FOLKLORE THEOREMS, IMPLICIT MAPS, AND INDIRECT INFERENCE
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FOLKLORE THEOREMS, IMPLICIT MAPS, AND INDIRECT INFERENCE

BY PETER C. B. PHILLIPS[1]

The delta method and continuous mapping theorem are among the most extensively used tools in asymptotic derivations in econometrics. Extensions of these methods are provided for sequences of functions that are commonly encountered in applications and where the usual methods sometimes fail. Important examples of failure arise in the use of simulation-based estimation methods such as indirect inference. The paper explores the application of these methods to the indirect inference estimator (IIE) in first order autoregressive estimation. The IIE uses a binding function that is sample size dependent. Its limit theory relies on a sequence-based delta method in the stationary case and a sequence-based implicit continuous mapping theorem in unit root and local to unity cases. The new limit theory shows that the IIE achieves much more than (partial) bias correction. It changes the limit theory of the maximum likelihood estimator (MLE) when the autoregressive coefficient is in the locality of unity, reducing the bias and the variance of the MLE without affecting the limit theory of the MLE in the stationary case. Thus, in spite of the fact that the IIE is a continuously differentiable function of the MLE, the limit distribution of the IIE is not simply a scale multiple of the MLE, but depends implicitly on the full binding function mapping. The unit root case therefore represents an important example of the failure of the delta method and shows the need for an implicit mapping extension of the continuous mapping theorem.

KEYWORDS: Binding function, delta method, exact bias, implicit continuous maps, indirect inference, maximum likelihood.

1. INTRODUCTION

ONE OF THE FOLKLORE THEOREMS of statistics is the delta method, a rigorous treatment of which first appeared in Cramér's (1946) treatise, although the history of the method is certainly much more distant. Its use in asymptotic derivations in econometrics is now almost universal. Equally important in econometric asymptotics, especially since the uptake of function space limit theory in the 1980s, is the continuous mapping theorem, the history of which also stretches into antiquity, an early source being the Mann and Wald (1943) article on stochastic order notation.

While these methods appear almost everywhere in econometrics, there are some cases where the methods do not apply directly. Particularly important examples arise when a problem involves sample functions that depend on the sample size or when the quantity of interest appears in an implicit functional form. In some cases, the methods fail, but with some modification can be made

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to work. In other cases, a new theorem is required to obtain the limit theory. There appears to be no systematic discussion of these issues in the literature, although there is some discussion in the statistical literature of extensions to the continuous mapping theorem for sequences of functions.

The substantive motivation for the present work was to find the limit distribution of the indirect inference estimator (IIE) in a simple first order autoregression. The indirect inference procedure subsumes many simulation-based methods in econometrics, as discussed by Gouriéroux, Monfort, and Renault (1993), such as simulated method of moments, models with latent variables or measurement errors, mean unbiased estimation, median unbiased estimation, and problems involving dynamic optimization and real business cycle modeling in macroeconomics, where they were first systematically applied by Smith (1993). The IIE is particularly effective in bias correction, which can be a major problem in autoregression, so the method is of considerable interest in that context. There are also manifestations of this problem and indirect inference alternatives in continuous time finance in diffusion equation estimation. In that context, Phillips and Yu (2009) used indirect inference to price contingent claims in derivative markets, and showed that this method removes bias and often reduces the variance of option price estimates that are based on maximum likelihood.

Investigation of the autoregressive model implementation of indirect inference reveals that a modified version of the delta method gives the correct solutions in stationary and explosive cases, but that the method fails in unit root and near unit root cases. Since the latter cases are most important in practical work, the failure has major implications. The explanation for the failure lies partly in the sample size dependence of the functional that defines the indirect inference estimator, partly in the implicit functional form that the estimator takes, and partly in the breakdown of linear approximation. All these issues need to be confronted so as to obtain the correct limit theory.

The problem is of wider significance because of the growing use of simulation-based methods in the construction of extremum estimators in econometrics. Particularly when a new procedure depends on the sampling distribution (moments or quantiles of another estimator, as in the case of mean or median unbiased estimation), the new limit distribution may be fundamentally affected by the properties of the implied distributional transformation, much as its finite sample distribution is affected. It is, in effect, only when the transformation is locally linear in a suitably sized shrinking neighborhood that the limit distribution follows straightforwardly from usual rules such as the delta method.

The present paper introduces these issues, provides some discussion and limit results that extend the delta method and continuous mapping theorem, and applies the ideas in the context of indirect inference (II) limit theory for the first order autoregression. It is shown that the IIE has a new form of limit distribution when the autoregressive coefficient is in the locality of unity. Interestingly, the IIE removes bias in the maximum likelihood estimator and changes its limiting distributional shape in a way that reduces variance. This is an instance where the delta method completely fails in the region of unity, but a suitably extended version of the delta method applies in the stationary case.

The paper is organized as follows. Section 2 provides some new mapping theorems that extend the usual delta method to sequences of functions and the continuous mapping theorem to sequences of implicit mappings. Both results are useful in considering simulation-based estimation procedures where sample-based functionals appear in extremum estimation problems. Section 3 describes the indirect inference approach, and Section 4 analyzes the use of this method in a first order autoregression, gives the analytic form of the binding function, and gives asymptotic expansion formulae for stationary, near unit root, and explosive cases. Section 5 gives the limit distribution of the indirect inference estimator, applying an extended delta method in the stationary cases and an implicit continuous mapping theorem in the unit root and local to unity cases, showing that for these parameters, the limit theory is a nonlinear functional of the standard unit root and near unit root asymptotics. Section 6 concludes and discusses various extensions. Many details, derivations, and proofs, including some new integral asymptotic expansions, are available in the original version of the paper (Phillips (2010)) and in the Supplemental Material (Phillips (2012)).

2. MAPPING THEOREMS

2.1. *Extending the Delta Method to Sequences of Functions*

While the ideas underlying the delta method have a long history, it seems that the original rigorous development was presented by Cramér (1946). Cramér's discussion included moments (p. 353), the limit distribution (p. 366), the multivariate case (p. 358), and more notably, because it is seldom referenced, the case where the leading term fails because of a zero first derivative and the variance is of smaller order than $O(n^{-1})$, leading to possibly nonnormal limit theory and a higher rate of convergence. Some related failures of standard methods of expansion and linearization were considered by Sargan (1983). Simple examples of such cases are sometimes mentioned in texts on asymptotic statistical theory, for instance, van der Vaart (2000). Functional versions of the delta method are also commonly used in semiparametric and nonparametric applications.

For the purposes of this paper, it is sufficient to work in the finite dimensional case. To fix ideas, we use the framework of van der Vaart (2000, Chapter 3). Let T_n be a random sequence in $\mathbb{R}^m$ for which $d_n(T_n - \theta) \Rightarrow T$ as $n \to \infty$ for some numerical sequence $d_n \to \infty$. In the usual case, $d_n = \sqrt{n}$ and T is Gaussian. Let $\varphi : \mathbb{R}^m \to \mathbb{R}^p$ be a map that is continuously differentiable at θ with derivative matrix φ'_θ. Then

$$(1) \qquad d_n(\varphi(T_n) - \varphi(\theta)) \Rightarrow \varphi'_\theta T.$$

In effect, $d_n(\varphi(T_n) - \varphi(\theta))$ behaves asymptotically as $n \to \infty$ like the linear functional $\varphi'_\theta T$, which van der Vaart wrote as the linear map $\varphi'_\theta(T)$. The validity of the result relies critically on the validity of a linear approximation at θ as $n \to \infty$. The same critical condition applies in the function space case. In simple applications, the matrix φ'_θ has full row rank and the distribution of T is nondegenerate in the sense that its support has positive Lebesgue measure in $\mathbb{R}^m$. Rank deficiencies lead to different rates of convergence and different limit results in the null subspaces. The limit results may be further complicated by the presence of different rates of convergence in the elements of T_n. These types of complications have been extensively studied in time series econometrics.

What happens when the function $\varphi = \varphi_n$ also depends on the sample size n? Some very important cases of this type arise in econometrics with the use of simulation-based estimators. In this case, an extended delta method for sequences seems well within reach. I could find no general reference in the statistical literature, but such results have almost certainly been used before in asymptotic arguments. A formal statement seems worthwhile, especially given its relevance to simulation-based estimator asymptotics in econometrics.

Consider the special case of a sequence of scalar functions φ_n of a single random sequence T_n. This case is sufficient for our purposes in the present work, but can be substantially generalized. If the functions φ_n are continuously differentiable and their derivatives φ'_n behave with regular variation in the vicinity of the limit θ (relative to the rate of convergence d_n of T_n), then we might expect some version of (1) with a rescaled rate of convergence to hold. The following result is verified by a direct mean value argument.

THEOREM 1: *Suppose φ_n has continuous derivatives φ'_n with $\varphi'_n(\theta) \neq 0$ for all n. Suppose also that the sequence $\{\varphi'_n\}$ is asymptotically locally relatively equicontinuous at θ in the sense that given $\delta > 0$, there exists a sequence $s_n \to \infty$ such that $\frac{s_n}{d_n} \to 0$, and for which, as $n \to \infty$,*

$$\sup_{|s_n(x-\theta)|<\delta} \left| \frac{\varphi'_n(x) - \varphi'_n(\theta)}{\varphi'_n(\theta)} \right| \to 0. \tag{2}$$

Then

$$\frac{d_n}{\varphi'_n(\theta)}(\varphi_n(T_n) - \varphi_n(\theta)) \Rightarrow T. \tag{3}$$

As the proof of Theorem 1 shows, the conditions effectively require that we may standardize and center the sequence of functions $\varphi_n(T_n)$ of T_n so that $\frac{d_n}{\varphi'_n(\theta)}(\varphi_n(T_n) - \varphi_n(\theta))$ is asymptotically linear in T_n in a wide enough neighborhood of θ. The multivariate case where φ_n and T_n are vectors is handled in a similar way. If $\Phi_n(\theta) = \partial\varphi_n(\theta)/\partial\theta'$ and F_n is a sequence of standardizing matrices for which $F_n \Phi_n(\theta) d_n^{-1}$ converges to a finite matrix F_φ and $\sup_{\|s_n(x-\theta)\|<\delta} \|F_n(\Phi_n(x) - \Phi_n(\theta)) d_n^{-1}\| \to 0$, where $\|s_n d_n^{-1}\| \to 0$, then $F_n(\varphi_n(T_n) - \varphi_n(\theta)) \Rightarrow F_\varphi T$. Here d_n is a matrix standardizing sequence for T_n for which (1) holds and $\|s_n(x-\theta)\| < \delta$ defines a shrinking neighborhood system of θ for some numerical matrix sequence $s_n \to \infty$. Under these conditions, we have the effective linearization

$$F_n(\varphi_n(T_n) - \varphi_n(\theta)) \sim F_n \Phi_n(\theta) d_n^{-1} d_n(T_n - \theta) \sim F_\varphi T$$

that produces the limit theory for $\varphi_n(T_n)$. In the scalar case, $F_n = d_n/\varphi_n'(\theta)$.

If the linearization condition fails, then we need to take further shape characteristics into account in determining the limit theory for $\varphi_n(T_n)$, just as in the usual delta method asymptotics. The original version of the paper (Phillips (2010)) gave several examples of this type of failure, showing that in some cases, a full power series expansion is required and that this is equivalent to the use of the continuous mapping theorem, leading to a limit that is a nonlinear function of T. The autoregression considered later provides a further illustration of this failure.

According to the definition of asymptotic local relative equicontinuity, the shrinking neighborhood system of θ may depend on φ_n so that (2) holds. In particular, the condition requires the existence of a shrinking neighborhood system

$$N_\delta^{s_n}(\theta) = \{x \in \mathbb{R} : |s_n(x-\theta)| < \delta, \delta > 0\} \tag{4}$$

for which (2) holds. So the rate of shrinkage around θ generally depends on the asymptotic behavior of the sequence of functions φ_n. The width of $N_\delta^{s_n}(\theta)$ is $O(s_n^{-1})$ and must be large enough to include the local region around θ of $O_p(d_n^{-1})$ which contains T_n, at least as $n \to \infty$, which is assured by the rate condition $s_n/d_n \to 0$.

The requirement (2) is stronger than the continuity of φ_n' at θ and different from equicontinuity of $\{\varphi_n'\}$ at θ, which requires a fixed rather than shrinking neighborhood of θ and ignores relative behavior. There is no requirement in the theorem that either $\varphi_n(x)$ or $\varphi_n'(x)$ converges. However, as a consequence of (2), the ratio $\frac{\varphi_n'(x)-\varphi_n'(\theta)}{\varphi_n'(\theta)}$ converges to zero uniformly in a local shrinking neighborhood of θ. Notably, the rate of convergence of $\varphi_n(T_n) - \varphi_n(\theta)$ is modified by the nonrandom sequence $\varphi_n'(\theta)$. When $\varphi_n = \varphi$ for all n, the limit result reduces to the usual delta method formula where $\varphi_n'(\theta) = \varphi'(\theta)$ is a simple slope coefficient. In the general case, the role of $\varphi_n'(\theta)$ changes to that of a slope coefficient combined with a rate of convergence adjustment that takes into account the dependence of the sequence φ_n on n. Just as the usual delta method requires a nondegenerate slope, Theorem 1 also requires that $\varphi_n'(\theta) \neq 0$, at least for large enough n. If this condition does not hold, then a higher order version of the result (3) may hold, but this case is not needed or pursued here.

