rho}_{r_1,r_2} - 1\right) = \tau_w\left(\delta_T - 1\right) + \tau_w\left(\hat{\rho}_{r_1,r_2} - \delta_T\right).$$

(1) When $\tau_1 \in N_0$ and $\tau_2 \in B$,

$$DF^z_{r_1,r_2} = r_w c T^{1-\alpha} + o_p(1) \to \infty;$$

(2) when $\tau_1 \in B$ and $\tau_2 \in N_1$,

$$DF^z_{r_1,r_2} = -c r_w T^{1-\alpha} \to -\infty;$$

(3) when $\tau_1 \in N_0$ and $\tau_2 \in N_1$,

$$DF^z_{r_1,r_2} = -c r_w T^{1-\alpha} \to -\infty.$$

This implies that when $\tau_1 \in N_0$ and $\tau_2 \in B$,

$$\hat{\rho}_{r_1,r_2} - 1 \sim_a T^{-\alpha}c \text{ and } T^{\alpha}\left(\hat{\rho}_{r_1,r_2} - 1\right) \xrightarrow{L} c,$$

and for the other two cases,

$$\hat{\rho}_{r_1,r_2} - 1 \sim_a -T^{-\alpha}c \text{ and } T^{\alpha}\left(\hat{\rho}_{r_1,r_2} - 1\right) \xrightarrow{L} -c.$$

In order to obtain the asymptotic distributions of the Dickey–Fuller t-statistic, we first obtain the equation standard error of the regression over $[r_1, r_2]$, which is

$$\hat{\sigma}_{r_1r_2} = \left\{\tau_w^{-1}\sum_{j=\tau_1}^{\tau_2}\left(\tilde{X}_j - \hat{\rho}_{r_1,r_2}\tilde{X}_{j-1}\right)^2\right\}^{1/2}.$$

LEMMA A7.

(1) *When* $\tau_1 \in N_0$ *and* $\tau_2 \in B$,

$$\hat{\sigma}^2_{r_1r_2} \sim_a T^{-1}\delta_T^{2(\tau_2-\tau_e)}\frac{r_e - r_1}{r_w^3 c^{-1}}B(r_e).$$

(2) *When* $\tau_1 \in B$ *and* $\tau_2 \in N_1$,

$$\hat{\sigma}^2_{r_1r_2} \sim_a \frac{1}{r_w}\delta_T^{2(\tau_f-\tau_e)}B(r_e)^2.$$

(3) *When* $\tau_1 \in N_0$ *and* $\tau_2 \in N_1$,

$$\hat{\sigma}^2_{r_1r_2} \sim_a \frac{\delta_T^{2(\tau_f-\tau_e)}}{r_w}B(r_e)^2.$$

PROOF. (1) When $\tau_1 \in N_0$ and $\tau_2 \in B$,

$$\begin{aligned}
\hat{\sigma}^2_{r_1 r_2} &= \tau_w^{-1} \sum_{j=\tau_1}^{\tau_2} \left(\tilde{X}_j - \hat{\delta}_{r_1,r_2} \tilde{X}_{j-1}\right)^2 \\
&= \tau_w^{-1} \left[\sum_{j=\tau_1}^{\tau_e - 1} \left[\varepsilon_j - \left(\hat{\delta}_{r_1,r_2} - 1\right) \tilde{X}_{j-1}\right]^2 + \sum_{j=\tau_e}^{\tau_2} \left[\varepsilon_j - \left(\hat{\delta}_{r_1,r_2} - \delta_T\right) \tilde{X}_{j-1}\right]^2 \right] \\
&= \tau_w^{-1} \sum_{j=\tau_1}^{\tau_2} \varepsilon_j^2 + \left(\hat{\delta}_{r_1,r_2} - 1\right)^2 \tau_w{}^{-1} \sum_{j=\tau_1}^{\tau_e - 1} \tilde{X}^2_{j-1} + \left(\hat{\delta}_{r_1,r_2} - \delta_T\right)^2 \tau_w^{-1} \sum_{j=\tau_e}^{\tau_2} \tilde{X}^2_{j-1} \\
&\quad -2\left(\hat{\delta}_{r_1,r_2} - 1\right) \tau_w^{-1} \sum_{j=\tau_1}^{\tau_e - 1} \tilde{X}_{j-1}\varepsilon_j - 2\left(\hat{\delta}_{r_1,r_2} - \delta_T\right) \tau_w^{-1} \sum_{j=\tau_e}^{\tau_2} \tilde{X}_{j-1}\varepsilon_j \\
&= \left(\hat{\delta}_{r_1,r_2} - 1\right)^2 \tau_w{}^{-1} \sum_{j=\tau_1}^{\tau_e - 1} \tilde{X}^2_{j-1} \\
&\sim_a T^{-1} \delta_T^{2(\tau_2 - \tau_e)} \frac{r_e - r_1}{r_w^3 c^{-1}} B(r_e).
\end{aligned}$$

The term $(\hat{\delta}_{r_1,r_2} - 1)^2 \tau_w{}^{-1} \sum_{j=\tau_1}^{\tau_e - 1} \tilde{X}^2_{j-1}$ dominates the other terms due to the fact that

$$\begin{aligned}
\left(\hat{\delta}_{r_1,r_2} - 1\right)^2 \tau_w{}^{-1} \sum_{j=\tau_1}^{\tau_e - 1} \tilde{X}^2_{j-1} &= O_p\left(T^{-2\alpha}\right) O_p\left(T^{2\alpha - 1}\delta_T^{2(\tau_2 - \tau_e)}\right) = O_p\left(T^{-1}\delta_T^{2(\tau_2 - \tau_e)}\right), \\
\left(\hat{\delta}_{r_1,r_2} - \delta_T\right)^2 \tau_w^{-1} \sum_{j=\tau_e}^{\tau_2} \tilde{X}^2_{j-1} &= O_p\left(\frac{1}{T^{2\alpha}\delta_T^{2(\tau_2 - \tau_e)}}\right) O_p\left(T^{\alpha}\delta_T^{2(\tau_2 - \tau_e)}\right) = O_p\left(T^{-\alpha}\right), \\
2\left(\hat{\delta}_{r_1,r_2} - 1\right) \tau_w^{-1} \sum_{j=\tau_1}^{\tau_e - 1} \tilde{X}_{j-1}\varepsilon_j &= O_p\left(T^{-\alpha}\right) O_p\left(\frac{\delta_T^{\tau_2 - \tau_e}}{T^{1-\alpha}}\right) = O_p\left(T^{-1}\delta_T^{\tau_2 - \tau_e}\right), \\
2\left(\hat{\delta}_{r_1,r_2} - \delta_T\right) \tau_w^{-1} \sum_{j=\tau_e}^{\tau_2} \tilde{X}_{j-1}\varepsilon_j &= O_p\left(\frac{1}{T^{\alpha}\delta_T^{\tau_2 - \tau_e}}\right) O_p\left(\frac{\delta_T^{\tau_2 - \tau_e}}{T^{(1-\alpha)/2}}\right) = O_p\left(T^{-(1+3\alpha)/2}\right).
\end{aligned}$$

(2) When $\tau_1 \in B$ and $\tau_2 \in N_1$,

$$\begin{aligned}
\tilde{X}_{\tau_f + 1} - \hat{\delta}_{r_1,r_2}\tilde{X}_{\tau_f} &= \frac{\delta_T^{\tau_f - \tau_1}}{r_w c T^{1-\alpha}} X_{\tau_e} - \tilde{X}_{\tau_f} - \left[\hat{\delta}_{r_1,r_2} - 1\right]\tilde{X}_{\tau_f} \\
&= O_p(T^{\alpha - 1/2}\delta_T^{\tau_f - \tau_1}) - O_p(T^{1/2}\delta_T^{\tau_f - \tau_e}) - O_p(T^{-\alpha}) O_p(T^{1/2}\delta_T^{\tau_f - \tau_e}) \\
&= -\tilde{X}_{\tau_f} = -\delta_T^{\tau_f - \tau_e} X_{\tau_e}\left\{1 + o_p(1)\right\},
\end{aligned}$$

using the fact that

$$\tilde{X}_{\tau_f} = \left[\delta_T^{\tau_f - \tau_e} - \frac{\delta_T^{\tau_f - \tau_1}}{r_w c T^{1-\alpha}}\right] X_{\tau_e}\left\{1 + o_p(1)\right\} = \delta_T^{\tau_f - \tau_e} X_{\tau_e}\left\{1 + o_p(1)\right\}.$$

Therefore,

$$\begin{aligned}
\hat{\sigma}^2_{r_1 r_2} &= \tau_w^{-1} \sum_{j=\tau_1}^{\tau_2} \left(\tilde{X}_j - \hat{\delta}_{r_1,r_2} \tilde{X}_{j-1}\right)^2 \\
&= \tau_w^{-1} \left\{ \sum_{j=\tau_f+2}^{\tau_2} \left[\varepsilon_j - \left(\hat{\delta}_{r_1,r_2} - 1\right) \tilde{X}_{j-1}\right]^2 + \sum_{j=\tau_1}^{\tau_f} \left[\varepsilon_j - \left(\hat{\delta}_{r_1,r_2} - \delta_T\right) \tilde{X}_{j-1}\right]^2 \right. \\
&\quad \left. + \left[\tilde{X}_{\tau_f+1} - \hat{\delta}_{r_1,r_2} \tilde{X}^2_{\tau_f} - \varepsilon_{\tau_f+1} + \varepsilon_{\tau_f+1}\right]^2 \right\} \\
&= \tau_w^{-1} \sum_{j=\tau_1}^{\tau_2} \varepsilon_j^2 + \left(\hat{\delta}_{r_1,r_2} - 1\right)^2 \tau_w{}^{-1} \sum_{j=\tau_f+2}^{\tau_2} \tilde{X}^2_{j-1} + \left(\hat{\delta}_{r_1,r_2} - \delta_T\right)^2 \tau_w^{-1} \sum_{j=\tau_1}^{\tau_f} \tilde{X}^2_{j-1} \\
&\quad - 2\left(\hat{\delta}_{r_1,r_2} - 1\right) \tau_w^{-1} \sum_{j=\tau_f+2}^{\tau_2} \tilde{X}_{j-1}\varepsilon_j - 2\left(\hat{\delta}_{r_1,r_2} - \delta_T\right) \tau_w^{-1} \sum_{j=\tau_1}^{\tau_f} \tilde{X}_{j-1}\varepsilon_j + \tau_w^{-1} \tilde{X}^2_{\tau_f} \\
&= \tau_w^{-1} \tilde{X}^2_{\tau_f} = \tau_w^{-1} \delta_T^{2(\tau_f - \tau_e)} X^2_{\tau_e} \left\{1 + o_p(1)\right\} \sim_a \frac{1}{r_w} \delta_T^{2(\tau_f - \tau_e)} B(r_e)^2 .
\end{aligned}$$