2.2. *Extending the Continuous Mapping Theorem to Implicit Maps*

If X_n is a random sequence for which $X_n \Rightarrow X$ on a certain probability space and if g is a measurable mapping on that space that is continuous except for a set D_g for which the limit measure $P(X \in D_g) = 0$, then $Y_n = g(X_n) \Rightarrow g(X)$. There are well known extensions of this theorem that hold for sequences of functions g_n for which $g_n(X_n) \Rightarrow g(X)$. The result is to be expected if g_n converges uniformly to g. Topsoe (1967) gave a simple and powerful result due to Rubin (undated) according to which if the set $E = \{x : g_n(x_n) \to g(x)\ \forall x_n \to x\}$ has probability 1 under the limit measure P, then $X_n \Rightarrow X$ implies $g_n(X_n) \Rightarrow g(X)$. See Billingsley (1968) and van der Vaart and Wellner (2000, Theorem 1.11.1) for a precise statement, related results, and some discussion. The Rubin condition corresponds to a form of asymptotic equicontinuity of $\{g_n\}$ almost everywhere under the limit measure; see van der Vaart and Wellner (2000). For probability measures on $\mathbb{R}$, if $E = \mathbb{R}$ and the functions g_n are continuous and converge uniformly to g, then $g_n(x_n) \to g(x)\ \forall x_n \to x$ and $P(E) = 1$, so Rubin's condition is assured by uniform convergence on compact sets of $\mathbb{R}$.

In some problems, the limit distribution of a random sequence is determined inversely by sequences of equations of the form

$$X_n = f_n(Y_n) \tag{5}$$

or determined implicitly by sequences of functions such as

$$h_n(X_n, Y_n) = 0. \tag{6}$$

To my knowledge, there are no limit results for such implicitly defined sequences in the literature, although implicit transformations are used in the literature (e.g., Stock and Watson (1998)). However, given the Rubin–Topsoe result, a limit theory follows provided a sequence of globally unique inverse functions exists for (5) and a corresponding sequence of globally unique implicit functions exists for (6), and these sequences are asymptotically equicontinuous in the Rubin sense.

Conditions for global inverse and global implicit functions have been determined in the mathematics literature since Hadamard (1906). Global results are known for quite general functions on normed spaces (see, for example, Cristea (2007)). A variety of conditions can be used to ensure univalence, including monotonicity and P matrix conditions on the Jacobian (see Parthasarathy (1983), for a review of results up to the early 1980s). For the present purposes in this paper, it is sufficient to employ results for the real line, where monotonicity is a sufficient condition. The following result uses a one dimensional global implicit function theorem (Ge and Wang (2002)) and often is convenient in econometric applications. It has been extended by Zhang and Ge (2006) using a Gerschgorin bound condition on the Jacobian to give a global implicit function theorem for mappings in Euclidean spaces of arbitrary dimension.

LEMMA 2: *Assume* $h:\mathbb{R}^{m+1}\to\mathbb{R}$ *is continuously differentiable and there exists a constant* $d>0$ *such that* $|\frac{\partial}{\partial y}h(x,y))|>d$ *for all* $(x,y)\in\mathbb{R}^m\times\mathbb{R}$. *Then there exists a unique continuously differentiable function* $g:\mathbb{R}^m\to\mathbb{R}$ *such that* $h(x,g(x))=0$.

For such an implicit function h with unique solution $y=g(x)$, an implied continuous mapping theorem follows immediately, namely $Y_n=g(X_n)\Rightarrow g(X)$ whenever $X_n\Rightarrow X$. More generally, suppose we have a sequence of continuously differentiable implicit functions $h_n(x,y)$ which satisfy the monotonicity condition $|\frac{\partial}{\partial y}h_n(x,y)|>d_n$ for all $(x,y)\in\mathbb{R}^m\times\mathbb{R}$ and some $d_n>0$. Then there exists a corresponding sequence of unique continuously differentiable solution functions g_n. If these functions satisfy the Rubin asymptotic equicontinuity condition, then $X_n\Rightarrow X$ implies $g_n(X_n)\Rightarrow g(X)$. This type of limit theory is relevant when estimators are defined implicitly using a sequence of sample size dependent functions as are typical for indirect inference procedures.

3. INDIRECT INFERENCE ESTIMATION

The idea of indirect inference is to use simulated data to determine characteristics such as population moments and to map their dependence on underlying parameters of interest in a manner that is useful in econometric estimation and inference. Like the delta method, this idea has a long history.[2] Practical implementation became possible with advances in computational capability that enabled a sufficiently large number of data generations and replications of a statistical procedure to capture parameter dependencies well enough for them to be used to improve estimation and inference.

Typical uses are to estimate parameters indirectly via their dependence on other parameters, which may be easier to estimate, or to use the simulations indirectly for calibration purposes, for example, in measuring and partially correcting bias in estimation. Mean unbiased estimation and median unbiased estimation are both examples of indirect inference, as are many other procedures which depend on finite sample characteristics like moments or quantiles that are estimated by simulation methods and functionalized on the parameters. The method is particularly useful in cases where the true likelihood function is intractable but may be available for a related model involving an auxiliary set of parameters. Smith (1993) applied the method to a real business cycle model in macroeconomics.

To fix ideas, a parametric model is simulated to produce H synthetic data trajectories $\{\tilde{y}^h(\theta)\}_{h=1}^H$ for a given parametric value θ. The number of observations

[2]For instance, Durbin indicated early consideration of such possibilities in an *Econometric Theory* interview (Phillips (1988)) and Sargan (1976) mentioned ideas of Barnard related to the bootstrap, both in the 1950s.

in each trajectory $\tilde{y}^h(\theta)$ is chosen to be the same as the number of observations in the observed data set to ensure finite sample calibration accuracy. Suppose $Q_n(\beta; y)$ is an objective function constructed from the actual data (y) for the estimation of some pseudoparameter β by means of the extremum criterion

$$\hat{\beta}_n = \arg\min_{\beta} Q_n(\beta; y).$$

The corresponding estimator based on the hth simulated path for some given θ is

$$\tilde{\beta}_n^h(\theta) = \arg\min_{\beta} Q_n(\beta; \tilde{y}^h(\theta)).$$

Indirect inference estimation of the original parameter θ proceeds by way of calibrating θ to $\hat{\beta}_n$ (or some function of $\hat{\beta}_n$) according to an additional criterion of the form

$$\breve{\theta}_{n,H} = \arg\min_{\theta} \left\| \hat{\beta}_n - \frac{1}{H} \sum_{h=1}^{H} \tilde{\beta}_n^h(\theta) \right\| \tag{7}$$

for some metric $\|\cdot\|$. As $H \to \infty$, we anticipate that $H^{-1}\sum_{h=1}^{H} \tilde{\beta}_n^h(\theta) \to_p \mathrm{E}\tilde{\beta}_n^h(\theta) =: b_n(\theta)$. Since H can be made arbitrarily large in implementation, the procedure effectively amounts to calibrating $b_n(\theta)$, which is called the binding function, so that

$$\breve{\theta}_n = \arg\min_{\theta} \|\hat{\beta}_n - b_n(\theta)\|. \tag{8}$$

If $b_n(\theta)$ is invertible, then we have $\breve{\theta}_n = b_n^{-1}(\hat{\beta}_n) =: f_n(\hat{\beta}_n)$. The estimator $\hat{\theta}_n$ is therefore determined indirectly by way of the binding function $b_n(\theta)$ and the estimator $\hat{\beta}_n$. In some applications of indirect inference, such as the one considered in the next section of this paper, the pseudoparameter β corresponds to the original parameter θ and the procedure seeks to adjust the estimator according to some aspect of its sampling properties such as its mean as in (7).

Importantly, the dependence of the estimator $\breve{\theta}_n$ on the data is via $\hat{\beta}_n$ and the sequence of binding functions b_n. In general, b_n depends on the finite sample distribution of the data through the exact finite sample functional involved in the criterion. In the case above, the functional is the finite sample mean function $\mathrm{E}\tilde{\beta}_n^h(\theta)$, but it could be another characteristic of the distribution like the median or some other quantile with appropriate modification of (7). The implicit dependence of $\breve{\theta}_n$ on the sequence of functions b_n means that the asymptotic distribution of $\breve{\theta}_n$ cannot be deduced simply by the delta method. As shown above, it is necessary to take into account the properties of the sequence b_n in determining the rate of convergence and the limit theory. The remainder of this paper studies a special case that shows how the mapping sequence can play a critical role in shaping the limit theory.

4. FIRST ORDER AUTOREGRESSION

4.1. *Bias and Bias Correction*

Suppose we wish to estimate the parameter ρ in the simple autoregression

$$y_t = \rho y_{t-1} + u_t, \quad t = 1, \dots, n, \tag{9}$$

from observations $y = \{y_t\}_{t=0}^n$, where u_t is independent and identically distributed $N(0, \sigma^2)$. Various conditions may be placed on the initial value y_0, and these affect finite sample behavior and may also affect the limit theory when ρ is in the neighborhood of unity or in the explosive region (see Phillips and Magdalinos (2009), for a recent treatment and the references therein). Such initialization effects are not the concern of the present paper, so we simply assume that $y_0 = 0$. However, the indirect inference approach is easily adapted to take into account different initializations. Also, it is often convenient to focus on the case where $\rho > 0$, since analogous mirror image results hold for $\rho < 0$.

Standard estimation procedures such as maximum likelihood (ML) and least squares (LS) produce downward biased coefficient estimators of ρ in finite samples when $\rho > 0$. These biases can be corrected using indirect inference via synthetic samples and a criterion such as (8). Let $\hat{\rho}_n = \sum_{t=1}^n y_t y_{t-1} / \sum_{t=1}^n y_{t-1}^2$ be the ML estimate of ρ under Gaussianity, assuming the initialization y_0 is fixed. The assumption of Gaussianity (or some other distribution) enables the generation of synthetic samples following (9), on which the practical implementation of indirect inference depends. But the asymptotic theory of $\hat{\rho}_n$ applies much more generally under standard central limit theory and functional central limit theory when the autoregressive coefficient satisfies $|\rho| \leq 1$ or local to unity $\rho = 1 + \frac{c}{n}$ parameterizations (with c constant), in which case the distribution of u_t in (9) does not matter. In this event, indirect inference in the limit depends on the (robust) asymptotic theory for $\hat{\rho}_n$ and the nature of the (quasi) binding function implied by the Gaussian assumption used in synthetic sampling. The results in the present paper cover this situation, so that the asymptotic theory derived for the indirect inference estimator here applies equally under more general conditions, such as martingale difference errors, on u_t in (9). On the other hand, for fixed $|\rho| > 1$, invariance principles do not apply and the limit behavior of $\hat{\rho}_n$ (and hence the indirect inference estimator) is distribution dependent. It is the combination of the behavior of the (quasi maximum likelihood) estimator ($\hat{\rho}_n$) and the binding function (under specific distributional assumptions) that needs to be explored in deriving the correct limit theory. The present paper makes this connection analytically. Using simulations, Andrews (1993) showed that, in the case of median unbiased estimation, the finite sample performance of the procedure was robust to substantial deviation from Gaussianity.

Among others, White (1961), Shenton and Johnson (1965; hereafter SJ), and Shenton and Vinod (1995; hereafter SV) gave asymptotic expansions of the bias in terms of powers of n^{-1} as $n \to \infty$. Different expansions were obtained for the case $|\rho| < 1$ and the case $\rho = 1$. This research has a bearing on the estimation of continuous time models from discrete data, where similar problems of estimation bias for the mean reversion parameter arise but can be more severe (Tang and Chen (2009)). This bias is particularly important because of its implications for derivative pricing in finance (Phillips and Yu (2005, 2009)).

The next section gives comprehensive bias expressions for $\hat{\rho}_n$ and asymptotic representations that cover stationary, unit root, and explosive ρ. The development is needed because the asymptotic formulae required to characterize the limit theory of the indirect inference estimator of ρ rely on full analytic specification of the binding function over all potential values of ρ. Figure 1 shows the bias function $\mathrm{E}(\hat{\rho}_n|\rho) - \rho$ of the ML estimate of ρ in the Gaussian model (9) for various sample sizes n. The downward bias for $\rho > 0$ is evidently greatest near unity and rapidly diminishes as ρ exceeds unity. The bias function is clearly highly nonlinear, as noted by MacKinnon and Smith (1998), who performed simulations in the case of an (autoregressive) AR(1) model with a fitted intercept. Most importantly, it has a rapidly changing derivative in the vicinity of unity.

The indirect inference method for fitting ρ takes this bias function into account and was explored in Gouriéroux, Renault, and Touzi (2000), Gouriéroux, Monfort, and Renault (2010) by Monte Carlo methods. As explained above, the approach uses simulations to calibrate the bias function and requires neither an explicit form of the bias nor a bias expansion formula. The simulation

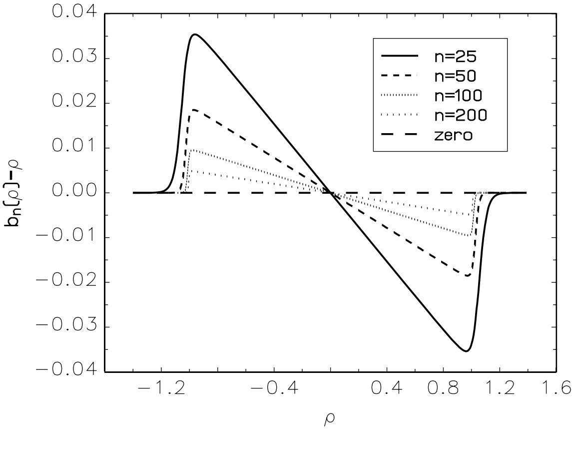

FIGURE 1.—The exact bias function $b_n(\rho) - \rho = \mathrm{E}(\hat{\rho}_n|\rho) - \rho$ of the ML estimator $\hat{\rho}_n$ for various n based on (16) and (17).

results reported in Gouriéroux, Renault, and Touzi (2000) showed that the indirect inference method works as well as the median unbiased estimator of Andrews (1993) when $H = 15{,}000$ and the calibration estimator is the maximum likelihood estimator (MLE). Both methods are dependent on the validity of the assumed data distribution for correct calibration through the finite sample binding formula, although some robustness in this calibration may be expected, as discussed above.