The term $\tau_w^{-1} \tilde{X}^2_{\tau_f}$ dominates the other terms due to the fact that

$$\begin{aligned}
\left(\hat{\delta}_{r_1,r_2} - 1\right)^2 \tau_w{}^{-1} \sum_{j=\tau_f+2}^{\tau_2} \tilde{X}^2_{j-1} &= O_p\left(T^{-2\alpha}\right)\left(T^{2\alpha-1}\delta_T^{2(\tau_f-\tau_1)}\right) = O_p\left(\frac{\delta_T^{2(\tau_f-\tau_1)}}{T}\right), \\
\left(\hat{\delta}_{r_1,r_2} - \delta_T\right)^2 \tau_w^{-1} \sum_{j=\tau_1}^{\tau_f} \tilde{X}^2_{j-1} &= O_p\left(\frac{1}{T^{2\alpha}}\right) O_p\left(T^{\alpha}\delta_T^{2(\tau_f-\tau_e)}\right) = O_p\left(\frac{\delta_T^{2(\tau_f-\tau_e)}}{T^{\alpha}}\right), \\
2\left(\hat{\delta}_{r_1,r_2} - 1\right) \tau_w^{-1} \sum_{j=\tau_f+2}^{\tau_2} \tilde{X}_{j-1}\varepsilon_j &= O_p\left(T^{-\alpha}\right) O_p\left(T^{\alpha-1}\delta_T^{\tau_f-\tau_1}\right) = O_p\left(\frac{\delta_T^{\tau_f-\tau_1}}{T}\right), \\
2\left(\hat{\delta}_{r_1,r_2} - \delta_T\right) \tau_w^{-1} \sum_{j=\tau_1}^{\tau_f} \tilde{X}_{j-1}\varepsilon_j &= O_p\left(\frac{1}{T^{\alpha}}\right) O_p\left(T^{(\alpha-1)/2}\delta_T^{\tau_f-\tau_e}\right) = O_p\left(\frac{\delta_T^{\tau_f-\tau_e}}{T^{(1+\alpha)/2}}\right), \\
\tau_w^{-1}\tilde{X}^2_{\tau_f} &= O_p\left(\delta_T^{2(\tau_f-\tau_e)}\right).
\end{aligned}$$

(3) When $\tau_1 \in N_0$ and $\tau_2 \in N_1$,

$$\begin{aligned}
\tilde{X}_{\tau_f+1} - \hat{\delta}_{r_1,r_2}\tilde{X}_{\tau_f} - \varepsilon_{\tau_f+1} &= -\frac{\delta_T^{\tau_f-\tau_e}}{r_w c T^{1-\alpha}} X_{\tau_e} - \tilde{X}_{\tau_f} - \left[\hat{\delta}_{r_1,r_2} - 1\right]\tilde{X}_{\tau_f} \\
&= -O_p\left(T^{\alpha-1/2}\delta_T^{\tau_f-\tau_e}\right) - O_p\left(T^{1/2}\delta_T^{\tau_f-\tau_e}\right) - O_p\left(T^{-\alpha}\right) O_p\left(T^{1/2}\delta_T^{\tau_f-\tau_e}\right) \\
&= -\tilde{X}_{\tau_f} = -\delta_T^{\tau_f-\tau_e} X_{\tau_e}\left\{1 + o_p(1)\right\},
\end{aligned}$$

using the fact that

$$\tilde{X}_{\tau_f} = \left[\delta_T^{\tau_f-\tau_e} - \frac{\delta_T^{\tau_f-\tau_e}}{r_w c T^{1-\alpha}}\right] X_{\tau_e}\left\{1+o_p(1)\right\} = \delta_T^{\tau_f-\tau_e} X_{\tau_e}\left\{1+o_p(1)\right\}.$$

$$\begin{aligned}
\hat{\sigma}^2_{r_1 r_2} &= \tau_w^{-1}\sum_{j=\tau_1}^{\tau_2}\left(\tilde{X}_j - \hat{\delta}_{r_1,r_2}\tilde{X}_{j-1}\right)^2 \\
&= \tau_w^{-1}\left\{\sum_{j=\tau_f+2}^{\tau_2}\left[\varepsilon_j - \left(\hat{\delta}_{r_1,r_2}-1\right)\tilde{X}_{j-1}\right]^2 + \sum_{j=\tau_1}^{\tau_e-1}\left[\varepsilon_j - \left(\hat{\delta}_{r_1,r_2}-1\right)\tilde{X}_{j-1}\right]^2\right. \\
&\qquad \left. + \sum_{j=\tau_e}^{\tau_f}\left[\varepsilon_j - \left(\hat{\delta}_{r_1,r_2}-\delta_T\right)\tilde{X}_{j-1}\right]^2 + \tilde{X}_{\tau_f+1} - \hat{\delta}_{r_1,r_2}\tilde{X}^2_{\tau_f}\right\}^2 \\
&= \tau_w^{-1}\sum_{j=\tau_1}^{\tau_2}\varepsilon_j^2 + \left(\hat{\delta}_{r_1,r_2}-1\right)^2 \tau_w{}^{-1}\left[\sum_{j=\tau_f+2}^{\tau_2}\tilde{X}^2_{j-1} + \sum_{j=\tau_1}^{\tau_e-1}\tilde{X}^2_{j-1}\right] + \left(\hat{\delta}_{r_1,r_2}-\delta_T\right)^2\tau_w^{-1}\sum_{j=\tau_e}^{\tau_f}\tilde{X}^2_{j-1} \\
&\quad - 2\left(\hat{\delta}_{r_1,r_2}-1\right)\tau_w^{-1}\left[\sum_{j=\tau_f+2}^{\tau_2}\tilde{X}_{j-1}\varepsilon_j + \sum_{j=\tau_1}^{\tau_e-1}\tilde{X}_{j-1}\varepsilon_j\right] - 2\left(\hat{\delta}_{r_1,r_2}-\delta_T\right)\tau_w^{-1}\sum_{j=\tau_e}^{\tau_f}\tilde{X}_{j-1}\varepsilon_j + \tau_w^{-1}\tau_f^2 \\
&= \tau_w^{-1}\tilde{X}^2_{\tau_f} = \frac{\delta_T^{2(\tau_f-\tau_e)}}{\tau_w}X^2_{\tau_e}\left\{1+o_p(1)\right\} \sim_a \frac{\delta_T^{2(\tau_f-\tau_e)}}{r_w}B(r_e)^2.
\end{aligned}$$

The term $\tau_w^{-1}\tilde{X}^2_{\tau_f}$ dominates the other terms due to the fact that

$$\left(\hat{\delta}_{r_1,r_2}-1\right)^2\frac{1}{\tau_w}\left[\sum_{j=\tau_f+2}^{\tau_2}\tilde{X}^2_{j-1} + \sum_{j=\tau_1}^{\tau_e-1}\tilde{X}^2_{j-1}\right] = O_p\left(\frac{\delta_T^{2(\tau_f-\tau_e)}}{T}\right),$$

$$\left(\hat{\delta}_{r_1,r_2}-\delta_T\right)^2\frac{1}{\tau_w}\sum_{j=\tau_e}^{\tau_f}\tilde{X}^2_{j-1} = O_p\left(\frac{\delta_T^{2(\tau_f-\tau_e)}}{T^\alpha}\right),$$

$$2\left(\hat{\delta}_{r_1,r_2}-1\right)\frac{1}{\tau_w}\left[\sum_{j=\tau_f+2}^{\tau_2}\tilde{X}_{j-1}\varepsilon_j + \sum_{j=\tau_1}^{\tau_e-1}\tilde{X}_{j-1}\varepsilon_j\right] = O_p\left(\frac{\delta_T^{\tau_f-\tau_e}}{T}\right),$$

$$2\left(\hat{\delta}_{r_1,r_2}-\delta_T\right)\frac{1}{\tau_w}\sum_{j=\tau_e}^{\tau_f}\tilde{X}_{j-1}\varepsilon_j = O_p\left(\frac{\delta_T^{\tau_f-\tau_e}}{T^{(1+\alpha)/2}}\right),$$

$$\tau_w^{-1}\tilde{X}^2_{\tau_f} = O_p\left(\delta_T^{2(\tau_f-\tau_e)}\right). \qquad \blacksquare$$

The asymptotic distributions of the t-statistic can be calculated as follows.

$$DF^t_{r_1,r_2} = \left(\frac{\sum_{j=\tau_1}^{\tau_2}\tilde{X}^2_{j-1}}{\hat{\sigma}^2_{r_1 r_2}}\right)^{1/2}\left(\hat{\rho}_{r_1,r_2}-1\right).$$

REMARK 3. (1) When $\tau_1 \in N_0$ and $\tau_2 \in B$,

$$DF^t_{r_1,r_2} \sim_a T^{1-\alpha/2}\frac{r_w^{3/2}}{\sqrt{2(r_e-r_1)}} \to \infty.$$

(2) When $\tau_1 \in B$ and $\tau_2 \in N_1$,

$$DF^t_{r_1,r_2} \sim_a -\left(\frac{1}{2}cr_w\right)^{1/2} T^{(1-\alpha)/2} \to -\infty.$$

(3) When $\tau_1 \in N_0$ and $\tau_2 \in N_1$,

$$DF^t_{r_1,r_2} \sim_a -\left(\frac{1}{2}cr_w\right)^{1/2} T^{(1-\alpha)/2} \to -\infty.$$

Taken together with (11) and (12), these results establish the limit behavior of the unit root statistics DF_r and $BSDF_r(r_0)$ in Theorem 1 (see also (A.10) below).

The PWY strategy. The origination of the bubble expansion and the termination of the bubble collapse based on the DF test are identified as

$$\hat{r}^e = \inf_{r\in[r_0,1]}\left\{r : DF_r > cv^{\beta_T}\right\} \text{ and } \hat{r}^f = \inf_{r\in[\hat{r}_e+L_T,1]}\left\{r : DF_r < cv^{\beta_T}\right\}.$$

We know that when $\beta_T \to 0$, $cv^{\beta_T} \to \infty$.

For the PWY strategy, we have $r_1 = 0$ and $r_2 = r_w = r$. The asymptotic distributions of the DF statistic under the alternative hypothesis are

$$DF_r \sim_a \begin{cases} F_r(W) & \text{if } r \in N_0 \\ T^{1-\alpha/2}\dfrac{r^{3/2}}{\sqrt{2(r_e - r_1)}} \to \infty & \text{if } r \in B \\ -T^{(1-\alpha)/2}\left(\dfrac{1}{2}cr\right)^{1/2} \to -\infty & \text{if } r \in N_1 \end{cases}.$$

It is obvious that if $r \in N_0$,

$$\lim_{T\to\infty} \Pr\left\{DF_r > cv^{\beta_T}\right\} = \Pr\left\{F_r(W) = \infty\right\} = 0.$$

If $r \in B$, $\lim_{T\to\infty} \Pr\left\{DF_r > cv^{\beta_T}\right\} = 1$ provided that $\frac{cv^{\beta_T}}{T^{1-\alpha/2}} \to 0$. If $r \in N_1$, $\lim_{T\to\infty} \Pr\{DF_r < cv_r^{\beta_T}\} = \lim_{T\to\infty} \Pr\{-T^{(1-\alpha)/2}(\frac{1}{2}cr)^{1/2} < cv^{\beta_T}\} = 1$.

It follows that for any $\eta, \gamma > 0$,

$$\Pr\{\hat{r}_e > r_e + \eta\} \to 0 \text{ and } \Pr\left\{\hat{r}_f < r_f - \gamma\right\} \to 0$$

due to the fact that $\Pr\left\{DF_{r_e+a_\eta} > cv^{\beta_T}\right\} \to 1$ for all $0 < a_\eta < \eta$ and $\Pr\left\{DF_{r_f-a_\gamma} > cv^{\beta_T}\right\} \to 1$ for all $0 < a_\gamma < \gamma$. Since $\eta, \gamma > 0$ is arbitrary and $\Pr\{\hat{r}_e < r_e\} \to 0$ and $\Pr\left\{\hat{r}_f > r_f\right\} \to 0$, we deduce that $\Pr\{|\hat{r}_e - r_e| > \eta\} \to 0$ and $\Pr\left\{|\hat{r}_f - r_f| > \gamma\right\} \to 0$ as $T \to \infty$, provided that

$$\frac{1}{cv^{\beta_T}} + \frac{cv^{\beta_T}}{T^{1-\alpha/2}} \to 0.$$

Therefore, the PWY date detectors $\hat{r}_e$ and $\hat{r}_f$ are consistent estimators of r_e and r_f. This proves Theorem 2.

The PSY algorithm. The origination of the bubble expansion and the termination of the bubble collapse based on the backward sup DF test are identified as

$$\hat{r}^e = \inf_{r \in [r_0, 1]} \left\{ r : BSDF_r(r_0) > scv^{\beta_T} \right\} \text{ and } \hat{r}^f = \inf_{r \in [\hat{r}_e + L_T, 1]} \left\{ r_2 : BSDF_r(r_0) < scv^{\beta_T} \right\}.$$

We know that when $\beta_T \to 0, scv^{\beta_T} \to \infty$.

Given that $r_1 \in [0, r - r_0]$, $r_2 = r$, and $r_w = r - r_1$, the asymptotic behavior of the backward sup DF statistic under the alternative hypothesis is characterized as

$$BSDF_r(r_0) \sim_a \begin{cases} F_r(W, r_0) & \text{if } r \in N_0 \\ T^{1-\alpha/2} \sup_{r_1 \in [0, r - r_0]} \left\{ \dfrac{(r - r_1)^{3/2}}{\sqrt{2(r_e - r_1)}} \right\} & \text{if } r \in B \\ -T^{(1-\alpha)/2} \sup_{r_1 \in [0, r - r_0]} \left\{ \left[\dfrac{1}{2} c(r - r_1) \right]^{1/2} \right\} & \text{if } r \in N_1 \end{cases}. \tag{A.10}$$

It is obvious that if $r \in N_0$,

$$\lim_{T \to \infty} \Pr \left\{ BSDF_r(r_0) > scv^{\beta_T} \right\} = \Pr \left\{ F_{r_2}(W, r_0) = \infty \right\} = 0.$$

If $r \in B$, $\lim_{T\to\infty} \Pr \left\{ BSDF_r(r_0) > scv^{\beta_T} \right\} = 1$ provided that $\frac{scv^{\beta_T}}{T^{1-\alpha/2}} \to 0$. If $r \in N_1$, $\lim_{T\to\infty} \Pr\{BSDF_r(r_0) < scv_r^{\beta_T}\} = 1$.

It follows that for any $\eta, \gamma > 0$,

$$\Pr \{\hat{r}_e > r_e + \eta\} \to 0 \text{ and } \Pr \left\{ \hat{r}_f < r_f - \gamma \right\} \to 0,$$

since $\Pr \left\{ BSDF_{r_e + a_\eta}(r_0) > scv^{\beta_T} \right\} \to 1$ for all $0 < a_\eta < \eta$ and $\Pr \left\{ BSDF_{r_f - a_\gamma}(r_0) > scv^{\beta_T} \right\} \to 1$ for all $0 < a_\gamma < \gamma$. Since $\eta, \gamma > 0$ is arbitrary and $\Pr \{\hat{r}_e < r_e\} \to 0$ and $\Pr \left\{ \hat{r}_f > r_f \right\} \to 0$, we deduce that $\Pr \{|\hat{r}_e - r_e| > \eta\} \to 0$ and $\Pr \left\{ |\hat{r}_f - r_f| > \gamma \right\} \to 0$ as $T \to \infty$, provided that

$$\frac{1}{scv^{\beta_T}} + \frac{scv^{\beta_T}}{T^{1-\alpha/2}} \to 0.$$

Therefore, the PSY date detectors $\hat{r}_e$ and $\hat{r}_f$ are consistent estimators of r_e and r_f. This proves Theorem 3.

B. *The Dating Algorithms (Two Bubbles)*

Section B.1 provides preliminary results that characterize the limit behavior of the regression components over subperiods of the data. Section B.2 provides test asymptotics and gives proofs of Theorems 4–9, which describe the consistency properties of the PWY, PSY, and sequential PWY dating strategies.

B.1. *notation and lemmas.*

- The two bubble periods are $B_1 = [\tau_{1e}, \tau_{1f}]$ and $B_2 = [\tau_{2e}, \tau_{2f}]$, where $\tau_{1e} = \lfloor Tr_{1e} \rfloor$, $\tau_{1f} = \lfloor Tr_{1f} \rfloor$, $\tau_{2e} = \lfloor Tr_{2e} \rfloor$, and $\tau_{2f} = \lfloor Tr_{2f} \rfloor$.
- The normal periods are $N_0 = [1, \tau_{1e})$, $N_1 = (\tau_{1f}, \tau_{2e})$, $N_2 = (\tau_{2f}, T]$.

We use the data-generating process

$$X_t = \begin{cases} X_{t-1} + \varepsilon_t & \text{for } t \in N_0 \\ \delta_T X_{t-1} + \varepsilon_t & \text{for } t \in B_i \text{ with } i = 1, 2 \\ X_{\tau_{if}}^* + \sum_{k=\tau_{if}+1}^{t} \varepsilon_k & \text{for } t \in N_i \text{ with } i = 1, 2 \end{cases}, \tag{A.11}$$

where $\delta_T = 1 + cT^{-\alpha}$ with $c > 0$ and $\alpha \in (0, 1)$, $\varepsilon_t \overset{i.i.d.}{\sim} (0, \sigma^2)$ and $X_{\tau_{if}}^* = X_{\tau_{ie}} + X_i^*$ with $X_i^* = O_p(1)$ for $i = 1, 2$. We state the following lemmas whose proofs follow arguments closely related to those given in the proofs of Lemmas A1–A6. They are provided in full in the technical supplement (Phillips et al., 2014; Lemmas S1–S6).

LEMMA A8. *Under the data-generating process,*

(1) *For* $t \in N_0$, $X_{t=\lfloor Tr \rfloor} \sim_a T^{1/2} B(r)$.
(2) *For* $t \in B_i$ *with* $i = 1, 2$, $X_{t=\lfloor Tr \rfloor} = \delta_T^{t-\tau_{ie}} X_{\tau_{ie}} \{1 + o_p(1)\} \sim_a T^{1/2} \delta_T^{t-\tau_{ie}} B(r_{ie})$.
(3) *For* $t \in N_i$ *with* $i = 1, 2$, $X_{t=\lfloor Tr \rfloor} \sim_a T^{1/2}[B(r) - B(r_{if}) + B(r_{ie})]$.

LEMMA A9. *Under the data-generating process,*

(1) *For* $\tau_1 \in N_{i-1}$ *and* $\tau_2 \in B_i$ *with* $i = 1, 2$,

$$\frac{1}{\tau_w} \sum_{j=\tau_1}^{\tau_2} X_j = \frac{T^\alpha \delta_T^{\tau_2 - \tau_{ie}}}{\tau_w c} X_{\tau_{ie}} \{1 + o_p(1)\} \sim_a T^{\alpha - 1/2} \delta_T^{\tau_2 - \tau_{ie}} \frac{1}{r_w c} B(r_{ie}).$$

(2) *For* $\tau_1 \in B_i$ *and* $\tau_2 \in N_i$ *with* $i = 1, 2$,

$$\frac{1}{\tau_w} \sum_{j=\tau_1}^{\tau_2} X_j = \frac{T^\alpha \delta_T^{\tau_{if} - \tau_1}}{\tau_w c} X_{\tau_{ie}} \{1 + o_p(1)\} \sim_a T^{\alpha - 1/2} \delta_T^{\tau_{if} - \tau_1} \frac{1}{r_w c} B(r_{ie}).$$

(3) *For* $\tau_1 \in N_{i-1}$ *and* $\tau_2 \in N_i$ *with* $i = 1, 2$,

$$\frac{1}{\tau_w} \sum_{j=\tau_1}^{\tau_2} X_j = X_{\tau_{ie}} \frac{T^\alpha \delta_T^{\tau_{if} - \tau_{ie}}}{\tau_w c} \{1 + o_p(1)\} \sim_a T^{\alpha - 1/2} \delta_T^{\tau_{if} - \tau_{ie}} \frac{1}{r_w c} B(r_{ie}).$$

(4) *For* $\tau_1 \in N_0$ *and* $\tau_2 \in N_2$, *if* $r_{1f} - r_{1e} > r_{2f} - r_{2e}$

$$\frac{1}{\tau_w} \sum_{j=\tau_1}^{\tau_2} X_j = \frac{T^\alpha \delta_T^{\tau_{1f} - \tau_{1e}}}{\tau_w c} X_{\tau_{1e}} \{1 + o_p(1)\} \sim_a T^{\alpha - 1/2} \delta_T^{\tau_{1f} - \tau_{1e}} \frac{1}{r_w c} B(r_{1e})$$

and if $r_{1f} - r_{1e} \le r_{2f} - r_{2e}$

$$\frac{1}{\tau_w} \sum_{j=\tau_1}^{\tau_2} X_j = \frac{T^\alpha \delta_T^{\tau_{2f} - \tau_{2e}}}{\tau_w c} X_{\tau_{2e}} \{1 + o_p(1)\} \sim_a T^{\alpha - 1/2} \delta_T^{\tau_{2f} - \tau_{2e}} \frac{1}{r_w c} B(r_{2e}).$$

(5) *For* $\tau_1 \in B_1$ *and* $\tau_2 \in B_2$, *if* $r_{1f} - r_1 > r_2 - r_{2e}$

$$\frac{1}{\tau_w} \sum_{j=\tau_1}^{\tau_2} X_j = \frac{T^\alpha \delta_T^{\tau_{1f} - \tau_1}}{\tau_w c} X_{\tau_{1e}} \{1 + o_p(1)\} \sim_a T^{\alpha - 1/2} \delta_T^{\tau_{1f} - \tau_1} \frac{1}{r_w c} B(r_{1e});$$

if $r_{1f} - r_1 \le r_2 - r_{2e}$

$$\frac{1}{\tau_w}\sum_{j=\tau_1}^{\tau_2} X_j = \frac{T^{\alpha}\delta_T^{\tau_2-\tau_{2e}}}{\tau_w c} X_{\tau_{2e}}\left\{1+o_p(1)\right\} \sim_a T^{\alpha-1/2}\delta_T^{\tau_2-\tau_{2e}}\frac{1}{r_w c}B(r_{2e}).$$