As in (8), when the number of replications $H \to \infty$, the indirect inference estimator of ρ satisfies

$$\breve{\rho}_n = \arg\min_{\rho} \left|\hat{\rho}_n - E(\tilde{\rho}_n^h(\rho))\right| = \arg\min_{\rho} |\hat{\rho}_n - b_n(\rho)|, \tag{10}$$

where $b_n(\rho) = E(\tilde{\rho}_n^h(\rho))$ is the binding function for the MLE $\hat{\rho}_n$. When b_n is invertible,

$$\breve{\rho}_n = b_n^{-1}(\hat{\rho}_n) := f_n(\hat{\rho}_n). \tag{11}$$

From SJ, the binding function is known to have the asymptotic expansion, as $n \to \infty$,

$$b_n(\rho) = \begin{cases} \rho - \dfrac{2\rho}{n} + O(n^{-2}), & |\rho| < 1, \\ \rho - \dfrac{1.7814}{n} + O(n^{-2}), & \rho = 1, \end{cases} \tag{12}$$

which is evidently discontinuous at $\rho = 1$. The numerical value -1.7814 is the mean of the limit distribution of $n(\hat{\rho}_n - 1)$ and bias persists in the limit when $\rho = 1$. The discontinuity in (12) reflects the discontinuity in the asymptotic distribution theory around unity and manifests this deeper issue in the asymptotics. In contrast, the binding function $b_n(\rho)$ itself is continuous and indeed continuously differentiable for all n.

Figure 2 shows the binding function for $n = 5000$ in a narrow band around unity to indicate the behavior of the function in this vicinity for very large values of n. The function is below the 45° line for all ρ with a slope that is less than unity for stationary ρ, but that increases and exceeds unity for ρ around unity while rapidly returning to virtually coincide with the 45° line for explosive ρ. To accomplish this smooth transition, the derivative of the binding function is below unity for $\rho < 1$, virtually unity for $\rho > 1$, and greater than unity in the immediate vicinity of $\rho = 1$. As is apparent from Figure 2, the binding function $b_n(\rho)$ is monotone and invertible. An analytic proof of this property for large n is given in the Supplemental Material, so that $\breve{\rho}_n$ is well defined by (11). Figure 1 shows that the bias function $b_n(\rho) - \rho$ has a derivative that quickly changes sign in the neighborhood of unity. So a linear approximation to $b_n(\rho)$ is completely inadequate around unity, even for very large n.

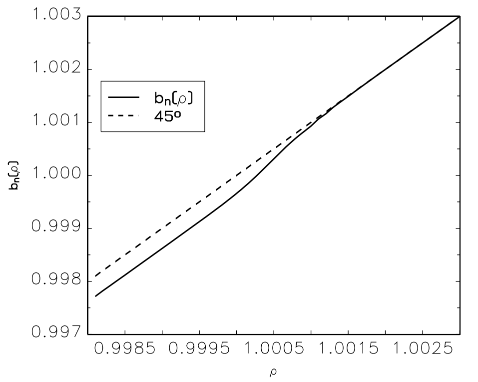

FIGURE 2.—The binding function $b_n(\rho)$ of the MLE $\hat{\rho}_n$ around unity for $n = 5000$.

Since the inverse binding function f_n is continuously differentiable with a nonzero first derivative at ρ, routine application of the delta method that ignores the sample size dependence of f_n suggests that

$$d_n(\breve{\rho}_n - \rho) \sim f_n'(\rho) d_n(\hat{\rho}_n - \rho). \tag{13}$$

When $|\rho| < 1$, we have $d_n = \sqrt{n}$ and, by standard theory, $\sqrt{n}(\hat{\rho}_n - \rho) \Rightarrow N(0, 1 - \rho^2)$. In this case, as shown in the following section, it turns out that we also have $\sqrt{n}(\breve{\rho}_n - \rho) \Rightarrow N(0, 1 - \rho^2)$ by virtue of Theorem 1. The main effect of indirect inference in the stationary AR(1) case therefore is to provide a finite sample bias correction to the estimator, while the asymptotic distribution of $\breve{\rho}_n$ is identical to the MLE.

However, when ρ is in the local vicinity of unity as $n \to \infty$, the linear representation (13) breaks down and it is necessary to take into account the precise features of the binding function $b_n(\rho)$ around unity to determine the correct limit theory. The asymptotics are complex and require much more detailed asymptotic representations of $b_n(\rho)$. These are given in the following sections.

4.2. *The Binding Function Formula*

The following theorem describes the binding function for the regions $|\rho| \le 1$ and $|\rho| > 1$.

THEOREM 3: *For model* (9) *the binding function* $b_n(\rho) = E(\hat{\rho}_n)$ *for the ML estimator* $\hat{\rho}_n$ *is given by*

$$b_n(\rho) = \begin{cases} \rho + \dfrac{1}{2}\dfrac{\partial}{\partial\rho}\left\{\displaystyle\int_0^1 x^{(n-5)/2}(1-\rho^2x^2)^{3/2}F_n^{-1/2}\,\mathrm{d}x\right\}, & |\rho| \le 1, \\ \rho + \dfrac{1}{2}\dfrac{\partial}{\partial\rho}\left\{\displaystyle\int_1^\infty x^{(n-5)/2}(\rho^2x^2-1)^{3/2}G_n^{-1/2}\,\mathrm{d}x\right\}, & |\rho| > 1, \end{cases} \tag{14}$$

where

$$F_n = F_n(x;\rho) = 1 - \rho^2 x + (1-x)x^{2n-1}\rho^{2n},$$
$$G_n = G_n(x;\rho) = \rho^2 x - 1 + (x-1)x^{2n-1}\rho^{2n}.$$

REMARKS:

(i) The proof of Theorem 3 follows SJ and SV in using results for ratios of quadratic forms in normal variates. SV developed the integral representation (14) for the case $|\rho| \le 1$. The present result extends that work to the explosive case and provides explicit representations of the bias for $|\rho| \le 1$ and $|\rho| > 1$. These representations are used to develop a complete set of asymptotic expansions which facilitate the development of the limit theory for the indirect inference estimator.

(ii) Explicit formulae for (14) are derived in the proof of Theorem 3, which is given in the Supplemental Material. We find that

$$b_n(\rho) = b_n(\rho; |\breve{\rho}| \le 1) + b_n(\rho; |\rho| > 1), \tag{15}$$

where, for $|\rho| \le 1$,

$$\begin{aligned} b_n(\rho; |\rho| \le 1) = \rho &- \frac{3\rho}{2}\int_0^1 x^{(n-1)/2}(1-\rho^2x^2)^{1/2}F_n^{-1/2}\,\mathrm{d}x \\ &+ \frac{\rho}{2}\int_0^1 x^{(n-3)/2}(1-\rho^2x^2)^{3/2}F_n^{-3/2}\,\mathrm{d}x \\ &- \frac{n\rho^{2n-1}}{2}\int_0^1 x^{(5n-7)/2}(1-\rho^2x^2)^{3/2}F_n^{-3/2}(1-x)\,\mathrm{d}x, \end{aligned} \tag{16}$$

and, for $|\rho| > 1$,

$$\begin{aligned} b_n(\rho; |\rho| > 1) = \rho &+ \frac{3\rho}{2}\int_1^\infty x^{(n-1)/2}(\rho^2x^2-1)^{1/2}G_n^{-1/2}\,\mathrm{d}x \\ &- \frac{\rho}{2}\int_1^\infty x^{(n-3)/2}(\rho^2x^2-1)^{3/2}G_n^{-3/2}\,\mathrm{d}x \\ &- \frac{n\rho^{2n-1}}{2}\int_1^\infty x^{(5n-7)/2}(\rho^2x^2-1)^{3/2}G_n^{-3/2}(x-1)\,\mathrm{d}x. \end{aligned} \tag{17}$$

These bias expressions are continuous through $\rho = 1$, as shown in Figure 2. Importantly, (16) and (17) have three integral components, the last of which involves the multiplicative factor ρ^{2n-1}. The component factor ρ^{2n} also appears in both F_n and G_n. The presence of this factor plays an important role in determining the bias (and the form of the bias expansion as $n \to \infty$) when ρ is in the vicinity of unity. In particular, when n becomes large, components that involve

ρ^{2n} are negligible when $|\rho| < 1$, relevant when ρ is in the vicinity of unity, and dominant when $\rho > 1$. Thus, while the finite sample expressions are continuous in ρ, asymptotic approximations are discontinuous as ρ passes through unity, because different components of (16) and (17) figure in the approximations, depending on the value of ρ.

4.3. *Asymptotic Bias Expansions*

The discontinuities in asymptotic expansions of the bias function (12) reflect fundamental differences in the limit theory as $n \to \infty$. As indicated above, the technical reason for the discontinuities is the presence of terms that involve ρ^{2n} in $b_n(\rho)$, which behave differently depending on whether $|\rho| < 1$, $|\rho| > 1$, or $\rho = 1 + c/n$. To provide a comprehensive analysis, we consider each of these domains separately. The following result summarizes the main cases of interest.

THEOREM 4: *For fixed* ρ,

$$b_n(\rho) = \begin{cases} \rho - \frac{2\rho}{n} + O(n^{-2}), & |\rho| < 1, \\ \pm 1 \mp \dfrac{1.7814}{n} + O(n^{-2}), & \rho = \pm 1, \\ \rho + O(|\rho|^{-n}), & |\rho| > 1. \end{cases} \tag{18}$$

For $\rho = 1 + c/n$ *with* $c < 0$ *and* $|\rho| < 1$,

$$b_n\left(1 + \frac{c}{n}\right) = 1 + \frac{c}{n} + \frac{1}{n} g^-(c)\left\{1 + O\left(\frac{c}{n}\right)\right\}, \tag{19}$$

where

$$g^-(c) = -\frac{3}{4}\int_0^1 y^{-3/4}\ell(y, c)^{-1/2}\, dy + \frac{1}{4}\int_0^1 y^{-3/4}\ell(y, c)^{-3/2}\, dy + \frac{e^{2c}}{8}\int_0^1 y^{1/4}\ell(y, c)^{-3/2}\log y\, dy.$$

For $\rho = 1 + c/n$ *with* $c > 0$,

$$b_n\left(1 + \frac{c}{n}\right) = 1 + \frac{c}{n} + \frac{1}{n} g^+(c)\left\{1 + O\left(\frac{c}{n}\right)\right\}, \tag{20}$$

where

$$g^+(c) = \frac{3}{4}\int_0^\infty e^{(1/4)w} k^+(w; c)^{1/2}\, dw - \frac{1}{4}\int_0^\infty e^{(1/4)w} k^+(w; c)^{3/2}\, dw - \frac{e^{2c}}{8}\int_0^\infty e^{(5/4)w} k^+(w; c)^{3/2} w\, dw. \tag{21}$$

In these formulae,

$$\ell(y,c) := \frac{4c + \log y + ye^{2c}\log y}{4c + 2\log y} \quad \textit{and} \tag{22}$$

$$k^{+}(w;c) := \frac{4c + 2w}{4c + w + e^{2c}we^{w}}.$$

Formulae (19) *and* (20) *provide valid approximations to the binding function for* $\rho = 1 + \frac{c}{n}$ *when* $c \to \mp\infty$, *respectively, and* $c = o(n)$ *as* $n \to \infty$. *The binding function* $b_n(1 + \frac{c}{n})$ *is approximately linear as* $n \to \infty$ *when* $|c| \to \infty$ *with* $c = o(n)$. *In particular*

$$b_n(\rho) = \begin{cases} \rho - \dfrac{2\rho}{n}\left\{1 + O\left(\dfrac{c}{n}\right)\right\}, & \textit{for } c \to -\infty, \\ \rho + O\left(\dfrac{1}{ne^{c}}\right), & \textit{for } c \to \infty. \end{cases} \tag{23}$$

REMARKS:

(i) The results for $|\rho| < 1$ and $\rho = \pm 1$ are well known. The result for $|\rho| > 1$ appears to be new, as are the results for the local to unity cases. The latter results are particularly useful in deriving the limit distribution of the indirect inference estimator. The distinction between $c < 0$ and $c > 0$ arises because of the formulation of the binding function $b_n(\rho)$ in these two cases and the manner in which the asymptotic expansions are obtained.

(ii) When $c \nearrow 0$ and $c \searrow 0$, the bias function expansions (19) and (20) converge to the same value, so this local formulation, just like the function b_n, is continuous. In particular, $g^{+}(0) = g^{-}(0) = -1.7814$ and

$$\lim_{c \nearrow 0} b_n\left(1 + \frac{c}{n}\right) = \lim_{c \searrow 0} b_n\left(1 + \frac{c}{n}\right) = 1 - \frac{1.7814}{n} + O(n^{-2}). \tag{24}$$

(iii) As indicated in (23), when $\rho = 1 + c/n$ and $|\rho| < 1$ with $c \searrow -\infty$, and $\frac{c}{n} \to 0$ as $n \to \infty$, we have

$$\begin{aligned} b_n(\rho) &= \rho - \left\{\frac{3\rho}{4n}\int_0^1 y^{-3/4}\,dy - \frac{\rho}{4n}\int_0^1 y^{-3/4}\right\}\left\{1 + O\left(\frac{c}{n}\right)\right\} \\ &= \rho - \frac{2\rho}{n}\{1 + o(1)\}, \end{aligned}$$

corresponding to the case of fixed $|\rho| < 1$. When $\rho = 1 + c/n$ with $c > 0$ and $c \nearrow \infty$, we have

$$b_n(\rho) = \rho + O\left(\frac{1}{ne^{c}}\right),$$

corresponding to the fixed $\rho > 1$ case. Observe that in both cases, $b_n(\rho)$ is linear in $\rho = 1 + \frac{c}{n}$ at the limits of the domain of definition and

$$(25) \qquad g^-(c) \to -2, \quad \text{as} \quad c \to -\infty \quad \text{and} \quad g^+(c) \to 0, \quad \text{as} \quad c \to \infty.$$

Further discussion of the properties of $g^-(c)$, $g^+(c)$, and their derivatives is given in the Supplemental Material.