(6) *For* $\tau_1 \in B_1$ *and* $\tau_2 \in N_2$, *if* $r_{1f} - r_1 > r_{2f} - r_{2e}$,

$$\frac{1}{\tau_w}\sum_{j=\tau_1}^{\tau_2} X_j = \frac{T^{\alpha}\delta_T^{\tau_{1f}-\tau_1}}{\tau_w c} X_{\tau_{1e}}\left\{1+o_p(1)\right\} \sim_a T^{\alpha-1/2}\delta_T^{\tau_{1f}-\tau_1}\frac{1}{r_w c}B(r_1)$$

and if $r_{1f} - r_1 \le r_{2f} - r_{2e}$,

$$\frac{1}{\tau_w}\sum_{j=\tau_1}^{\tau_2} X_j = \frac{T^{\alpha}\delta_T^{\tau_{2f}-\tau_{2e}}}{\tau_w c} X_{\tau_{2e}}\left\{1+o_p(1)\right\} \sim_a T^{\alpha-1/2}\delta_T^{\tau_{2f}-\tau_{2e}}\frac{1}{r_w c}B(r_{2e}).$$

(7) *For* $\tau_1 \in N_0$ *and* $\tau_2 \in B_2$, *if* $r_{1f} - r_{1e} > r_2 - r_{2e}$,

$$\frac{1}{\tau_w}\sum_{j=\tau_1}^{\tau_2} X_j = \frac{T^{\alpha}\delta_T^{\tau_{1f}-\tau_{1e}}}{\tau_w c} X_{\tau_{1e}}\left\{1+o_p(1)\right\} \sim_a T^{\alpha-1/2}\delta_T^{\tau_{1f}-\tau_{1e}}\frac{1}{r_w c}B(r_{1e})$$

and if $r_{1f} - r_{1e} \le r_2 - r_{2e}$,

$$\frac{1}{\tau_w}\sum_{j=\tau_1}^{\tau_2} X_j = \frac{T^{\alpha}\delta_T^{\tau_2-\tau_{2e}}}{\tau_w c} X_{\tau_{2e}}\left\{1+o_p(1)\right\} \sim_a T^{\alpha-1/2}\delta_T^{\tau_2-\tau_{2e}}\frac{1}{r_w c}B(r_{2e}).$$

LEMMA A10. *Define the centered quantity* $\tilde{X}_t = X_t - \tau_w^{-1}\sum_{j=\tau_1}^{\tau_2} X_j$.

(1) *For* $\tau_1 \in N_{i-1}$ *and* $\tau_2 \in B_i$ *with* $i = 1, 2$,

$$\tilde{X}_t = \begin{cases} -\dfrac{T^{\alpha}\delta_T^{\tau_2-\tau_{ie}}}{\tau_w c} X_{\tau_{ie}}\left\{1+o_p(1)\right\} & \text{if } t \in N_{i-1} \\ \left[\delta_T^{t-\tau_{ie}} - \frac{T^{\alpha}\delta_T^{\tau_2-\tau_{ie}}}{\tau_w c}\right] X_{\tau_{ie}}\left\{1+o_p(1)\right\} & \text{if } t \in B_i \end{cases}.$$

(2) *For* $\tau_1 \in B_i$ *and* $\tau_2 \in N_i$ *with* $i = 1, 2$,

$$\tilde{X}_t = \begin{cases} \left[\delta_T^{t-\tau_{ie}} - \frac{T^{\alpha}\delta_T^{\tau_{if}-\tau_1}}{\tau_w c}\right] X_{\tau_{ie}}\left\{1+o_p(1)\right\} & \text{if } t \in B_i \\ -\frac{T^{\alpha}\delta_T^{\tau_{if}-\tau_1}}{\tau_w c} X_{\tau_{ie}}\left\{1+o_p(1)\right\} & \text{if } t \in N_i \end{cases}.$$

(3) *For* $\tau_1 \in N_{i-1}$ *and* $\tau_2 \in N_i$ *with* $i = 1, 2$,

$$\tilde{X}_t = \begin{cases} -\frac{T^{\alpha}\delta_T^{\tau_{if}-\tau_{ie}}}{\tau_w c} X_{\tau_{ie}}\left\{1+o_p(1)\right\} & \text{if } t \in N_{i-1} \cup N_i \\ \left[\delta_T^{t-\tau_{ie}} - \frac{T^{\alpha}\delta_T^{\tau_{if}-\tau_{ie}}}{\tau_w c}\right] X_{\tau_{ie}}\left\{1+o_p(1)\right\} & \text{if } t \in B_i \end{cases}.$$

(4) *For* $\tau_1 \in N_0$ *and* $\tau_2 \in N_2$, *if* $r_{1f} - r_{1e} > r_{2f} - r_{2e}$

$$\tilde{X}_t = \begin{cases} -\frac{T^\alpha \delta_T^{\tau_{1f}-\tau_{1e}}}{\tau_w c} X_{\tau_{1e}} \left\{1 + o_p(1)\right\} & \text{if } t \in N_i, \\ \left[\delta_T^{t-\tau_{ie}} X_{\tau_{ie}} - \frac{T^\alpha \delta_T^{\tau_{1f}-\tau_{1e}}}{\tau_w c} X_{\tau_{1e}}\right]\left\{1 + o_p(1)\right\} & \text{if } t \in B_i, i = 1, 2, \end{cases}$$

and if $r_{1f} - r_{1e} \leq r_{2f} - r_{2e}$

$$\tilde{X}_t = \begin{cases} -\frac{T^\alpha \delta_T^{\tau_{2f}-\tau_{2e}}}{\tau_w c} X_{\tau_{2e}} \left\{1 + o_p(1)\right\} & \text{if } t \in N_i, \\ \left[\delta_T^{t-\tau_{ie}} X_{\tau_{ie}} - \frac{T^\alpha \delta_T^{\tau_{2f}-\tau_{2e}}}{\tau_w c} X_{\tau_{2e}}\right]\left\{1 + o_p(1)\right\} & \text{if } t \in B_i, i = 1, 2 \end{cases}.$$

(5) *For* $\tau_1 \in B_1$ *and* $\tau_2 \in B_2$, *if* $r_{1f} - r_1 > r_2 - r_{2e}$,

$$\tilde{X}_t = \begin{cases} \left[\delta_T^{t-\tau_{ie}} X_{\tau_{ie}} - \frac{T^\alpha \delta_T^{\tau_{1f}-\tau_1}}{\tau_w c} X_{\tau_{1e}}\right]\left\{1 + o_p(1)\right\} & \text{if } t \in B_i, i = 1, 2, \\ -\frac{T^\alpha \delta_T^{\tau_{1f}-\tau_1}}{\tau_w c} X_{\tau_{1e}} \left\{1 + o_p(1)\right\} & \text{if } t \in N_1 \end{cases}$$

and if $\tau_{1f} - \tau_1 \leq r_2 - r_{2e}$

$$\tilde{X}_t = \begin{cases} \left[\delta_T^{t-\tau_{ie}} X_{\tau_{ie}} - \frac{T^\alpha \delta_T^{\tau_2-\tau_{2e}}}{\tau_w c} X_{\tau_{2e}}\right]\left\{1 + o_p(1)\right\} & \text{if } t \in B_i, i = 1, 2, \\ -\frac{T^\alpha \delta_T^{\tau_2-\tau_{2e}}}{\tau_w c} X_{\tau_{2e}} \left\{1 + o_p(1)\right\} & \text{if } t \in N_1 \end{cases}.$$

(6) *For* $\tau_1 \in B_1$ *and* $\tau_2 \in N_2$, *if* $r_{1f} - r_1 > r_{2f} - r_{2e}$,

$$\tilde{X}_t = \begin{cases} \left[\delta_T^{t-\tau_{ie}} X_{\tau_{ie}} - \frac{T^\alpha \delta_T^{\tau_{1f}-\tau_1}}{\tau_w c} X_{\tau_{1e}}\right]\left\{1 + o_p(1)\right\} & \text{if } t \in B_i, i = 1, 2, \\ -\frac{T^\alpha \delta_T^{\tau_{1f}-\tau_1}}{\tau_w c} X_{\tau_{1e}} \left\{1 + o_p(1)\right\} & \text{if } t \in N_i, i = 1, 2, \end{cases}$$

and if $\tau_{1f} - \tau_1 \leq r_{2f} - r_{2e}$,

$$\tilde{X}_t = \begin{cases} \left[\delta_T^{t-\tau_{ie}} X_{\tau_{ie}} - \frac{T^\alpha \delta_T^{\tau_{2f}-\tau_{2e}}}{\tau_w c} X_{\tau_{2e}}\right]\left\{1 + o_p(1)\right\} & \text{if } t \in B_i, i = 1, 2, \\ -\frac{T^\alpha \delta_T^{\tau_{2f}-\tau_{2e}}}{\tau_w c} X_{\tau_{2e}} \left\{1 + o_p(1)\right\} & \text{if } t \in N_i, i = 1, 2 \end{cases}.$$

(7) *For* $\tau_1 \in N_0$ *and* $\tau_2 \in B_2$, *if* $r_{1f} - r_{1e} > r_2 - r_{2e}$

$$\tilde{X}_t = \begin{cases} -\frac{T^\alpha \delta_T^{\tau_{1f}-\tau_{1e}}}{\tau_w c} X_{\tau_{1e}} \left\{1 + o_p(1)\right\} & \text{if } t \in N_i, i = 1, 2, \\ \left[\delta_T^{t-\tau_{ie}} X_{\tau_{ie}} - \frac{T^\alpha \delta_T^{\tau_{1f}-\tau_{1e}}}{\tau_w c} X_{\tau_{1e}}\right]\left\{1 + o_p(1)\right\} & \text{if } t \in B_i, i = 1, 2 \end{cases}$$

and if $r_{1f} - r_{1e} \leq r_2 - r_{2e}$

$$\tilde{X}_t = \begin{cases} -\frac{T^\alpha \delta_T^{\tau_2-\tau_{2e}}}{\tau_w c} X_{\tau_{2e}} \left\{1 + o_p(1)\right\} & \text{if } t \in N_i, i = 1, 2, \\ \left[\delta_T^{t-\tau_{ie}} X_{\tau_{ie}} - \frac{T^\alpha \delta_T^{\tau_2-\tau_{2e}}}{\tau_w c} X_{\tau_{2e}}\right]\left\{1 + o_p(1)\right\} & \text{if } t \in B_i, i = 1, 2 \end{cases}.$$

LEMMA A11. *The sample variance of $\tilde{X}_t$ has the following limit form:*

(1) *For $\tau_1 \in N_{i-1}$ and $\tau_2 \in B_i$ with $i = 1, 2$,*

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}^2 = \frac{T^\alpha \delta_T^{2(\tau_2-\tau_{ie})}}{2c} X_{\tau_{ie}}^2 \left\{1 + o_p(1)\right\} \sim_a \frac{T^{1+\alpha} \delta_T^{2(\tau_2-\tau_{ie})}}{2c} B(r_{ie})^2 .$$

(2) *For $\tau_1 \in B_i$ and $\tau_2 \in N_i$ with $i = 1, 2$,*

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}^2 = \frac{T^\alpha \delta_T^{2(\tau_{if}-\tau_{ie})}}{2c} X_{\tau_{ie}}^2 \left\{1 + o_p(1)\right\} \sim_a \frac{T^{1+\alpha} \delta_T^{2(\tau_{if}-\tau_{ie})}}{2c} B(r_{ie})^2 .$$

(3) *For $\tau_1 \in N_{i-1}$ and $\tau_2 \in N_i$ with $i = 1, 2$,*

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}^2 = \frac{T^\alpha \delta_T^{2(\tau_{if}-\tau_{ie})}}{2c} X_{\tau_{ie}}^2 \left\{1 + o_p(1)\right\} \sim_a \frac{T^{1+\alpha} \delta_T^{2(\tau_{if}-\tau_{ie})}}{2c} B(r_{ie})^2 .$$

(4) *For $\tau_1 \in N_0$ and $\tau_2 \in N_2$,*

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}^2 = \begin{cases} \frac{T^\alpha \delta_T^{2\left(\tau_{1f}-\tau_{1e}\right)}}{2c} X_{\tau_{1e}}^2 \left\{1 + o_p(1)\right\} \sim_a \frac{T^{\alpha+1} \delta_T^{2\left(\tau_{1f}-\tau_{1e}\right)}}{2c} B(r_{1e})^2 & \text{if } r_{1f} - r_{1e} > r_{2f} - r_{2e} \\ \frac{T^\alpha \delta_T^{2\left(\tau_{2f}-\tau_{2e}\right)}}{2c} X_{\tau_{2e}}^2 \left\{1 + o_p(1)\right\} \sim_a \frac{T^{\alpha+1} \delta_T^{2\left(\tau_{2f}-\tau_{2e}\right)}}{2c} B(r_{2e})^2 & \text{if } r_{1f} - r_{1e} \le r_{2f} - r_{2e} \end{cases} .$$

(5) *For $\tau_1 \in B_1$ and $\tau_2 \in B_2$,*

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}^2 = \begin{cases} \frac{T^\alpha \delta_T^{2\left(\tau_{1f}-\tau_{1e}\right)}}{2c} X_{\tau_{1e}}^2 \left\{1 + o_p(1)\right\} \sim_a \frac{T^{\alpha+1} \delta_T^{2\left(\tau_{1f}-\tau_{1e}\right)}}{2c} B(r_{1e})^2 & \text{if } r_{1f} - r_{1e} > r_2 - r_{2e} \\ \frac{T^\alpha \delta_T^{2(\tau_2-\tau_{2e})}}{2c} X_{\tau_{2e}}^2 \left\{1 + o_p(1)\right\} \sim_a T^{\alpha+1} \delta_T^{2(\tau_2-\tau_{2e})} \frac{1}{2c} B(r_{2e})^2 & \text{if } r_{1f} - r_{1e} \le r_2 - r_{2e} \end{cases} .$$

(6) *For $\tau_1 \in B_1$ and $\tau_2 \in N_2$,*

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}^2 = \begin{cases} \frac{T^\alpha \delta_T^{2\left(\tau_{1f}-\tau_{1e}\right)}}{2c} X_{\tau_{1e}}^2 \left\{1 + o_p(1)\right\} \sim_a \frac{T^{\alpha+1} \delta_T^{2\left(\tau_{1f}-\tau_{1e}\right)}}{2c} B(r_{1e})^2 & \text{if } r_{1f} - r_{1e} > r_{2f} - r_{2e} \\ \frac{T^\alpha \delta_T^{2\left(\tau_{2f}-\tau_{2e}\right)}}{2c} X_{\tau_{2e}}^2 \left\{1 + o_p(1)\right\} \sim_a \frac{T^{\alpha+1} \delta_T^{2\left(\tau_{2f}-\tau_{2e}\right)}}{2c} B(r_{2e})^2 & \text{if } r_{1f} - r_{1e} \le r_{2f} - r_{2e} \end{cases} .$$

(7) *For $\tau_1 \in N_0$ and $\tau_2 \in B_2$,*

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}^2 = \begin{cases} \frac{T^\alpha \delta_T^{2\left(\tau_{1f}-\tau_{1e}\right)}}{2c} X_{\tau_{1e}}^2 \left\{1 + o_p(1)\right\} \sim_a \frac{T^{\alpha+1} \delta_T^{2\left(\tau_{1f}-\tau_{1e}\right)}}{2c} B(r_{1e})^2 & \text{if } r_{1f} - r_{1e} > r_2 - r_{2e} \\ \frac{T^\alpha \delta_T^{2(\tau_2-\tau_{2e})}}{2c} X_{\tau_{2e}}^2 \left\{1 + o_p(1)\right\} \sim_a \frac{T^{\alpha+1} \delta_T^{2(\tau_2-\tau_{2e})}}{2c} B(r_{2e})^2 & \text{if } r_{1f} - r_{1e} \le r_2 - r_{2e} \end{cases} .$$

LEMMA A12. *The sample covariance of $\tilde{X}_t$ and ε_t has the following limit form:*

(1) *For $\tau_1 \in N_{i-1}$ and $\tau_2 \in B_i$ with $i = 1, 2$,*

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1} \varepsilon_j \sim_a T^{(\alpha+1)/2} \delta_T^{\tau_2-\tau_{ie}} X_c B(r_{ie}) .$$

(2) *For $\tau_1 \in B_i$ and $\tau_2 \in N_i$ with $i = 1, 2$,*

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}\varepsilon_j \sim_a T^{(\alpha+1)/2}\delta_T^{\tau_{if}-\tau_{ie}} X_c B(r_{ie}).$$

(3) *For $\tau_1 \in N_{i-1}$ and $\tau_2 \in N_i$ with $i = 1, 2$,*

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}\varepsilon_j \sim_a T^{(\alpha+1)/2}\delta_T^{\tau_{if}-\tau_{ie}} X_c B(r_{ie}).$$

(4) *For $\tau_1 \in N_0$ and $\tau_2 \in N_2$,*

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}\varepsilon_j \sim_a \begin{cases} T^{(1+\alpha)/2}\delta_T^{\tau_{1f}-\tau_{1e}} X_c B(r_{1e}) & \text{if } r_{1f} - r_{1e} > r_{2f} - r_{2e} \\ T^{(1+\alpha)/2}\delta_T^{\tau_{2f}-\tau_{2e}} X_c B(r_{2e}) & \text{if } r_{1f} - r_{1e} \le r_{2f} - r_{2e} \end{cases}.$$

(5) *For $\tau_1 \in B_1$ and $\tau_2 \in B_2$,*

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}\varepsilon_j \sim_a \begin{cases} T^{(\alpha+1)/2}\delta_T^{\tau_{1f}-\tau_{1e}} X_c B(r_{1e}) & \text{if } r_{1f} - r_{1e} > r_2 - r_{2e} \\ T^{(\alpha+1)/2}\delta_T^{\tau_2-\tau_{2e}} X_c B(r_{2e}) & \text{if } r_{1f} - r_{1e} \le r_2 - r_{2e} \end{cases}.$$

(6) *For $\tau_1 \in B_1$ and $\tau_2 \in N_2$,*

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}\varepsilon_j \sim_a \begin{cases} T^{(1+\alpha)/2}\delta_T^{\tau_{1f}-\tau_{1e}} X_c B(r_{1e}) & \text{if } r_{1f} - r_{1e} > r_{2f} - r_{2e} \\ T^{(1+\alpha)/2}\delta_T^{\tau_{2f}-\tau_{2e}} X_c B(r_{2e}) & \text{if } r_{1f} - r_{1e} \le r_{2f} - r_{2e} \end{cases}.$$

(7) *For $\tau_1 \in N_0$ and $\tau_2 \in B_2$,*

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}\varepsilon_j \sim_a \begin{cases} T^{(\alpha+1)/2}\delta_T^{\tau_{1f}-\tau_{1e}} X_c B(r_{1e}) & \text{if } r_{1f} - r_{1e} > r_2 - r_{2e} \\ T^{(\alpha+1)/2}\delta_T^{\tau_2-\tau_{2e}} X_c B(r_{2e}) & \text{if } r_{1f} - r_{1e} \le r_2 - r_{2e} \end{cases}.$$

LEMMA A13. *The sample covariance of $\tilde{X}_{j-1}$ and $X_j - \delta_T X_{j-1}$ has the following limit form:*

(1) *For $\tau_1 \in N_{i-1}$ and $\tau_2 \in B_i$ with $i = 1, 2$,*

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}(X_j - \delta_T X_{j-1}) \sim_a \frac{r_{ie} - r_1}{r_w} T\delta_T^{\tau_2-\tau_{ie}} B(r_{ie}) \int_{r_1}^{r_{ie}} B(s)\,ds.$$

(2) *For $\tau_1 \in B_i$ and $\tau_2 \in N_i$ with $i = 1, 2$,*

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}(X_j - \delta_T X_{j-1}) \sim_a -T\delta_T^{2(\tau_{if}-\tau_{ie})} B(r_{ie})^2.$$

(3) *For $\tau_1 \in N_{i-1}$ and $\tau_2 \in N_i$ with $i = 1, 2$,*

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}(X_j - \delta_T X_{j-1}) \sim_a \sim_a -T\delta_T^{2(\tau_{if}-\tau_{ie})} B(r_{ie})^2.$$

(4) *For* $\tau_1 \in N_0$ *and* $\tau_2 \in N_2$,

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}(X_j - \delta_T X_{j-1}) \sim_a \begin{cases} -T\delta_T^{2(\tau_{1f}-\tau_{1e})} B(r_{1e})^2 & \text{if } r_{1f} - r_{1e} > r_{2f} - r_{2e} \\ -T\delta_T^{2(\tau_{2f}-\tau_{2e})} B(r_{2e})^2 & \text{if } r_{1f} - r_{1e} \le r_{2f} - r_{2e} \end{cases}.$$

(5) *For* $\tau_1 \in B_1$ *and* $\tau_2 \in B_2$,

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}(X_j - \delta_T X_{j-1}) \sim_a \begin{cases} -T\delta_T^{2(\tau_{1f}-\tau_{1e})} B(r_{1e})^2 & \text{if } r_{1f} - r_{1e} > r_2 - r_{2e} \\ T^{\alpha}\delta_T^{\tau_2-\tau_{2e}+\tau_{1f}-\tau_{1e}} \frac{1}{r_w c} B(r_{2e}) B(r_{1e}) & \text{if } r_{1f} - r_{1e} \le r_2 - r_{2e} \end{cases}.$$