(iv) The following alternative form of the leading term in the binding function when $\rho = 1 + c/n$ and $c < 0$ is useful in both analytical and computational work and is established in the Supplemental Material:

$$(26) \qquad g^-(c) = -\frac{3}{4}\int_0^\infty e^{-(1/4)v} k^-(v; c)^{1/2}\, dv + \frac{1}{4}\int_0^\infty e^{-(1/4)v} k^-(v; c)^{3/2}\, dv - \frac{e^{2c}}{8}\int_0^\infty e^{-(5/4)v} k^-(v; c)^{3/2} v\, dv,$$

where

$$(27) \qquad k^-(v; c) := \frac{4c - 2v}{4c - v - e^{2c} v e^{-v}} = \frac{2v - 4c}{v + e^{2c} v e^{-v} - 4c}.$$

(v) Observe that $k^-(v; c) > 0$ for all $c \leq 0$ over $v \in (0, \infty)$ and that $k^+(w; c) > 0$ for all $c > 0$ over $w \in (0, \infty)$. Hence, the integrands in (26) and (21) are real and well defined.

(vi) The function $g(c)$ is continuous at $c = 0$. In particular,

$$(28) \qquad g^+(0) - g^-(0) = \frac{2^{3/2}}{8}\int_0^\infty e^{-(5/4)w}(1 + e^{-w})^{-3/2}(4 + w + e^w(4 - w))\, dw = 0$$

by analytic evaluation of the integral as shown in the Supplemental Material.

5. INDIRECT INFERENCE LIMIT THEORY

The indirect inference estimator of ρ is defined implicitly in terms of the binding function of the ML estimator $\hat{\rho}_n$, so that $\hat{\rho}_n = b_n(\breve{\rho})$, where b_n is given by (16) and (17). We write this implicit relationship for $\breve{\rho}$ as

$$\hat{\rho}_n = b_n(\breve{\rho}) = b_n(\breve{\rho}; |\breve{\rho}| \leq 1) + b_n(\breve{\rho}; |\breve{\rho}| > 1).$$

To find the limit distribution of $\breve{\rho}$, we use asymptotic formulae for the binding function $b_n(\rho)$ and its derivatives. We consider the stationary and near unit root cases separately.

5.1. *Stationary Case*

When $|\rho| < 1$, the extended delta method of Theorem 1 is applicable. To show this, we consider the first derivative of the binding function. As is clear from (16), when $|\rho| < 1$, the final term in the binding function expression is $O(\rho^n)$. The first derivative of this function is of the same order and since it is dominated by the other terms, it can be neglected. The calculations for the remaining terms involve expanding the integrals as $n \to \infty$ (see the Supplemental Material for details), leading to the following result for $|\rho| < 1$:

$$b_n'(\rho) = 1 + O(n^{-1}). \tag{29}$$

It follows that given $|\rho| < 1$, for all $\delta > 0$ and any sequence $s_n \to \infty$ for which $s_n/n^{1/2} \to 0$, we have

$$\sup_{s_n|r-\rho|<\delta} \left| \frac{b_n'(\rho) - b_n'(r)}{b_n'(r)} \right| \to 0.$$

Writing $\breve{\rho} = b_n^{-1}(\hat{\rho}_n) = f_n(\hat{\rho}_n)$ and using the fact that $f_n'(r) = 1/b_n'(r)$ and

$$\frac{f_n'(r) - f_n'(\rho)}{f_n'(\rho)} = \frac{b_n'(\rho) - b_n'(r)}{b_n'(r)},$$

it follows that

$$\sup_{s_n|r-\rho|<\delta} \left| \frac{f_n'(r) - f_n'(\rho)}{f_n'(\rho)} \right| \to 0.$$

Hence, by Theorem 1,

$$\sqrt{n}(\breve{\rho} - \rho) \sim \frac{1}{b_n'(\rho)} \sqrt{n}(\hat{\rho}_n - \rho) \sim \sqrt{n}(\hat{\rho}_n - \rho) \Rightarrow N(0, 1 - \rho^2).$$

5.2. *Unit Root Case*

The unit root case is considerably more complex because of the implicit determination of $\breve{\rho}$ via the mapping $\hat{\rho}_n = b_n(\breve{\rho})$. No explicit functional form for the inverse mapping $\breve{\rho} = b_n^{-1}(\hat{\rho}_n)$ is available, although series expressions can be obtained using Lagrange inversion. Instead of an explicit inverse map, we can directly manipulate the expression to obtain an explicit relationship between the standardized and centered versions of the ML estimator $\xi_n^{\text{ml}} = n(\hat{\rho}_n - \rho)$ and the II estimator $\xi_n^{\text{ii}} = n(\breve{\rho} - \rho)$. The transformed mapping can be used to deduce the limit theory for ξ_n^{ii} via an implicit function version of the continuous mapping theorem. This approach is applicable when $\rho = 1$ and when $\rho = 1 + c/n$.

We start with the bias expressions for $\hat{\rho}_n$ in the near integrated case $\rho = 1 + c/n$. We need to allow for both $c \leq 0$ and $c > 0$, so we combine (19) with (20). Then, for $\rho = 1 + c/n$, we have

$$b_n(\rho) = \rho + \frac{1}{n} g(c) + R_n(c), \tag{30}$$

where $R_n(c) = O(n^{-2})$ uniformly for c in compact sets of $\mathbb{R}$ and $R_n(c) = O(\frac{c}{n^2})$ for $|c| \to \infty$ with $c = o(n)$ as $n \to \infty$. In (30),

$$g(c) = g^-(c) 1_{\{c \leq 0\}} + g^+(c) 1_{\{c > 0\}}, \tag{31}$$

where $g^-(c)$ and $g^+(c)$ are defined in (26) and (21). As shown earlier, when $c \to \pm 0$, we get $g^-(0) = g^+(0) = -1.7814$ and the function $g(c)$ is continuous through $c = 0$ (see the last remark above). In view of (25), $\lim_{c \to -\infty} g(c) = -2$ and $\lim_{c \to \infty} g(c) = 0$. The first derivative $g'(c)$ is also continuous and well defined at the limits of its domain of definition with $\lim_{|c| \to \infty} g'(c) = 0$, and the function $h(c) := c + g(c)$ is monotonic over $c \in (-\infty, \infty)$. The reader is referred to the Supplemental Material for further details.

The derivatives of the binding function $b_n(\rho)$ in the vicinity of unity have a different form from when $|\rho| < 1$. In particular, terms involving ρ^{2n} in (16) and (17) are no longer exponentially small. Calculations reveal that for $\rho = 1 + c/n$, the derivatives take the form

$$b_n^{(j)}(\rho) = \begin{cases} 1 + g'(n(\rho - 1)) + O(n^{-1}), & j = 1, \\ n^{j-1}\big(1 + g^{(j)}(n(\rho - 1))\big)\{1 + o(1)\}, & j > 1, \end{cases}$$

and therefore satisfy $b_n^{(j)}(1) = O(n^{j-1})$, so that second and higher derivatives are unbounded at $\rho = 1$ as $n \to \infty$. This corresponds to the rapidly changing form of the bias function $b_n(\rho) - \rho$ in the vicinity of unity that is evident in Figure 1. As a result, the extended delta method fails for ρ in the immediate vicinity of unity. Note, in particular, that for some intermediate value $\tilde{r}$ between r and ρ, we have

$$\sup_{s_n |r - \rho| < \delta} \left| \frac{b_n'(\rho) - b_n'(r)}{b_n'(r)} \right| = \sup_{s_n |r - \rho| < \delta} \left| \frac{b_n^{(2)}(\tilde{r})(\rho - r)}{b_n'(r)} \right| = O_p\left(n \times \frac{\delta}{s_n}\right),$$

which is divergent for all sequences $s_n \to \infty$ for which $s_n/n \to 0$. Hence the sequence b_n (and by implication b_n^{-1}) is not asymptotically locally relatively equicontinuous and Theorem 1 does not apply. One way to address this failure in the delta method is to attempt a full Taylor representation of $b_n(\rho)$ and Lagrange inversion, as in the examples discussed in Section 2 of Phillips (2010). However, a more direct approach turns out to be possible using the relation (30).

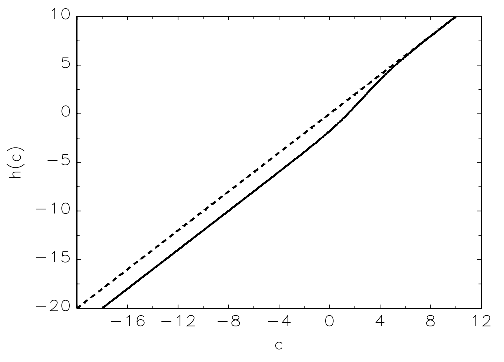

FIGURE 3.—The limit function h (solid line) in the implicit map $\xi^{\mathrm{ml}} = h(\xi^{\mathrm{ii}})$ relating the indirect inference limiting variate ξ^{ii} to the limiting ML variate $\xi^{\mathrm{ml}} = \int W\,dW/\int W^2$, shown against the 45° line (broken line).

In particular, since $\hat{\rho}_n = b_n(\breve{\rho})$, by direct substitution of the centered and scaled estimates $\xi_n^{\mathrm{ml}} = n(\hat{\rho}_n - \rho)$ and $\xi_n^{\mathrm{ii}} = n(\breve{\rho} - \rho)$ into (30), we have the relationship

$$\xi_n^{\mathrm{ml}} = \xi_n^{\mathrm{ii}} + g(\xi_n^{\mathrm{ii}}) + nR_n(\xi_n^{\mathrm{ii}}) = h(\xi_n^{\mathrm{ii}}) + nR_n(\xi_n^{\mathrm{ii}}), \tag{32}$$

where $h(c) = c + g(c)$ and $g(c)$ is given by (31). Equation (32) defines a sequence of implicit mappings that determine ξ_n^{ii} in terms of ξ_n^{ml}. By Theorem 4 the remainder in (30) satisfies $nR_n(c) = O(\frac{c}{n}) = o(1)$ for all $c = o(n)$ and $nR_n(c) = O(n^{-1})$ uniformly over c in compact subsets of $\mathbb{R}$. The binding function relationship $\hat{\rho}_n = b_n(\breve{\rho})$ is linear at the limits of its domain of definition, continuous and invertible with first derivative bounded, and bounded away from zero as $n \to \infty$, so that $\breve{\rho} - \rho \to_p 0$ follows from $\hat{\rho}_n - \rho \to_p 0$ and hence $\xi_n^{\mathrm{ii}} = n(\breve{\rho} - \rho) = o_p(n)$. It follows that $nR_n(\xi_n^{\mathrm{ii}}) = O_p(\frac{\xi_n^{\mathrm{ii}}}{n}) = o_p(1)$ and then (32) is

$$\xi_n^{\mathrm{ml}} = \xi_n^{\mathrm{ii}} + g(\xi_n^{\mathrm{ii}}) + o_p(1) = h(\xi_n^{\mathrm{ii}}) + o_p(1).$$

The functions g and h are independent of n. In the limit as $n \to \infty$, we therefore have $\xi_n^{\mathrm{ml}} \sim h(\xi_n^{\mathrm{ii}})$, so the limit function h implicitly determines the limit distribution of ξ_n^{ii}.

The limit function h is graphed in Figure 3. This function is monotonic and continuous (including the point $c = 0$), and a continuous inverse function h^{-1} exists by virtue of Lemma 2. Details of these properties and the derivative of h are given in the Supplemental Material. The shape of the limit function h is remarkably similar to that of the binding function b_n shown in Figure 3.

The remaining argument is straightforward. According to standard theory (Phillips (1987)), $\xi_n^{\mathrm{ml}} \Rightarrow \xi^{\mathrm{ml}} = \int_0^1 W\,dW/\int_0^1 W^2$, where W is a standard Brownian motion. By the Skorohod representation theorem, we can enlarge the probability space with distributionally equivalent random sequences for which $\xi_n^{\mathrm{ml}} \to_{\mathrm{a.s.}} \xi^{\mathrm{ml}}$. On this space, by the continuity of the inverse map h^{-1}, we deduce that $\xi_n^{\mathrm{ii}} \to_{\mathrm{a.s.}} \xi^{\mathrm{ii}}$, where ξ^{ii} is the solution of $\xi^{\mathrm{ml}} = h(\xi^{\mathrm{ii}})$. Hence, in the original space, we have $\xi_n^{\mathrm{ii}} \Rightarrow \xi^{\mathrm{ii}}$ as $n \to \infty$ by the implicit continuous mapping theorem. Thus, the limit distribution of ξ^{ii} in the unit root case is given by

$$n(\breve{\rho} - 1) \Rightarrow h^{-1}\biggl(\int_0^1 W\,dW \Big/ \int_0^1 W^2\biggr), \tag{33}$$

where h^{-1} is the inverse function of $h(c) = c + g(c)$ and $g(c)$ is given in (31).

The distribution of the centered and scaled IIE ξ_n^{ii} is shown in Figure 4 against that of the MLE ξ_n^{ml}. The differences are immediately evident from the figure. The distribution of ξ_n^{ii} is much less biased than ξ_n^{ml}, as we would expect from the criterion function, but it is also much more concentrated than that of ξ_n^{ml}. Whereas the bulk of the distribution of ξ_n^{ml} is to the left of the origin, the distribution of ξ_n^{ii} leans to the explosive side of the origin while still retaining a long left hand tail. Thus, the functional transformation h^{-1} changes the shape as well as the location of the limit distribution of the ML estimator.