(6) *For* $\tau_1 \in B_1$ *and* $\tau_2 \in N_2$,

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}(X_j - \delta_T X_{j-1}) \sim_a \begin{cases} -T\delta_T^{2(\tau_{1f}-\tau_{1e})} B(r_{1e})^2 & \text{if } r_{1f} - r_{1e} > r_{2f} - r_{2e} \\ -T\delta_T^{2(\tau_{2f}-\tau_{2e})} B(r_{2e})^2 & \text{if } r_{1f} - r_{1e} \le r_{2f} - r_{2e} \end{cases}.$$

(7) *For* $\tau_1 \in N_0$ *and* $\tau_2 \in B_2$,

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}(X_j - \delta_T X_{j-1}) \sim_a \begin{cases} -T\delta_T^{2(\tau_{1f}-\tau_{1e})} B(r_{1e})^2 & \text{if } r_{1f} - r_{1e} > r_2 - r_{2e} \\ T^{\alpha}\delta_T^{\tau_2-\tau_{2e}+\tau_{1f}-\tau_{1e}} \frac{1}{r_w c} B(r_{2e}) B(r_{1e}) & \text{if } r_{1f} - r_{1e} \le r_2 - r_{2e} \end{cases}.$$

B.2. *Test asymptotics and proofs of Theorems 4–9.* The fitted regression model for the recursive unit root tests is

$$X_t = \hat{\alpha}_{r_1,r_2} + \hat{\rho}_{r_1,r_2} X_{t-1} + \hat{\varepsilon}_t,$$

where as in (A.9) above the intercept $\hat{\alpha}_{r_1,r_2}$ and slope coefficient $\hat{\rho}_{r_1,r_2}$ are obtained using data over the subperiod $[r_1, r_2]$.

REMARK 4. Based on Lemma A11 and Lemma A13, we can obtain the limit distribution of $\hat{\rho}_{r_1,r_2} - \delta_T$ using

$$\hat{\rho}_{r_1,r_2} - \delta_T = \frac{\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}(X_j - \delta_T X_{j-1})}{\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}^2}.$$

(1) When $\tau_1 \in N_{i-1}$ and $\tau_2 \in B_i$ with $i = 1, 2$,

$$\hat{\rho}_{r_1,r_2} - \delta_T \sim_a T^{-\alpha}\delta_T^{-(\tau_2-\tau_{ie})} \frac{\frac{r_{ie}-r_1}{r_w}\int_{r_1}^{r_{ie}} B(s)\,ds}{B(r_{ie})};$$

(2) when $\tau_1 \in B_i$ and $\tau_2 \in N_i$ with $i = 1, 2$,

$$\hat{\rho}_{r_1,r_2} - \delta_T \sim_a -2T^{-\alpha}c;$$

(3) when $\tau_1 \in N_{i-1}$ and $\tau_2 \in N_i$ with $i = 1, 2$,

$$\hat{\rho}_{r_1,r_2} - \delta_T \sim_a -2T^{-\alpha}c;$$

(4) when $\tau_1 \in N_0$ and $\tau_2 \in N_2$,

$$\hat{\rho}_{r_1,r_2} - \delta_T \sim_a -2T^{-\alpha}c;$$

(5) when $\tau_1 \in B_1$ and $\tau_2 \in B_2$,

$$\hat{\rho}_{r_1,r_2} - \delta_T \sim_a \begin{cases} -2T^{-\alpha}c & \text{if } r_{1f} - r_{1e} > r_2 - r_{2e} \\ T^{-1}\delta_T^{-(\tau_2-\tau_{2e})+(\tau_{1f}-\tau_{1e})} \frac{2B(r_{1e})}{r_w B(r_{2e})} & \text{if } r_{1f} - r_{1e} \leq r_2 - r_{2e} \end{cases};$$

(6) when $\tau_1 \in B_1$ and $\tau_2 \in N_2$,

$$\hat{\rho}_{r_1,r_2} - \delta_T \sim_a -2T^{-\alpha}c;$$

(7) when $\tau_1 \in N_0$ and $\tau_2 \in B_2$,

$$\hat{\rho}_{r_1,r_2} - \delta_T \sim_a \begin{cases} -2T^{-\alpha}c & \text{if } r_{1f} - r_{1e} > r_2 - r_{2e} \\ T^{-1}\delta_T^{-(\tau_2-\tau_{2e})+(\tau_{1f}-\tau_{1e})} \frac{2B(r_{1e})}{r_w B(r_{2e})} & \text{if } r_{1f} - r_{1e} \leq r_2 - r_{2e} \end{cases}.$$

REMARK 5. The asymptotic distributions of the unit root coefficient Z-statistics can be calculated using

$$DF^z_{r_1,r_2} = \tau_w\left(\hat{\rho}_{r_1,r_2} - 1\right) = \tau_w\left(\delta_T - 1\right) + \tau_w\left(\hat{\rho}_{r_1,r_2} - \delta_T\right).$$

(1) When $\tau_1 \in N_{i-1}$ and $\tau_2 \in B_i$ with $i = 1, 2$,

$$DF^z_{r_1,r_2} = r_w c T^{1-\alpha} + o_p(1) \to \infty.$$

(2) When $\tau_1 \in B_i$ and $\tau_2 \in N_i$ with $i = 1, 2$,

$$DF^z_{r_1,r_2} = -cr_w T^{1-\alpha} \to -\infty.$$

(3) When $\tau_1 \in N_{i-1}$ and $\tau_2 \in N_i$ with $i = 1, 2$,

$$DF^z_{r_1,r_2} = -cr_w T^{1-\alpha} \to -\infty.$$

(4) When $\tau_1 \in N_0$ and $\tau_2 \in N_2$,

$$DF^z_{r_1,r_2} = -cr_w T^{1-\alpha} \to -\infty.$$

(5) When $\tau_1 \in B_1$ and $\tau_2 \in B_2$,

$$DF^z_{r_1,r_2} = \begin{cases} -cr_w T^{1-\alpha} \to -\infty & \text{if } r_{1f} - r_{1e} > r_2 - r_{2e} \\ cr_w T^{1-\alpha} \to \infty & \text{if } r_{1f} - r_{1e} \leq r_2 - r_{2e} \end{cases}.$$

(6) When $\tau_1 \in B_1$ and $\tau_2 \in N_2$,

$$DF^z_{r_1,r_2} = -cr_w T^{1-\alpha} \to -\infty.$$

(7) When $\tau_1 \in N_0$ and $\tau_2 \in B_2$,

$$DF^z_{r_1,r_2} = \begin{cases} -cr_w T^{1-\alpha} \to -\infty & \text{if } r_{1f} - r_{1e} > r_2 - r_{2e} \\ cr_w T^{1-\alpha} \to \infty & \text{if } r_{1f} - r_{1e} \leq r_2 - r_{2e} \end{cases}.$$

To obtain the asymptotic distributions of the t-statistics, we first obtain the equation standard error of the regression over $[r_1, r_2]$, which is

$$\hat{\sigma}_{r_1 r_2} = \left\{ \tau_w^{-1} \sum_{j=\tau_1}^{\tau_2} \left(\tilde{X}_j - \hat{\rho}_{r_1,r_2} \tilde{X}_{j-1} \right)^2 \right\}^{1/2}.$$

LEMMA A14. (1) *When $\tau_1 \in N_{i-1}$ and $\tau_2 \in B_i$ with $i = 1, 2$,*

$$\hat{\sigma}^2_{r_1 r_2} \sim_a T^{-1} \delta_T^{2(\tau_2 - \tau_{ie})} \frac{r_{ie} - r_1}{c^{-1} r_w^3} B(r_{ie})^2 .$$

(2) *When $\tau_1 \in B_i$ and $\tau_2 \in N_i$ with $i = 1, 2$,*

$$\hat{\sigma}^2_{r_1 r_2} \sim_a \frac{1}{r_w} \delta_T^{2(\tau_{if} - \tau_{ie})} B(r_{ie})^2 .$$

(3) *When $\tau_1 \in N_{i-1}$ and $\tau_2 \in N_i$ with $i = 1, 2$,*

$$\hat{\sigma}^2_{r_1 r_2} \sim_a \frac{\delta_T^{2(\tau_{if} - \tau_{ie})}}{r_w} B(r_{ie})^2 .$$

(4) *When $\tau_1 \in N_0$ and $\tau_2 \in N_2$,*

$$\hat{\sigma}^2_{r_1 r_2} \sim_a \begin{cases} r_w^{-1} \delta_T^{2(\tau_{1f} - \tau_{1e})} B(r_{1e})^2 & \text{if } r_{1f} - r_{1e} > r_{2f} - r_{2e} \\ r_w^{-1} \delta_T^{2(\tau_{2f} - \tau_{2e})} B(r_{2e})^2 & \text{if } r_{1f} - r_{1e} \le r_{2f} - r_{2e} \end{cases}.$$

(5) *When $\tau_1 \in B_1$ and $\tau_2 \in B_2$,*

$$\hat{\sigma}^2_{r_1 r_2} \sim_a \begin{cases} \delta_T^{2(\tau_{1f} - \tau_{1e})} r_w^{-1} B(r_{1e})^2 & \text{if } r_{1f} - r_{1e} > r_2 - r_{2e} \\ T^{-1} \delta_T^{2(\tau_2 - \tau_{2e})} \frac{r_{2e} - r_{1f}}{r_w^3} B(r_{2e})^2 & \text{if } r_{1f} - r_{1e} \le r_2 - r_{2e} \end{cases}.$$

(6) *When $\tau_1 \in B_1$ and $\tau_2 \in N_2$,*

$$\hat{\sigma}^2_{r_1 r_2} \sim_a \begin{cases} \delta_T^{2(\tau_{1f} - \tau_{1e})} \frac{1}{r_w} B(r_{1e})^2 & \text{if } r_{1f} - r_{1e} > r_{2f} - r_{2e} \\ \delta_T^{2(\tau_{2f} - \tau_{2e})} \frac{1}{r_w} B(r_{2e})^2 & \text{if } r_{1f} - r_{1e} \le r_{2f} - r_{2e} \end{cases}.$$

(7) *When $\tau_1 \in N_0$ and $\tau_2 \in B_2$,*

$$\hat{\sigma}^2_{r_1 r_2} \sim_a \begin{cases} \delta_T^{2(\tau_{1f} - \tau_{1e})} \frac{1}{r_w} B(r_{1e})^2 & \text{if } r_{1f} - r_{1e} > r_2 - r_{2e} \\ T^{-1} \delta_T^{2(\tau_2 - \tau_{2e})} \frac{r_{1e} - r_1 + r_{2e} - r_{1f}}{r_w^3} B(r_{2e})^2 & \text{if } r_{1f} - r_{1e} \le r_2 - r_{2e} \end{cases}.$$