Since the binding function b_n and limit function h are monotonic, tests and confidence intervals based on the IIE $\breve{\rho}$ and the MLE $\hat{\rho}$ are asymptotically equivalent. But in finite samples, there are differences and they can be important in practice. For example, when $|\rho| < 1$, $\hat{\rho}$ and $\breve{\rho}$ have the same $N(\rho, \frac{1-\rho^2}{n})$ asymptotic distribution, but tests of $\rho = \rho^0$ and confidence intervals for ρ based on the nominal asymptotics differ. A similar point applies in the case of mildly explosive asymptotics (Phillips and Magdalinos (2007, 2008)). Unit root tests based on the test statistics $Z_{\breve{\rho}} = n(\breve{\rho} - 1)$ and $Z_{\hat{\rho}} = n(\hat{\rho} - 1)$ are also asymptotically equivalent, as are confidence intervals constructed by inverting these tests using the local to unit limit theory, as in Stock (1991). But finite sample tests and confidence intervals based on the nominal asymptotics differ.[3]

5.3. *Local to Unity Case*

In this case, the true value is $\rho = 1 + c/n$ and $\xi_n^{\mathrm{ml}} = n(\hat{\rho}_n - \rho) \Rightarrow \xi^{\mathrm{ml}} := \int_0^1 J_c\,dW / \int_0^1 J_c^2$, where W is a standard Brownian motion and $J_c(\cdot) = \int_0^{\cdot} e^{c(\cdot - s)}\,dW(s)$ is a linear diffusion (Phillips (1987), Chan and Wei (1987)). Since (30) continues to hold for all c, we again have $\hat{\rho}_n = b_n(\breve{\rho})$ with $b_n(\rho) = \rho + \frac{1}{n}g(c) + R_n(c)$ and $R_n(c) = O(c/n^2)$. Then setting $\breve{c} = n(\breve{\rho} - 1)$, we have

[3]For example, if $f_{L,\alpha}^{\mathrm{ii}}$ is the lower α percentile of the limit variate $\xi^{\mathrm{ii}} = h^{-1}(\int_0^1 W\,dW / \int_0^1 W^2)$, then a one sided nominal $100\alpha\%$ test will reject if $Z_{\breve{\rho}} < f_{L,\alpha}^{\mathrm{ii}}$, that is, if $\hat{\rho} < b_n(1 + f_{L,\alpha}^{\mathrm{ii}}/n)$ or $Z_{\hat{\rho}} < n\{b_n(1 + f_{L,\alpha}^{\mathrm{ii}}/n) - 1\} = f_{L,\alpha}^{\mathrm{ii}} + g(f_{L,\alpha}^{\mathrm{ii}}) + O(n^{-1}) = h(f_{L,\alpha}^{\mathrm{ii}}) + O(n^{-1}) = f_{L,\alpha}^{\mathrm{ml}} + O(n^{-1})$, where $f_{L,\alpha}^{\mathrm{ml}}$ is the lower α percentile of the distribution of the limit variate ξ^{ml}. So tests and confidence intervals differ by $O(n^{-1})$.

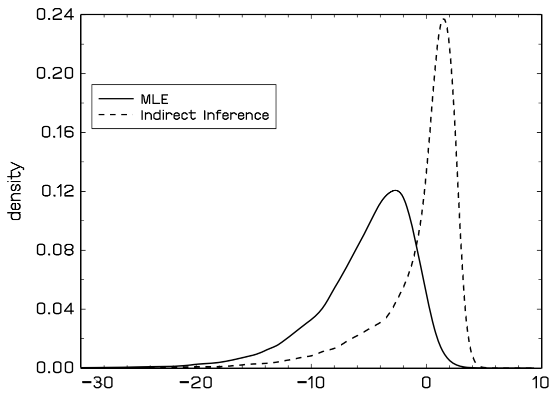

FIGURE 4.—Densities of $n(\xi_n^{\mathrm{ml}} - 1)$ (solid line) and $n(\xi_n^{\mathrm{ii}} - 1)$ (broken line) for $n = 500$.

$\hat{\rho}_n = b_n(\breve{\rho}) = \breve{\rho} + \frac{1}{n}g(\breve{c}) + R_n(\breve{c})$ and by a similar argument to that when $c = 0$, we have, for $\rho = 1 + c/n$,

$$\begin{aligned} n(\hat{\rho}_n - \rho) &= n(\breve{\rho} - \rho) + g(n(\breve{\rho} - \rho) + n(\rho - 1)) + nR_n(n(\breve{\rho} - 1)) \\ &= n(\breve{\rho} - \rho) + g(n(\breve{\rho} - \rho) + c) + o_p(1). \end{aligned} \tag{34}$$

Substituting $\xi_n^{\mathrm{ml}} = n(\hat{\rho}_n - \rho)$ and $\xi_n^{\mathrm{ii}} = n(\breve{\rho} - \rho)$ into (34), we find that

$$\xi_n^{\mathrm{ml}} = \xi_n^{\mathrm{ii}} + g(\xi_n^{\mathrm{ii}} + c) + o_p(1)$$

and hence

$$\xi_n^{\mathrm{ml}} + c = (\xi_n^{\mathrm{ii}} + c) + g(\xi_n^{\mathrm{ii}} + c) + o_p(1) = h(\xi_n^{\mathrm{ii}} + c) + o_p(1).$$

Proceeding as in the unit root case, we deduce that

$$n(\breve{\rho} - \rho) \Rightarrow h^{-1}\left(\int_0^1 J_c\,dW \Big/ \int_0^1 J_c^2 + c\right) - c,$$

so the limit distribution of the indirect inference estimator is given by the inverse of the same implicit mapping h as in the unit root case. Only the intercept $(-c)$ and argument functional $\int_0^1 J_c\,dW / \int_0^1 J_c^2 + c$ of h^{-1} change according to the value of the localizing coefficient c.

6. CONCLUSIONS AND EXTENSIONS

Simulation-based estimation procedures like indirect inference can complicate limit theory by virtue of the presence of sample-sized dependent functionals in the estimators. These functionals serve an important role because of the manner in which they intentionally capture and correct for (possibly undesirable) finite sample features of more basic estimation procedures like maximum likelihood or quasi maximum likelihood. A resulting complication is that conventional delta method arguments may fail because the estimator relies on a sequence of functions and stronger conditions are required to validate the standard approach. Another complication is that the estimator may be determined implicitly, so that it is necessary to work with implicit mappings and global inversion to define the estimator sequence. A final complication and one that is potentially the most significant is that the sequence of functions may influence the limit distribution theory in a material way, affecting the shape characteristics of the distribution as well as simple matters such as location and scale. The indirect inference estimator of the autoregressive coefficient is shown to be affected in this way for values of the coefficient in the usual $O(n^{-1})$ vicinity of unity. The resulting limit theory provides both a bias correction and a variance reduction to the maximum likelihood estimator in this vicinity, opening the way to other procedures that have similar properties without compromising the limit theory for the stationary case, such as the fully aggregated estimator of Han, Phillips, and Sul (2012).

Given the prolific nature of simulation-based techniques in econometrics, it seems evident that in many cases, econometric estimators and inferential procedures will rely on sample-sized based functionals. In such cases, it will generally be necessary to use some version of the extended delta method in asymptotic derivations. These methods are likely to become more numerous in future econometric work as cases of greater complexity are studied using simulation-based methods. Of course, most of these applications will not involve the type of additional difficulties that arise in the limit binding function of the unit root case where nonlinearities in the function persist in the limit and implicit maps are involved. Nonetheless, these additional complexities may arise in some cases of practical importance where simulation-based methods are used in vector time series systems with some unit roots.

The AR(1) case considered here is the prototype for all models with an autoregressive unit root. Practical cases typically involve more variables, parameters, and deterministic trends. In such cases, it becomes necessary to deal with multivariate asymptotics, possible degeneracies in the limit theory, and the development of binding function algebra for vector autoregressive systems with possible unit roots. Generalization of the extended delta method to multivariate functions that allow for degeneracies therefore seems worthwhile. Similarly, the implicit mapping theorem may be usefully extended to multivariate and functional inverse and implicit function theorems. These may be necessary in dealing with nonstationary time series systems where indirect inference methods are used. More immediate applications of the results here are to dynamic panel data models and continuous time systems where indirect inference methods have been employed to correct bias and to price derivative securities. These various extensions and applications seem worthy of consideration in future research.

APPENDIX

Proofs of the main results are only outlined here. The reader is referred to the Supplemental Material for detailed derivations and some additional results on asymptotic expansions of integrals that are needed in the main arguments.

PROOF OF THEOREM 1: By the mean value theorem,

$$\varphi_n(T_n) - \varphi_n(\theta) = \varphi_n'(T_n^*)(T_n - \theta)$$

for some T_n^* on the line segment connecting T_n and θ. Hence

$$\frac{d_n}{\varphi_n'(\theta)}(\varphi_n(T_n) - \varphi_n(\theta)) = \left\{1 + \frac{\varphi_n'(T_n^*) - \varphi_n'(\theta)}{\varphi_n'(\theta)}\right\} d_n(T_n - \theta).$$

Since $|T_n^* - \theta| \le |T_n - \theta| = O_p(d_n^{-1})$ and $\frac{s_n}{d_n} \to 0$, it follows that $s_n|T_n^* - \theta| = o_p(1)$. Then

$$\frac{\varphi_n'(T_n^*) - \varphi_n'(\theta)}{\varphi_n'(\theta)} \to_p 0$$

by local relative equicontinuity (2) in a shrinking neighborhood of radius $O(s_n^{-1})$, giving the required result. *Q.E.D.*

For the proof of Lemma 2, see Ge and Wang (2002, Lemma 1). For the proof of Theorem 3, see the Supplemental Material.

PROOF OF THEOREM 4:

(i) *Case* $|\rho| < 1$. Using $F_n = 1 - \rho^2 x + (1 - x)x^{2n-1}\rho^{2n} = 1 - \rho^2 x + O(\rho^{2n})$ and $n\rho^n = o(n^{-2})$, we have for $|\rho| < 1$, after some calculations given in the Supplemental Material,

$$\begin{aligned} b_n(\rho) &= \rho - \frac{3\rho}{2}\int_0^1 x^{(n-1)/2}(1 - \rho^2 x^2)^{1/2}(1 - \rho^2 x)^{-1/2}\,dx \\ &\quad + \frac{\rho}{2}\int_0^1 x^{(n-3)/2}(1 - \rho^2 x^2)^{3/2}(1 - \rho^2 x)^{-3/2}\,dx + O(n^{-2}) \\ &= \rho - \frac{2\rho}{n} + O(n^{-2}), \end{aligned}$$

which is the well known asymptotic bias formula for $\hat{\rho}_n$ in the stationary case.

(ii) *Case* $\rho = \pm 1$. When $\rho = 1$, we have $F_n(x; 1) = 1 - x + (1 - x)x^{2n-1} = (1 - x)(1 + x^{2n-1})$ and, after some calculation and expansions (detailed in the

Supplemental Material), we obtain

$$
\begin{aligned}
(35)\qquad b_n(1) &= 1 - \frac{3}{2}\int_0^1 x^{(n-1)/2}\frac{(1+x)^{1/2}}{(1+x^{2n-1})^{1/2}}\,dx \\
&\quad + \frac{1}{2}\int_0^1 x^{(n-3)/2}\frac{(1+x)^{3/2}}{(1+x^{2n-1})^{3/2}}\,dx \\
&\quad - \frac{n}{2}\int_0^1 x^{(5n-7)/2}\frac{(1+x)^{3/2}}{(1+x^{2n-1})^{3/2}}(1-x)\,dx \\
&= 1 - \frac{1.7814}{n} + O(n^{-2})
\end{aligned}
$$

upon numerical evaluation of the integrals. Similar calculations apply when $\rho = -1$, in which case we have

$$
b_n(-1) = -1 + \frac{1.7814}{n} + O(n^{-2}),
$$

giving the mirror image of (35).

(iii) *Case* $\rho = 1 + \frac{c}{n}$, $c < 0$. This is the (stationary) local to unity case with $\rho = 1 + \frac{c}{n}$, $c < 0$, and $|\rho| < 1$. The following arguments allow c to be fixed and $c \to -\infty$ as $n \to \infty$ with $c = o(n)$. The relevant expression for the bias is

$$
\begin{aligned}
(36)\qquad b_n(\rho) &= \rho - \frac{3\rho}{2}\int_0^1 x^{(n-1)/2}(1-\rho^2x^2)^{1/2}F_n^{-1/2}\,dx \\
&\quad + \frac{\rho}{2}\int_0^1 x^{(n-3)/2}(1-\rho^2x^2)^{3/2}F_n^{-3/2}\,dx \\
&\quad - \frac{n\rho^{2n-1}}{2}\int_0^1 x^{(5n-7)/2}(1-\rho^2x^2)^{3/2}F_n^{-3/2}(1-x)\,dx.
\end{aligned}
$$

Set $y = x^{2n-1}$ so that $dy = (2n-1)x^{2n-2}\,dx = (2n-1)y^{(2n-2)/(2n-1)}\,dx = (2n-1)y^{1-1/(2n-1)}\,dx$. Then using $F_n = 1 - \rho^2 x + (1-x)x^{2n-1}\rho^{2n}$, we have, for the first integral in (36),

$$
\begin{aligned}
&\int_0^1 x^{(n-1)/2}(1-\rho^2x^2)^{1/2}F_n^{-1/2}\,dx \\
&\quad = \frac{1}{2n-1}\int_0^1 y^{(n+1)/(4n-4)-1}\big(1-\rho^2 y^{2/(2n-1)}\big)^{1/2} \\
&\qquad \times \big\{\big(1-\rho^2 y^{1/(2n-1)}\big) + \big(1-y^{1/(2n-1)}\big)y\rho^{2n}\big\}^{-1/2}\,dy \\
&\quad = \frac{1}{2n-1}\int_0^1 y^{-3/4+1/(2n-2)}\big(1-\rho^2 y^{2/(2n-1)}\big)^{1/2} \\
&\qquad \times \big\{\big(1-\rho^2 y^{1/(2n-1)}\big) + \big(1-y^{1/(2n-1)}\big)y\rho^{2n}\big\}^{-1/2}\,dy.
\end{aligned}
$$

Since $y^{b/(2n+a)} = 1 + \frac{b}{2n+a}\log y + O(n^{-2})$, $\rho^2 = 1 + \frac{2c}{n} + \frac{c^2}{n^2}$, $\rho^{2n} = (1+\frac{c}{n})^{2n} = e^{2c}\{1 + O(\frac{c^2}{n})\}$, and $c < 0$ with $\frac{c}{n} = o(1)$, we find after some calculations (see the Supplemental Material for details) that

$$\int_0^1 x^{(n-1)/2}(1-\rho^2x^2)^{1/2}F_n^{-1/2}\,dx = \frac{1}{2n}\int_0^1 y^{-3/4}\ell(y,c)^{-1/2}\,dy\left\{1+O\left(\frac{c}{n}\right)\right\}, \tag{37}$$

where

$$\ell(y,c) := \frac{4c+\log y}{4c+2\log y} + \frac{\log y}{4c+2\log y}ye^{2c}.$$

The remaining integrals can be transformed in the same fashion as shown in the Supplemental Material, giving

$$\int_0^1 x^{(n-3)/2}(1-\rho^2x^2)^{3/2}F_n^{-3/2}\,dx = \frac{1}{2n}\int_0^1 y^{-3/4}\ell(y,c)^{-3/2}\,dy\left\{1+O\left(\frac{c}{n}\right)\right\} \tag{38}$$

and

$$\int_0^1 x^{(5n-7)/2}(1-\rho^2x^2)^{3/2}F_n^{-3/2}(1-x)\,dx = \frac{1}{4n^2}\int_0^1 y^{1/4}\ell(y,c)^{-3/2}\log y\,dy\left\{1+O\left(\frac{c}{n}\right)\right\}. \tag{39}$$

Combining results (37)–(39) gives the following approximation to the binding function for $\rho = 1 + \frac{c}{n}$ with $c < 0$, $c = o(n)$ and $|\rho| < 1$:

$$\begin{aligned} b_n(\rho) = \rho &- \frac{3\rho}{4n}\int_0^1 y^{-3/4}\ell(y,c)^{-1/2}\,dy\left\{1+O\left(\frac{c}{n}\right)\right\} \\ &+ \frac{\rho}{4n}\int_0^1 y^{-3/4}\ell(y,c)^{-3/2}\,dy\left\{1+O\left(\frac{c}{n}\right)\right\} \\ &+ \frac{\rho^{2n-1}}{8n}\int_0^1 y^{1/4}\ell(y,c)^{-3/2}\log y\,dy\left\{1+O\left(\frac{c}{n}\right)\right\}. \end{aligned} \tag{40}$$

The approximation (40) holds with an error of $O(n^{-2})$ uniformly for all c in compact sets of $\mathbb{R}_- = (-\infty, 0)$ when $n \to \infty$. It also holds with a relative error of $O(\frac{c}{n})$ when $c \to -\infty$ provided $c = o(n)$. In this case, terms involving e^{2c} become exponentially small. The approximation to the binding function when

$n \to \infty$ and $c \to -\infty$ while $|\rho| = |1 + \frac{c}{n}| < 1$ and $c = o(n)$ as $n \to \infty$ has the form

$$(41) \qquad b_n(\rho) = \rho - \frac{3\rho}{4n}\int_0^1 y^{-3/4}\,dy + \frac{\rho}{4n}\int_0^1 y^{-3/4} + O\left(\frac{c}{n^2}\right)$$
$$= \rho - \frac{\rho}{2n}\left[\frac{y^{1/4}}{1/4}\right]_0^1 + O\left(\frac{c}{n^2}\right) = \rho - \frac{2\rho}{n} + O\left(\frac{c}{n^2}\right),$$

where the leading term in the approximation corresponds to the case of fixed ρ with $|\rho| < 1$. In this case, the binding function is linear in ρ as $n \to \infty$ and $c \to -\infty$ with $|\rho| < 1$.

(iv) *Case* $\rho = 1 + \frac{c}{n}$, $c > 0$. We allow for the case where $c \to \infty$ under the condition $c = o(n)$. The binding function formula for $\rho > 1$ is

$$(42) \qquad b_n(\rho) = \rho + \frac{3\rho}{2}\int_1^\infty x^{(n-1)/2}(\rho^2x^2 - 1)^{1/2}G_n^{-1/2}\,dx$$
$$- \frac{\rho}{2}\int_1^\infty x^{(n-3)/2}(\rho^2x^2 - 1)^{3/2}G_n^{-3/2}\,dx$$
$$- \frac{n\rho^{2n-1}}{2}\int_1^\infty x^{(n-5)/2}(\rho^2x^2 - 1)^{3/2}G_n^{-3/2}(x-1)x^{2n-1}\,dx.$$

We proceed by taking each term in turn. As before, set $y = x^{2n-1}$ so that

$$dy = (2n-1)x^{2n-2}\,dx = (2n-1)y^{(2n-2)/(2n-1)}\,dx$$
$$= (2n-1)y^{1-1/(2n-1)}\,dx,$$

and use the expansion $y^{1/(2n-1)} = 1 + \frac{1}{2n-1}\log y + O(n^{-2})$. In $G_n = \rho^2x - 1 + (x-1)x^{2n-1}\rho^{2n}$, we have $\rho = 1 + \frac{c}{n}$ with $c > 0$ and $c \to \infty$ such that $c = o(n)$. As before, $\rho^2 = 1 + \frac{2c}{n} + \frac{c^2}{n^2}$, and $\rho^{2n} = (1 + \frac{c}{n})^{2n} = e^{2c}\{1 + O(\frac{c^2}{n})\}$. The integral in the second term of (42) is found to be (see the Supplemental Material)

$$(43) \qquad \int_1^\infty x^{(n-1)/2}(\rho^2x^2 - 1)^{1/2}G_n^{-1/2}\,dx$$
$$= \frac{1}{2n}\int_1^\infty y^{-3/4}\left\{\frac{4c + 2\log y}{4c + \log y + \rho^{2n}y\log y}\right\}^{1/2} dy\left\{1 + O\left(\frac{c}{n}\right)\right\}$$
$$= \frac{1}{2n}\int_0^\infty e^{1/4w}k^+(w;c)^{1/2}\,dw\left\{1 + O\left(\frac{c}{n}\right)\right\},$$

using the transformation $w = \log y$ for $w \in [0, \infty)$ and where

$$k^+(w;c) := \frac{4c + 2w}{4c + w + e^{2c}we^w}.$$

Proceeding in the same way with the integral in the third term of (42), we find

$$\int_1^\infty x^{(n-3)/2}(\rho^2x^2-1)^{3/2}G_n^{-3/2}\,dx = \frac{1}{2n}\int_0^\infty e^{1/4w}k^+(w;c)^{3/2}\,dw\left\{1+O\left(\frac{c}{n}\right)\right\}, \tag{44}$$

and handling the integral in the fourth term of (42) in the same way, we have

$$\int_1^\infty x^{(n-5)/2}(\rho^2x^2-1)^{3/2}G_n^{-3/2}(x-1)x^{2n-1}\,dx = \left(\frac{1}{2n}\right)^2\int_0^\infty e^{5/4w}k^+(w;c)^{3/2}w\,dw\left\{1+O\left(\frac{c}{n}\right)\right\}. \tag{45}$$

Combining (43)–(45) in (42), we get, for $\rho = 1+\frac{c}{n}$ with $c>0$,

$$\begin{aligned} b_n(\rho) = \rho &+ \frac{3\rho}{4n}\int_0^\infty e^{1/4w}k^+(w;c)^{1/2}\,dw\left\{1+O\left(\frac{c}{n}\right)\right\} \\ &- \frac{\rho}{4n}\int_0^\infty e^{1/4w}k^+(w;c)^{3/2}\,dw\left\{1+O\left(\frac{c}{n}\right)\right\} \\ &- \frac{\rho^{2n-1}}{8n}\int_0^\infty e^{5/4w}k^+(w;c)^{3/2}w\,dw\left\{1+O\left(\frac{c}{n}\right)\right\}. \end{aligned} \tag{46}$$

Hence the bias function to $O(n^{-1})$ when ρ is local to unity on the explosive side is

$$\begin{aligned} b_n\left(1+\frac{c}{n}\right) = 1+\frac{c}{n} &+ \frac{3}{4n}\int_0^\infty e^{1/4w}k^+(w;c)^{1/2}\,dw\left\{1+O\left(\frac{c}{n}\right)\right\} \\ &- \frac{1}{4n}\int_0^\infty e^{1/4w}k^+(w;c)^{3/2}\,dw\left\{1+O\left(\frac{c}{n}\right)\right\} \\ &- \frac{e^{2c}}{8n}\int_0^\infty e^{(5/4)w}k^+(w;c)^{3/2}w\,dw\left\{1+O\left(\frac{c}{n}\right)\right\}. \end{aligned} \tag{47}$$

The approximation (47) holds with an error of $O(n^{-2})$ uniformly for all c in compact sets of $\mathbb{R}_+ = (0,\infty)$ when $n\to\infty$. The formula also produces a valid approximation to the binding function when $c\to\infty$ and $\rho = 1+\frac{c}{n} > 1$ as $n\to\infty$. In this case, noting the relative error order in (47) and the behavior of $k^+(w;c)$ as $c\to\infty$, the approximation to the binding function has the form

$$b_n\left(1+\frac{c}{n}\right) = 1+\frac{c}{n}+O\left(\frac{1}{ne^c}\right), \tag{48}$$

so that for large c, the binding function can be approximated by a linear function.

(v) *Case* $|\rho| > 1$. The bias expression is given by $b_n(\rho; |\rho| > 1)$ in (42). We examine the order of magnitude of each term in turn as $n \to \infty$. We find

$$\begin{aligned}
&\int_1^\infty x^{(n-1)/2}(\rho^2 x^2 - 1)^{1/2} G_n^{-1/2}\,dx \\
&\quad = \int_1^\infty x^{(n-1)/2}\left\{\frac{\rho^2 x^2 - 1}{\rho^2 x - 1 + (x-1)x^{2n-1}\rho^{2n}}\right\}^{1/2} dx \\
&\quad \le \frac{B}{\rho^n}\int_1^\infty \frac{1}{x^{n/2}}\,dx = O(n^{-1}\rho^{-n}), \\
&\int_1^\infty x^{(n-3)/2}(\rho^2 x^2 - 1)^{3/2} G_n^{-3/2}\,dx \\
&\quad = \int_1^\infty x^{(n-3)/2}\left\{\frac{\rho^2 x^2 - 1}{\rho^2 x - 1 + (x-1)x^{2n-1}\rho^{2n}}\right\}^{3/2} dx \\
&\quad = O(n^{-1}\rho^{-3n}),
\end{aligned}$$

and

$$\begin{aligned}
&\frac{n\rho^{2n-1}}{2}\int_1^\infty x^{(n-5)/2}(\rho^2 x^2 - 1)^{3/2} G_n^{-3/2}(x-1)x^{2n-1}\,dx \\
&\quad = \frac{n\rho^{-n-1}}{2}\int_1^\infty \frac{1}{x^{n/2+2}}\left\{\frac{\rho^2 x^2 - 1}{(x-1) + (\rho^2 x - 1)/(x^{2n-1}\rho^{2n})}\right\}^{3/2}(x-1)\,dx \\
&\quad = O(\rho^{-n}),
\end{aligned}$$

which therefore dominates the last three terms of (42). It follows that $b_n(\rho) = \rho + O(\rho^{-n})$, showing that the bias is exponentially small (and negative, in view of the sign of the final term of (42)) for $\rho > 1$. A similar result holds when $\rho < -1$, in which case the bias is exponentially small and positive. *Q.E.D.*

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Econometrica Supplementary Material

SUPPLEMENT TO "FOLKLORE THEOREMS, IMPLICIT MAPS, AND INDIRECT INFERENCE"

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BY PETER C. B. PHILLIPS

THIS SUPPLEMENT PROVIDES technical results and proofs.

A.1. *Some Useful Integral Asymptotic Expansions*

The following lemmas provide results on asymptotic expansions of integrals that are useful in the main arguments. In particular, these results are used to develop bias expansions for the binding function $b_n(\rho)$ (see equation (14) of the paper) for three separate fixed ρ cases ($|\rho|<1$, $\rho=\pm 1$, and $|\rho|>1$) and to show asymptotic behavior in the local to unity case where $\rho = 1+\frac{c}{n}$ for fixed c and for $|c|\to\infty$ with $c=o(n)$ as $n\to\infty$.

These integral asymptotic expansion formulae are likely to have applications in other contexts.