REMARK 6. The asymptotic distributions of the DF t-statistic can be calculated as

$$DF^t_{r_1,r_2} = \left(\frac{\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}^2}{\hat{\sigma}^2_{r_1 r_2}} \right)^{1/2} (\hat{\rho}_{r_1,r_2} - 1).$$

(1) When $\tau_1 \in N_{i-1}$ and $\tau_2 \in B_i$ with $i = 1, 2$,

$$DF^t_{r_1,r_2} \sim_a T^{1-\alpha/2} \frac{r_w^{3/2}}{\sqrt{2(r_{ie} - r_1)}} \to \infty;$$

(2) when $\tau_1 \in B_i$ and $\tau_2 \in N_i$ with $i = 1, 2$,

$$DF^t_{r_1,r_2} \sim_a -\left(\frac{1}{2} c r_w\right)^{1/2} T^{(1-\alpha)/2} \to -\infty;$$

(3) when $\tau_1 \in N_{i-1}$ and $\tau_2 \in N_i$ with $i = 1, 2$,

$$DF^t_{r_1,r_2} \sim_a -\left(\frac{1}{2} c r_w\right)^{1/2} T^{(1-\alpha)/2} \to -\infty;$$

(4) when $\tau_1 \in N_0$ and $\tau_2 \in N_2$,

$$DF^t_{r_1,r_2} \sim_a -\left(\frac{1}{2} c r_w\right)^{1/2} T^{(1-\alpha)/2} \to -\infty;$$

(5) when $\tau_1 \in B_1$ and $\tau_2 \in B_2$,

$$DF^t_{r_1,r_2} \sim_a \begin{cases} -\left(\frac{1}{2} c r_w\right)^{1/2} T^{(1-\alpha)/2} \to -\infty & \text{if } r_{1f} - r_{1e} > r_2 - r_{2e} \\ \left[\frac{c r_w^3}{2(r_{2c} - r_{1f})}\right]^{1/2} T^{1-\alpha/2} \to \infty & \text{if } r_{1f} - r_{1e} \leq r_2 - r_{2e} \end{cases};$$

(6) when $\tau_1 \in B_1$ and $\tau_2 \in N_2$,

$$DF^t_{r_1,r_2} \sim_a -\left(\frac{1}{2} c r_w\right)^{1/2} T^{(1-\alpha)/2} \to -\infty;$$

(7) when $\tau_1 \in N_0$ and $\tau_2 \in B_2$,

$$DF^t_{r_1,r_2} \sim_a \begin{cases} -\left(\frac{1}{2} c r_w\right)^{1/2} T^{(1-\alpha)/2} \to -\infty & \text{if } r_{1f} - r_{1e} > r_2 - r_{2e} \\ \left[\frac{c r_w^3}{2(r_{1e} - r_1 + r_{2e} - r_{1f})}\right]^{1/2} T^{1-\alpha/2} \to \infty & \text{if } r_{1f} - r_{1e} \leq r_2 - r_{2e} \end{cases}.$$

Taken together with (11) and (12), these results establish the limit behavior of the unit root statistics DF_r and $BSDF_r(r_0)$ in the two cases considered in theorems 4 and 5 (see also (A.13) below).

The PWY strategy. The origination of the bubble expansion r_{1e}, r_{2e} and the termination of the bubble collapse r_{1f}, r_{2f} based on the DF test are identified as

$$\hat{r}_{1e} = \inf_{r \in [r_0, 1]} \left\{ r_2 : DF_r > cv^{\beta_T} \right\} \text{ and } \hat{r}_{1f} = \inf_{r \in [\hat{r}_{1e} + L_T, 1]} \left\{ r_2 : DF_r < cv^{\beta_T} \right\},$$
$$\hat{r}_{2e} = \inf_{r \in (\hat{r}_{1f}, 1]} \left\{ r_2 : DF_r > cv^{\beta_T} \right\} \text{ and } \hat{r}_{2f} = \inf_{r \in [\hat{r}_{2e} + L_T, 1]} \left\{ r_2 : DF_r < cv^{\beta_T} \right\}.$$

We know that when $\beta_T \to 0$, $cv^{\beta_T} \to \infty$.

Case I. Suppose $r_{1f} - r_{1e} > r_{2f} - r_{2e}$. Given that $r_1 = 0$ and $r_2 = r_w = r$, the asymptotic distributions of the DF statistic under the alternative hypothesis are

$$DF_r \sim_a \begin{cases} F_r(W) & \text{if } r \in N_0 \\ T^{1-\alpha/2} \frac{r^{3/2}}{\sqrt{2(r_{1e}-r_1)}} & \text{if } r \in B_1 \\ -T^{(1-\alpha)/2}\left(\frac{1}{2}cr\right)^{1/2} & \text{if } r \in N_1 \cup B_2 \cup N_2 \end{cases}.$$

It is obvious that if $r \in N_0$,

$$\lim_{T\to\infty} \Pr\left\{DF_r > cv^{\beta_T}\right\} = \Pr\left\{F_{r_2}(W) = \infty\right\} = 0.$$

If $r \in B_1$, $\lim_{T\to\infty} \Pr\left\{DF_r > cv^{\beta_T}\right\} = 1$ provided that $\frac{cv^{\beta_T}}{T^{1-\alpha/2}} \to 0$. If $r \in N_1$, $\lim_{T\to\infty} \Pr\left\{DF_r < cv^{\beta_T}\right\} = 1$.

It follows that for any $\eta, \gamma > 0$,

$$\Pr\{\hat{r}_{1e} > r_{1e} + \eta\} \to 0 \text{ and } \Pr\left\{\hat{r}_{1f} < r_{1f} - \gamma\right\} \to 0,$$

due to the fact that $\Pr\left\{DF_{r_{1e}+a_\eta} > cv^{\beta_T}\right\} \to 1$ for all $0 < a_\eta < \eta$ and $\Pr\left\{DF_{r_{1f}-a_\gamma} > cv^{\beta_T}\right\} \to 1$ for all $0 < a_\gamma < \gamma$. Since $\eta, \gamma > 0$ is arbitrary, $\Pr\{\hat{r}_{1e} < r_{1e}\} \to 0$ and $\Pr\left\{\hat{r}_{1f} > r_{1f}\right\} \to 0$, we deduce that $\Pr\{|\hat{r}_{1e} - r_{1e}| > \eta\} \to 0$ and $\Pr\left\{|\hat{r}_{1f} - r_{1f}| > \gamma\right\} \to 0$ as $T \to \infty$, provided that

$$\frac{1}{cv^{\beta_T}} + \frac{cv^{\beta_T}}{T^{1-\alpha/2}} \to 0.$$

The strategy can therefore consistently estimate both r_{1e} and r_{1f}.

Since $\lim_{T\to\infty} \Pr\left\{DF_r < cv^{\beta_T}\right\} = 1$ when $r \in N_1 \cup B_2 \cup N_2$, the strategy cannot estimate r_{2e} and r_{2f} consistently when $r_{1f} - r_{1e} > r_{2f} - r_{2e}$. This proves Theorem 6.

Case II. Suppose $r_{1f} - r_{1e} \le r_{2f} - r_{2e}$. The asymptotic distributions of the DF statistic under the alternative hypothesis are

$$DF_r \sim_a \begin{cases} F_r(W) & \text{if } r \in N_0 \\ T^{1-\alpha/2} \frac{r^{3/2}}{\sqrt{2(r_{1e}-r_1)}} & \text{if } r \in B_1 \\ -T^{(1-\alpha)/2}\left(\frac{1}{2}cr\right)^{1/2} & \text{if } r \in N_1 \cup N_2 \\ -T^{(1-\alpha)/2}\left(\frac{1}{2}cr\right)^{1/2} & \text{if } r \in B_2 \text{ and } r_{1f} - r_{1e} > r - r_{2e} \\ T^{1-\alpha/2}\left[\frac{cr^3}{2(r_{1e}+r_{2e}-r_{1f})}\right]^{1/2} & \text{if } r \in B_2 \text{ and } r_{1f} - r_{1e} \le r - r_{2e} \end{cases}. \tag{A.12}$$

It is obvious that if $r \in N_0$,

$$\lim_{T\to\infty} \Pr\left\{DF_r > cv^{\beta_T}\right\} = \Pr\left\{F_r(W) = \infty\right\} = 0.$$

If $r \in B_1$, $\lim_{T\to\infty} \Pr\left\{DF_r > cv^{\beta_T}\right\} = 1$ provided that $\frac{cv^{\beta_T}}{T^{1-\alpha/2}} \to 0$. If $r \in N_1$, $\lim_{T\to\infty} \Pr\left\{DF_r < cv^{\beta_T}\right\} = 1$.

It follows that for any $\eta, \gamma > 0$,

$$\Pr\{\hat{r}_{1e} > r_{1e} + \eta\} \to 0 \text{ and } \Pr\left\{\hat{r}_{1f} < r_{1f} - \gamma\right\} \to 0,$$

due to the fact that $\Pr\left\{BDF_{r_{1e}+a_\eta} > cv^{\beta_T}\right\} \to 1$ for all $0 < a_\eta < \eta$ and $\Pr\left\{DF_{r_{1f}-a_\gamma} > cv^{\beta_T}\right\} \to 1$ for all $0 < a_\gamma < \gamma$. Since $\eta, \gamma > 0$ is arbitrary and $\Pr\{\hat{r}_{1e} < r_{1e}\} \to 0$ and $\Pr\left\{\hat{r}_{1f} > r_{1f}\right\} \to 0$, we deduce that $\Pr\{|\hat{r}_{1e} - r_{1e}| > \eta\} \to 0$ and $\Pr\left\{|\hat{r}_{1f} - r_{1f}| > \gamma\right\} \to 0$ as $T \to \infty$, provided that

$$\frac{1}{cv^{\beta_T}} + \frac{cv^{\beta_T}}{T^{1-\alpha/2}} \to 0.$$

The strategy therefore consistently estimates r_{1e} and r_{1f}.

If $r \in B_2$ and $r_{1f} - r_{1e} > r - r_{2e}$, $\lim_{T\to\infty} \Pr\left\{DF_r < cv^{\beta_T}\right\} = 1$ since $cv^{\beta_T} \to \infty$. If $r \in B_2$ and $r_{1f} - r_{1e} \leq r - r_{2e}$, $\lim_{T\to\infty} \Pr\left\{DF_r > cv^{\beta_T}\right\} = 1$ provided that $\frac{cv^{\beta_T}}{T^{1-\alpha/2}} \to 0$ in view of the final panel entry of (A.12). If $r \in N_1$, $\lim_{T\to\infty} \Pr\left\{DF_r < cv^{\beta_T}\right\} = 1$. This implies that the strategy cannot identify the second bubble when $r_{1f} - r_{1e} > r - r_{2e}$. However, when $r_{1f} - r_{1e} \leq r - r_{2e}$ it can identify the second bubble provided that

$$\frac{1}{cv^{\beta_T}} + \frac{cv^{\beta_T}}{T^{1-\alpha/2}} \to 0.$$

This suggests that estimated second bubble origination date $\hat{r}_{2e}$ will be biased, taking values of $r_{2e} + r_{1f} - r_{1e}$ (in view of the condition $r_{1f} - r_{1e} \leq r - r_{2e}$ under which the final panel entry of (A.12) holds). The termination point r_{2f} can be consistently estimated. This proves Theorem 7.