LEMMA 1: *Let* $F_n = F_n(x;\rho) = 1-\rho^2 x + (1-x)x^{2n-1}\rho^{2n}$ *and suppose* $a_1, a_2, \gamma > 0$. *Then as* $n\to\infty$,

$$\int_0^1 x^{a_1 n + a_4}(1-\rho^2 x^2)^{\alpha} F_n^{-\beta}\, dx = \frac{(1-\rho^2)^{\alpha-\beta}}{a_1 n} + O(n^{-2}), \quad |\rho|<1, \tag{A.1}$$

$$\int_0^1 x^{a_1 n + a_4}(1+x)^{\alpha}(1+\gamma x^{a_2 n + a_3})^{\beta}\, dx = \frac{2^{\alpha}}{a_2 n}\int_0^1 y^{(a_1-a_2)/a_2}(1+\gamma y)^{\beta}\, dy + O(n^{-2}), \tag{A.2}$$

$$n\int_0^1 x^{a_1 n + a_4}(1+x)^{\alpha}(1+\gamma x^{a_2 n + a_3})^{\beta}(1-x)\, dx = -\frac{2^{\alpha}}{a_2^2 n}\int_0^1 y^{(a_1-a_2)/a_2}(1+\gamma y)^{\beta}\log y\, dy + O(n^{-2}). \tag{A.3}$$

PROOF: To prove (A.1), note first that ρ^{2n} is exponentially small for $|\rho|<1$. Then $F_n = 1-\rho^2 x + O(\rho^{2n})$. Setting $y = x^{a_1 n + a_4}$, we have $dy = (a_1 n + a_4)x^{a_1 n + a_4 - 1}\, dx = (a_1 n + a_4) y^{(a_1 n + a_4 - 1)/(a_1 n + a_4)}\, dx$ and upon transforma-

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tion,

$$
\begin{aligned}
&\int_0^1 x^{a_1 n+a_4}(1-\rho^2 x^2)^\alpha F_n^{-\beta}\,dx \\
&\quad = \frac{1}{a_1 n + a_4}\int_0^1 y^{1-(a_1 n+a_4-1)/(a_1 n+a_4)}\big(1-\rho^2 y^{2/(a_1 n+a_4)}\big)^\alpha \\
&\qquad \times \big(1-\rho^2 y^{1/(a_1 n+a_4)}\big)^{-\beta}\,dy \\
&\quad = \frac{(1-\rho^2)^{\alpha-\beta}}{a_1 n + a_4}\int_0^1 y^{1/(a_1 n+a_4)}\,dy\{1+O(n^{-1})\} \\
&\quad = \frac{(1-\rho^2)^{\alpha-\beta}}{a_1 n} + O(n^{-2}),
\end{aligned}
$$

since $y^{b/(a_1 n+a_4)} = 1 + \frac{b}{a_1 n + a_4}\log y + O(n^{-2})$ for all $b \neq 0$ and $|\int_0^1 \log y\,dy| = 1$.

To prove (A.2), set $y = x^{a_2 n+a_3}$, so that $dy = (a_2 n + a_3)x^{a_2 n+a_3-1}\,dx = a_2 n y^{(a_2 n+a_3-1)/(a_2 n+a_3)}\,dx\{1+O(n^{-1})\}$ and upon transformation,

$$
\begin{aligned}
&\int_0^1 x^{a_1 n+a_4}(1+x)^\alpha(1+\gamma x^{a_2 n+a_3})^\beta\,dx \\
&\quad = \frac{1}{a_2 n + a_3}\int_0^1 y^{((a_1-a_2)n+a_4-a_3+1)/(a_2 n+a_3)} \\
&\qquad \times \big(1+y^{1/(a_2 n+a_3)}\big)^\alpha(1+\gamma y)^\beta\,dy \\
&\quad = \frac{2^\alpha}{a_2 n}\int_0^1 y^{(a_1-a_2)/a_2}(1+\gamma y)^\beta\,dy + O(n^{-2}),
\end{aligned}
$$

since $y^{b/(a_2 n+a_3)} = 1 + \frac{b}{a_2 n + a_3}\log y + O(n^{-2})$ for all $y > 0$.

To prove (A.3), the same approach leads to

$$
\begin{aligned}
\text{(A.4)}\qquad & n\int_0^1 x^{a_1 n+a_4}(1+x)^\alpha(1+\gamma x^{a_2 n+a_3})^\beta(1-x)\,dx \\
&\quad = \frac{n}{a_2 n + a_3}\int_0^1 y^{((a_1-a_2)n+a_4-a_3+1)/(a_2 n+a_3)}\big(1+y^{1/(a_2 n+a_3)}\big)^\alpha \\
&\qquad \times (1+\gamma y)^\beta\big(1-y^{1/(a_2 n+a_3)}\big)\,dy \\
&\quad = -\frac{2^\alpha}{a_2^2 n}\int_0^1 y^{(a_1-a_2)/a_2}(1+\gamma y)^\beta \log y\,dy + O(n^{-2}).
\end{aligned}
$$

Observe that, using the transformation $w = -\log y$, we have

$$\int_0^1 y^{a-1} |\log y|^b \, dy = \int_0^\infty e^{-aw} w^b \, dw < \infty \tag{A.5}$$

for all $a > 0$ and $b > -1$, which ensures that (A.4) is finite. *Q.E.D.*

LEMMA 2: *For $\rho = 1 + \frac{c}{n}$, with c fixed, $F_n = 1 - \rho^2 x + (1-x)x^{2n-1}\rho^{2n}$, $a_1 > 0$, and $\alpha - \beta > -1$, we have*

$$\int_0^1 x^{a_1 n + a_4}(1-\rho^2 x^2)^\alpha F_n^{-\beta} \, dx$$
$$= \begin{cases} \dfrac{2^\alpha}{2n} \displaystyle\int_0^1 y^{(a_1-2)/2}(1+e^{2c}y)^{-\beta} \, dy + O(n^{-2}), & \alpha = \beta, \\ \dfrac{2^\alpha}{4n} \displaystyle\int_0^1 y^{(a_1-2)/2}(1+e^{2c}y)^{-\beta}(-\log y)^{\alpha-\beta} \, dy + O(n^{-2}), & \alpha \neq \beta. \end{cases}$$

PROOF: Since $\rho^{2n} = (1 + \frac{c}{n})^{2n} = e^{2c}\{1 + O(n^{-1})\}$, we have

$$\begin{aligned} F_n(x, \rho) &= 1 - x + (1-x)x^{2n-1}e^{2c} + O(n^{-1}) \\ &= (1-x)(1 + e^{2c}x^{2n-1}) + O(n^{-1}). \end{aligned} \tag{A.6}$$

Using Lemma 1, we obtain

$$\int_0^1 x^{a_1 n + a_4}(1-\rho^2 x^2)^\alpha F_n^{-\beta} \, dx$$
$$= \int_0^1 x^{a_1 n + a_4}(1-x^2)^\alpha (1-x)^{-\beta}(1+e^{2c}x^{2n-1})^{-\beta} \, dx\{1 + O(n^{-1})\}$$
$$= \int_0^1 x^{a_1 n + a_4}(1+x)^\alpha (1-x)^{\alpha-\beta}(1+e^{2c}x^{2n-1})^{-\beta} \, dx\{1 + O(n^{-1})\}$$
$$= \begin{cases} \displaystyle\int_0^1 x^{a_1 n + a_4}(1+x)^\alpha (1+e^{2c}x^{2n-1})^{-\beta} \, dx\{1 + O(n^{-1})\}, \\ \alpha = \beta \\ \displaystyle\int_0^1 x^{a_1 n + a_4}(1+x)^\alpha (1-x)^{\alpha-\beta}(1+e^{2c}x^{2n-1})^{-\beta} \, dx\{1 + O(n^{-1})\}, \\ \alpha \neq \beta \end{cases}$$
$$= \begin{cases} \dfrac{2^\alpha}{2n} \displaystyle\int_0^1 y^{(a_1-2)/2}(1+e^{2c}y)^{-\beta} \, dy + O(n^{-2}), & \alpha = \beta, \\ \dfrac{2^\alpha}{4n} \displaystyle\int_0^1 y^{(a_1-2)/2}(1+e^{2c}y)^{-\beta}(-\log y)^{\alpha-\beta} \, dy + O(n^{-2}), & \alpha \neq \beta, \end{cases}$$

the final integral being finite in view of (A.5) when $\alpha - \beta > -1$. *Q.E.D.*

LEMMA 3: *If* $|\rho| < 1$ *and* $a_1 > 0$, *then as* $n \to \infty$,

$$\int_0^1 x^{a_1 n + a_2}(1-\rho^2 x^2)^{\alpha}(1-\rho^2 x)^{-\beta}\,dx = \frac{(1-\rho^2)^{\alpha-\beta}}{a_1 n} + O(n^{-2}).$$

PROOF: Integrating by parts, we have

$$\begin{aligned}
&\int_0^1 x^{a_1 n + a_2}(1-\rho^2 x^2)^{\alpha}(1-\rho^2 x)^{-\beta}\,dx \\
&= \left[\frac{x^{a_1 n + a_2 + 1}}{a_1 n + a_2 + 1}(1-\rho^2 x^2)^{\alpha}(1-\rho^2 x)^{-\beta}\right]_0^1 \\
&\quad + \frac{2\alpha\rho^2}{a_1 n + a_2 + 1}\int_0^1 x^{a_1 n + a_2 + 2}(1-\rho^2 x^2)^{\alpha-1}(1-\rho^2 x)^{-\beta}\,dx \\
&\quad - \frac{\beta\rho^2}{a_1 n + a_2 + 1}\int_0^1 x^{a_1 n + a_2 + 2}(1-\rho^2 x^2)^{\alpha-1}(1-\rho^2 x)^{-\beta-1}\,dx \\
&= \frac{(1-\rho^2)^{\alpha-\beta}}{a_1 n} + O(n^{-2}).
\end{aligned}$$

Q.E.D.

A.2. *Proof of Expression (25)*

We consider the first derivative of the binding function $b_n(\rho)$ when $|\rho| < 1$, namely

$$\begin{aligned}
&b_n(\rho; |\rho| \le 1) \\
&= \rho - \frac{3\rho}{2}\int_0^1 x^{(n-1)/2}(1-\rho^2 x^2)^{1/2} F_n^{-1/2}\,dx \\
&\quad + \frac{\rho}{2}\int_0^1 x^{(n-3)/2}(1-\rho^2 x^2)^{3/2} F_n^{-3/2}\,dx \\
&\quad - \frac{n\rho^{2n-1}}{2}\int_0^1 x^{(5n-7)/2}(1-\rho^2 x^2)^{3/2} F_n^{-3/2}(1-x)\,dx,
\end{aligned} \tag{A.7}$$

given in equation (16) in the paper. As is clear from (A.7), the final term in the binding function expression is $O(\rho^n)$. The first derivative of this function is of the same order and since it is dominated by the other terms, it can be neglected in the following calculations. For $|\rho| < 1$, we therefore have

$$
\begin{aligned}
\text{(A.8)} \qquad b_n'(\rho) = 1 &- \frac{\partial}{\partial\rho}\left\{\frac{3\rho}{2}\int_0^1 x^{(n-1)/2}(1-\rho^2x^2)^{1/2}F_n^{-1/2}\,dx\right\} \\
&+ \frac{\partial}{\partial\rho}\left\{\frac{\rho}{2}\int_0^1 x^{(n-3)/2}(1-\rho^2x^2)^{3/2}F_n^{-3/2}\,dx\right\} + O(\rho^n) \\
= 1 &- \frac{3}{2}\int_0^1 x^{(n-1)/2}(1-\rho^2x^2)^{1/2}F_n^{-1/2}\,dx \\
&+ \frac{3\rho^2}{2}\int_0^1 x^{(n-5)/2}(1-\rho^2x^2)^{-1/2}F_n^{-1/2}\,dx \\
&+ \frac{3\rho}{4}\int_0^1 x^{(n-1)/2}(1-\rho^2x^2)^{1/2}F_n^{-3/2}\frac{\partial}{\partial\rho}F_n\,dx \\
&+ \frac{1}{2}\int_0^1 x^{(n-3)/2}(1-\rho^2x^2)^{3/2}F_n^{-3/2}\,dx \\
&- \frac{3\rho^2}{2}\int_0^1 x^{(n-7)/2}(1-\rho^2x^2)^{1/2}F_n^{-3/2}\,dx \\
&- \frac{3\rho}{4}\int_0^1 x^{(n-3)/2}(1-\rho^2x^2)^{3/2}F_n^{-5/2}\frac{\partial}{\partial\rho}F_n\,dx + O(\rho^n).
\end{aligned}
$$

Now $F_n = 1-\rho^2x+(1-x)x^{2n-1}\rho^{2n}$ and

$$\text{(A.9)} \qquad \frac{\partial}{\partial\rho}F_n = -2\rho x + 2n(1-x)x^{2n-1}\rho^{2n-1} = -2\rho x + O(n\rho^{2n-1}),$$

so that substituting (A.9) into (A.8) and using (A.1) of Lemma 1 above, we deduce that for $|\rho| < 1$,

$$\text{(A.10)} \qquad b_n'(\rho) = 1 + O(n^{-1}),$$

as required.

A.3. *Proofs of the Main Results*

PROOF OF THEOREM 1: By the mean value theorem,

$$\varphi_n(T_n) - \varphi_n(\theta) = \varphi_n'(T_n^*)(T_n-\theta)$$

for some T_n^* on the line segment connecting T_n and θ. Hence

$$\frac{d_n}{\varphi_n'(\theta)}(\varphi_n(T_n)-\varphi_n(\theta)) = \left\{1+\frac{\varphi_n'(T_n^*)-\varphi_n'(\theta)}{\varphi_n'(\theta)}\right\}d_n(T_n-\theta).$$

Since $|T_n^* - \theta| \leq |T_n - \theta| = O_p(d_n^{-1})$ and $\frac{s_n}{d_n} \to 0$, it follows that $s_n|T_n^* - \theta| = o_p(1)$. Then

$$\frac{\varphi_n'(T_n^*) - \varphi_n'(\theta)}{\varphi_n'(\theta)} \to_p 0$$

by local relative equicontinuity (see equation (2) of the paper) in a shrinking neighborhood of radius $O(s_n^{-1})$, giving the required result. *Q.E.D.*

For the Proof of Lemma 2 of the paper, see Ge and Wang (2002, Lemma 1).