The PSY algorithm. The origination of the bubble expansion r_{1e}, r_{2e} and the termination of the bubble collapse r_{1f}, r_{2f} based on the backward sup DF test are identified as follows:

$$\hat{r}_{1e} = \inf_{r\in[r_0,1]}\left\{r : BSDF_r(r_0) > scv^{\beta_T}\right\} \text{ and } \hat{r}_{1f} = \inf_{r\in[\hat{r}_{1e}+L_T,1]}\left\{r : BSDF_r(r_0) < scv^{\beta_T}\right\},$$
$$\hat{r}_{2e} = \inf_{r\in(\hat{r}_{1f},1]}\left\{r : BSDF_r(r_0) > scv^{\beta_T}\right\} \text{ and } \hat{r}_{2f} = \inf_{r\in[\hat{r}_{2e}+L_T,1]}\left\{r : BSDF_r(r_0) < scv^{\beta_T}\right\}.$$

We know that when $\beta_T \to 0$, $scv^{\beta_T} \to \infty$.

Given that $r_1 \in [0, r - r_0]$, $r_2 = r$ and $r_w = r_2 - r_1$, the asymptotic distributions of the backward sup DF statistic under the alternative hypothesis are

$$\text{(A.13)} \qquad BSDF_r(r_0) \sim_a \begin{cases} F_r(W, r_0) & \text{if } r \in N_0 \\ T^{1-\alpha/2} \sup\limits_{r_1\in[0,r-r_0]} \left\{\frac{(r-r_1)^{3/2}}{\sqrt{2(r_{ie}-r_1)}}\right\} & \text{if } r \in B_i \\ -T^{(1-\alpha)/2} \sup\limits_{r_1\in[0,r-r_0]} \left[\frac{1}{2}c(r-r_1)\right]^{1/2} & \text{if } r \in N_1 \cup N_2 \end{cases}.$$

It is obvious that if $r \in N_0$,

$$\lim_{T\to\infty} \Pr\left\{BSDF_r(r_0) > scv^{\beta_T}\right\} = \Pr\left\{F_r(W, r_0) = \infty\right\} = 0.$$

If $r \in B_i$ with $i = 1, 2$, $\lim_{T\to\infty} \Pr\left\{BSDF_r(r_0) > scv^{\beta_T}\right\} = 1$ provided that $\frac{scv^{\beta_T}}{T^{1-\alpha/2}} \to 0$. If $r \in N_i$ with $i = 1, 2$, $\lim_{T\to\infty} \Pr\left\{BSDF_r(r_0) < scv^{\beta_T}\right\} = 1$.

It follows that for any $\eta, \gamma > 0$,

$$\Pr\{\hat{r}_{ie} > r_{ie} + \eta\} \to 0 \text{ and } \Pr\left\{\hat{r}_{if} < r_{if} - \gamma\right\} \to 0,$$

since $\Pr\left\{BSDF_{r_{ie}+a_\eta}(r_0) > scv^{\beta_T}\right\} \to 1$ for all $0 < a_\eta < \eta$ and $\Pr\left\{BSDF_{r_{if}-a_\gamma}(r_0) > scv^{\beta_T}\right\} \to 1$ for all $0 < a_\gamma < \gamma$. Since $\eta, \gamma > 0$ is arbitrary and $\Pr\{\hat{r}_{ie} < r_{ie}\} \to 0$ and $\Pr\left\{\hat{r}_{if} > r_{if}\right\} \to 0$, we deduce that $\Pr\{|\hat{r}_{ie} - r_{ie}| > \eta\} \to 0$ and $\Pr\left\{|\hat{r}_{if} - r_{if}| > \gamma\right\} \to 0$ as $T \to \infty$, provided that

$$\frac{1}{scv^{\beta_T}} + \frac{scv^{\beta_T}}{T^{1-\alpha/2}} \to 0.$$

Therefore, the date-stamping strategy based on the backward sup ADF test can consistently estimate r_{1e}, r_{1f}, r_{2e}, and r_{2f}. This proves Theorem 8.

The sequential PWY procedure. The origination of the bubble expansion r_{1e}, r_{2e} and the termination of the bubble collapse r_{1f}, r_{2f} based on the sequential DF test are identified as

$$\hat{r}_{1e} = \inf_{r\in[r_0,1]}\left\{r : DF_r > cv^{\beta_T}\right\} \text{ and } \hat{r}_{1f} = \inf_{r\in[\hat{r}_{1e}+L_T,1]}\left\{r : DF_r < cv^{\beta_T}\right\},$$
$$\hat{r}_{2e} = \inf_{r\in(\hat{r}_{1f}+r_0,1]}\left\{r :_{\hat{r}_{1f}} DF_r > cv^{\beta_T}\right\} \text{ and } \hat{r}_{2f} = \inf_{r\in[\hat{r}_{2e}+L_T,1]}\left\{r :_{\hat{r}_{1f}} DF_r < cv^{\beta_T}\right\},$$

where ${}_{\hat{r}_{1f}}DF_r$ is the DF statistic calculated over $(\hat{r}_{1f}, r]$. We know that when $\beta_T \to 0, cv^{\beta_T} \to \infty$.

The starting point of the regression r_1 takes value of zero for the DF_r statistic and the regression window $r_w = r$. The asymptotic distributions of the DF_r statistic under the alternative hypothesis are

$$DF_r \sim_a \begin{cases} F_r(W) & \text{if } r \in N_0 \\ T^{1-\alpha/2}\frac{r^{3/2}}{\sqrt{2(r_{1e}-r_1)}} & \text{if } r \in B_1 \\ -T^{(1-\alpha)/2}\left(\frac{1}{2}cr\right)^{1/2} & \text{if } r \in N_1 \end{cases}$$

It is obvious that if $r \in N_0$,

$$\lim_{T\to\infty} \Pr\left\{DF_r > cv^{\beta_T}\right\} = \Pr\left\{F_{r_2}(W) = \infty\right\} = 0.$$

If $r \in B_1$, $\lim_{T\to\infty}\Pr\left\{DF_r > cv^{\beta_T}\right\} = 1$ provided that $\frac{cv^{\beta_T}}{T^{1-\alpha/2}} \to 0$. If $r \in N_1$, $\lim_{T\to\infty}\Pr\left\{DF_r < cv^{\beta_T}\right\} = 1$. It follows that for any $\eta, \gamma > 0$,

$$\Pr\{\hat{r}_{1e} > r_{1e} + \eta\} \to 0 \text{ and } \Pr\left\{\hat{r}_{1f} < r_{1f} - \gamma\right\} \to 0,$$

since $\Pr\left\{DF_{r_{1e}+a_\eta} > cv^{\beta_T}\right\} \to 1$ for all $0 < a_\eta < \eta$ and $\Pr\left\{DF_{r_{1f}-a_\gamma} > cv^{\beta_T}\right\} \to 1$ for all $0 < a_\gamma < \gamma$. Since $\eta, \gamma > 0$ is arbitrary and $\Pr\{\hat{r}_{1e} < r_{1e}\} \to 0$ and $\Pr\left\{\hat{r}_{1f} > r_{1f}\right\} \to 0$, we deduce that $\Pr\{|\hat{r}_{1e} - r_{1e}| > \eta\} \to 0$ and $\Pr\left\{|\hat{r}_{1f} - r_{1f}| > \gamma\right\} \to 0$ as $T \to \infty$, provided that

$$\frac{1}{cv^{\beta_T}} + \frac{cv^{\beta_T}}{T^{1-\alpha/2}} \to 0.$$

Thus, this date-stamping strategy consistently estimates r_{1e} and r_{1f}.

For the ${}_{\hat{r}_{1f}}DF_r$ statistic, the starting point of the regression equals $\hat{r}_{1f}$ so that the regression window is $r_w = r - \hat{r}_{1f}$. Given that $\hat{r}_{1f} \xrightarrow{p} r_{1f}$, in the limit we have r_w equal to $r - r_{1f}$. The asymptotic distributions of the ${}_{\hat{r}_{1f}}DF_r$ statistic under the alternative hypothesis are

$$
{}_{\hat{r}_{1f}}DF_r \sim_a \begin{cases} F_r(W) & \text{if } r \in N_1 \\ T^{1-\alpha/2} \frac{(r-r_{1f})^{3/2}}{\sqrt{2(r_{2e}-r_1)}} & \text{if } r \in B_2 \\ -T^{(1-\alpha)/2} \left[\frac{1}{2}c(r-r_{1f})\right]^{1/2} & \text{if } r \in N_2 \end{cases} .
$$

If $r \in N_1$, $\lim_{T\to\infty} \Pr\{{}_{\hat{r}_{1f}}DF_r > cv^{\beta_T}\} = \Pr\{F_r(W) = \infty\} = 0$. If $r \in B_2$, $\lim_{T\to\infty} \Pr\{{}_{\hat{r}_{1f}}DF_r > cv^{\beta_T}\} = 1$ provided that $\frac{cv^{\beta_T}}{T^{1-\alpha/2}} \to 0$. If $r \in N_2$, $\lim_{T\to\infty} \Pr\left\{{}_{\hat{r}_{1f}}DF_r < cv^{\beta_T}\right\} = 1$.

For any $\phi, \kappa > 0$,

$$
\Pr\{\hat{r}_{2e} > r_{2e} + \phi\} \to 0 \text{ and } \Pr\left\{\hat{r}_{2f} < r_{2f} - \kappa\right\} \to 0,
$$

since $\Pr\left\{{}_{\hat{r}_{1f}}DF_{r_{2e}+a_\phi} > cv^{\beta_T}\right\} \to 1$ for all $0 < a_\phi < \phi$ and $\Pr\left\{{}_{\hat{r}_{1f}}DF_{r_{2f}-a_\kappa} > cv^{\beta_T}\right\} \to 1$ for all $0 < a_\kappa < \kappa$. Since $\phi, \kappa > 0$ is arbitrary and $\Pr\left\{r_{1f} < \hat{r}_{2e} < r_{2e}\right\} \to 0$ and $\Pr\left\{\hat{r}_{2f} > r_{2f}\right\} \to 0$, we deduce that $\Pr\{|\hat{r}_{2e} - r_{2e}| > \eta\} \to 0$ and $\Pr\left\{|\hat{r}_{2f} - r_{2f}| > \gamma\right\} \to 0$ as $T \to \infty$, provided that

$$
\frac{1}{cv^{\beta_T}} + \frac{cv^{\beta_T}}{T^{1-\alpha/2}} \to 0.
$$

Therefore, the alternative sequential implementation of the PWY procedure consistently estimates r_{2e} and r_{2f}. This proves Theorem 9.

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