PROOF OF THEOREM 3: The structure of the proof follows Shenton and Johnson (1965; SJ) and Shenton and Vinod (1995; SV) by considering ratios of quadratic forms in normal variates. The starting point is to write the density and moments of $\hat{\rho}_n = \sum_{t=1}^n y_t y_{t-1} / \sum_{t=1}^n y_{t-1}^2 = U/V$ in terms of the joint moment generating function $m(u, q)$ of the quadratic forms (U, V). White (1961)—see also Vinod and Shenton (1996)—showed that $m(u, q) = D_n^{-1/2}$, where $D_n = D_n(u, q)$ is a determinant that satisfies the second order difference equation

$$D_n = (1 + \rho^2 + 2q)D_{n-1} - (\rho + u)^2 D_{n-2}, \quad D_0 = D_1 = 1.$$

Then by direct calculation (see SJ, p. 3), we have the following expression for the bias function,

$$E(\hat{\rho}_n - \rho) = \int_0^\infty \frac{\partial}{\partial \rho} D_n(q)^{-1/2}\, dq, \tag{A.11}$$

where the determinant $D_n(q) = D_n(0, q)$ is evaluated explicitly as

$$D_n(q) = A\theta^n + (1 - A)\rho^{2n}\theta^{-n}, \quad A = \frac{\theta - \rho^2}{\theta^2 - \rho^2}, \tag{A.12}$$

$$\theta = \theta(q) = \big(1 + \rho^2 + 2q + \sqrt{\Delta}\big)/2,$$

$$\Delta = (1 + \rho^2 + 2q)^2 - 4\rho^2. \tag{A.13}$$

Observe that the inequalities

$$\begin{aligned}
&\theta = \theta(q) = \big(1 + \rho^2 + 2q + \sqrt{\Delta}\big)/2 \geq 0,\\
&\Delta = (1 - \rho^2)^2 + 4q^2 + 4q(1 + \rho^2) \geq (1 - \rho^2)^2 \geq 0,\\
&\theta - \rho^2 = \big(1 - \rho^2 + 2q + \sqrt{\Delta}\big)/2 \geq q \geq 0,\\
&\theta - \rho = (1 - \rho)^2 + 2q + \sqrt{\Delta} \geq 0,\\
&\theta + \rho = (1 + \rho)^2 + 2q + \sqrt{\Delta} \geq 0,\\
&\theta^2 - \rho^2 = (\theta - \rho)(\theta + \rho) \geq 0
\end{aligned}$$

hold for all $q \geq 0$. It follows that the determinant (A.12) is positive for all $q > 0$ and the integral (A.11) is defined for all ρ.

Write the binding function as

$$\begin{aligned} b_n(\rho) = E(\hat{\rho}_n) &= \rho + \int_0^\infty \frac{\partial}{\partial \rho} D_n(q)^{-1/2}\, dq \\ &= \rho - \frac{1}{2}\int_0^\infty D_n(q)^{-3/2} \frac{\partial D_n(q)}{\partial \rho}\, dq. \end{aligned}$$

Define $x = 1/\theta$ and $C = 1 + \rho^2 + 2q$, so that

$$x = \frac{2}{1+\rho^2+2q+\sqrt{\Delta}} = \frac{2}{C+\sqrt{\Delta}} = 2\frac{C-\sqrt{\Delta}}{C^2-\Delta} = \frac{C-\sqrt{\Delta}}{2\rho^2}, \tag{A.14}$$

since $\Delta = (1+\rho^2+2q)^2 - 4\rho^2 = C^2 - 4\rho^2$. It follows from (A.14) that

$$C + \Delta^{1/2} = \frac{2}{x} \quad \text{and} \quad C - \Delta^{1/2} = 2\rho^2 x,$$

so that $C = 1/x + \rho^2 x$, leading to

$$q = \frac{1}{2}(C - 1 - \rho^2) = \frac{1}{2}(1/x + \rho^2 x - 1 - \rho^2) = \frac{(1-x)(1-\rho^2 x)}{2x}.$$

We write

$$\begin{aligned} q &= \frac{(1-x)(1-\rho^2 x)}{2x} \\ &= \begin{cases} \dfrac{(1-x)(1-\rho^2 x)}{2x}, & x \in (0,1], |\rho| \leq 1, \\ \dfrac{(x-1)(x\rho^2 - 1)}{2x}, & x \in [1,\infty), |\rho| > 1, \end{cases} \end{aligned}$$

with derivative

$$\frac{dq}{dx} = -\frac{(1-\rho^2 x^2)}{2x^2} \begin{cases} < 0, & x \in (0,1], |\rho| \leq 1, \\ > 0, & x \in [1,\infty), |\rho| > 1, \end{cases} \tag{A.15}$$

so $q = q(x)$ is monotonic over the two domains of x in each case with $q \in [0,\infty)$. We may therefore change the variable of integration in (A.11) from q to x with corresponding changes in the domain of integration depending on the value of ρ as specified in (A.15). For $\rho = 1$, either domain may be used.

Using this change of variable, we have

$$A = \frac{\theta - \rho^2}{\theta^2 - \rho^2} = \frac{\dfrac{1}{x} - \rho^2}{\dfrac{1}{x^2} - \rho^2} = \frac{x - \rho^2 x^2}{1 - \rho^2 x^2} = \frac{x(1 - \rho^2 x)}{1 - \rho^2 x^2},$$

$$1 - A = 1 - \frac{x - \rho^2 x^2}{1 - \rho^2 x^2} = \frac{1 - x}{1 - \rho^2 x^2},$$

and then

$$\begin{aligned} D_n(q) &= \frac{(1 - \rho^2 x)}{1 - \rho^2 x^2} \frac{1}{x^{n-1}} + \frac{1 - x}{1 - \rho^2 x^2} \rho^{2n} x^n \\ &= \begin{cases} \dfrac{1 - \rho^2 x + (1 - x) x^{2n-1} \rho^{2n}}{(1 - \rho^2 x^2) x^{n-1}} = \dfrac{F_n(x : \rho)}{(1 - \rho^2 x^2) x^{n-1}}, & |\rho| \leq 1, \\ \dfrac{\rho^2 x - 1 + (x - 1) x^{2n-1} \rho^{2n}}{(\rho^2 x^2 - 1) x^{n-1}} = \dfrac{G_n(x : \rho)}{(\rho^2 x^2 - 1) x^{n-1}}, & |\rho| > 1, \end{cases} \end{aligned}$$

where

$$\begin{aligned} F_n(x; \rho) &:= 1 - \rho^2 x + (1 - x) x^{2n-1} \rho^{2n}, \\ G_n(x; \rho) &:= \rho^2 x - 1 + (x - 1) x^{2n-1} \rho^{2n}. \end{aligned}$$

For $|\rho| \leq 1$, we have

$$\begin{aligned} E(\hat{\rho}_n - \rho) &= \frac{\partial}{\partial \rho} \int_0^\infty D_n(q)^{-1/2} \, dq \\ &= \frac{\partial}{\partial \rho} \int_0^1 \left(\frac{F_n(x; \rho)}{(1 - \rho^2 x^2) x^{n-1}} \right)^{-1/2} \frac{(1 - \rho^2 x^2)}{2x^2} \, dx \\ &= \frac{1}{2} \frac{\partial}{\partial \rho} \left\{ \int_0^1 x^{(n-5)/2} (1 - \rho^2 x^2)^{3/2} F_n(x; \rho)^{-1/2} \, dx \right\}. \end{aligned} \tag{A.16}$$

To evaluate (A.16), note that

$$\begin{aligned} &\frac{\partial}{\partial \rho} \left\{ \int_0^1 x^{(n-5)/2} (1 - \rho^2 x^2)^{3/2} F_n^{-1/2} \, dx \right\} \\ &\quad = \frac{3}{2} \int_0^1 x^{(n-5)/2} (-2\rho x^2)(1 - \rho^2 x^2)^{1/2} F_n^{-1/2} \, dx \\ &\qquad - \frac{1}{2} \int_0^1 x^{(n-5)/2} (1 - \rho^2 x^2)^{3/2} \\ &\qquad \times F_n^{-3/2} \{-2\rho x + 2n(1 - x) x^{2n-1} \rho^{2n-1}\} \, dx, \end{aligned}$$

so that for $|\rho| \le 1$, we have

$$
\begin{aligned}
\text{(A.17)} \quad b_n(\rho) &= \rho + \frac{1}{2}\frac{\partial}{\partial\rho}\left\{\int_0^1 x^{(n-5)/2}(1-\rho^2x^2)^{3/2}F_n^{-1/2}(x;\rho)\,dx\right\} \\
&= \rho + \frac{3}{4}\int_0^1 x^{(n-5)/2}(-2\rho x^2)(1-\rho^2x^2)^{1/2}F_n^{-1/2}\,dx \\
&\quad - \frac{1}{4}\int_0^1 x^{(n-5)/2}(1-\rho^2x^2)^{3/2} \\
&\quad \times F_n^{-3/2}\{-2\rho x + 2n(1-x)x^{2n-1}\rho^{2n-1}\}\,dx \\
&= \rho - \frac{3\rho}{2}\int_0^1 x^{(n-1)/2}(1-\rho^2x^2)^{1/2}F_n^{-1/2}\,dx \\
&\quad + \frac{\rho}{2}\int_0^1 x^{(n-3)/2}(1-\rho^2x^2)^{3/2}F_n^{-3/2}\,dx \\
&\quad - \frac{n\rho^{2n-1}}{2}\int_0^1 x^{(5n-7)/2}(1-\rho^2x^2)^{3/2}F_n^{-3/2}(1-x)\,dx.
\end{aligned}
$$

For $|\rho| > 1$, we have

$$
\begin{aligned}
\text{(A.18)} \quad & E(\hat{\rho}_n - \rho) \\
&= \frac{\partial}{\partial\rho}\int_0^\infty D_n(q)^{-1/2}dq, \\
&= \frac{\partial}{\partial\rho}\int_1^\infty \left(\frac{G_n(x;\rho)}{(\rho^2x^2-1)x^{n-1}}\right)^{-1/2}\frac{(\rho^2x^2-1)}{2x^2}\,dx \\
&= \frac{1}{2}\frac{\partial}{\partial\rho}\int_1^\infty x^{(n-5)/2}(\rho^2x^2-1)^{3/2}G_n(x:\rho)^{-1/2}\,dx
\end{aligned}
$$

and, by direct evaluation,

$$
\begin{aligned}
&\frac{\partial}{\partial\rho}\left\{\int_1^\infty x^{(n-5)/2}(\rho^2x^2-1)^{3/2}G_n^{-1/2}\,dx\right\} \\
&= \frac{3}{2}\int_1^\infty x^{(n-5)/2}(2\rho x^2)(\rho^2x^2-1)^{1/2}G_n^{-1/2}\,dx \\
&\quad - \frac{1}{2}\int_1^\infty x^{(n-5)/2}(\rho^2x^2-1)^{3/2} \\
&\quad \times G_n^{-3/2}\{2\rho x + 2n(x-1)x^{2n-1}\rho^{2n-1}\}\,dx
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{3}{2}\int_1^\infty x^{(n-5)/2}(2\rho x^2)(\rho^2x^2-1)^{1/2}G_n^{-1/2}\,dx \\
&\quad - \rho\int_1^\infty x^{(n-3)/2}(\rho^2x^2-1)^{3/2}G_n^{-3/2}\,dx \\
&\quad - n\rho^{2n-1}\int_1^\infty x^{(n-5)/2}(\rho^2x^2-1)^{3/2}G_n^{-3/2}(x-1)x^{2n-1}\,dx.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
b_n(\rho) &= \rho + \frac{1}{2}\frac{\partial}{\partial\rho}\left\{\int_1^\infty x^{(n-5)/2}(\rho^2x^2-1)^{3/2}G_n^{-1/2}\,dx\right\} \\
&= \rho + \frac{3}{4}\int_1^\infty x^{(n-5)/2}(2\rho x^2)(\rho^2x^2-1)^{1/2}G_n^{-1/2}\,dx \\
&\quad - \frac{\rho}{2}\int_1^\infty x^{(n-3)/2}(\rho^2x^2-1)^{3/2}G_n^{-3/2}\,dx \\
&\quad - \frac{n\rho^{2n-1}}{2}\int_1^\infty x^{(n-5)/2}(\rho^2x^2-1)^{3/2}G_n^{-3/2}(x-1)x^{2n-1}\,dx.
\end{aligned}
$$

Hence, the binding formula for $|\rho| > 1$ is

$$
\begin{aligned}
\text{(A.19)}\quad b_n(\rho) &= \rho + \frac{3\rho}{2}\int_1^\infty x^{(n-1)/2}(\rho^2x^2-1)^{1/2}G_n^{-1/2}\,dx \\
&\quad - \frac{\rho}{2}\int_1^\infty x^{(n-3)/2}(\rho^2x^2-1)^{3/2}G_n^{-3/2}\,dx \\
&\quad - \frac{n\rho^{2n-1}}{2}\int_1^\infty x^{(5n-7)/2}(\rho^2x^2-1)^{3/2}G_n^{-3/2}(x-1)\,dx.
\end{aligned}
$$

Transforming using $y = 1/x$, and noting that

$$
\begin{aligned}
G_n\left(\frac{1}{y};\rho\right) &= \rho^2\frac{1}{y} - 1 + \left(\frac{1}{y}-1\right)y^{-2n+1}\rho^{2n} \\
&= \frac{(\rho^2-y)y^{2n-1} + (1-y)\rho^{2n}}{y^{2n}} =: \frac{H_n(y;\rho)}{y^{2n}},
\end{aligned}
$$

we have the alternate form

$$
\begin{aligned}
\text{(A.20)}\quad b_n(\rho) &= \rho + \frac{3\rho}{2}\int_0^1 y^{-(n-1)/2}\frac{(\rho^2-y^2)^{1/2}}{y}H_n^{-1/2}(y;\rho)y^{n-2}\,dy \\
&\quad - \frac{\rho}{2}\int_0^1 y^{-(n-3)/2}\frac{(\rho^2-y^2)^{3/2}}{y^3}H_n^{-3/2}(y;\rho)y^{3n-2}\,dy
\end{aligned}
$$

$$- \frac{n\rho^{2n-1}}{2} \int_0^1 y^{-(5n-7)/2} \frac{(\rho^2 - y^2)^{3/2}}{y^3} H_n^{-3/2}(y;\rho) y^{3n-3}(1-y)\,dy$$

$$= \rho + \frac{3\rho}{2} \int_0^1 y^{(n-5)/2}(\rho^2 - y^2)^{1/2} H_n^{-1/2}(y;\rho)\,dy$$

$$- \frac{\rho}{2} \int_0^1 y^{(5n-7)/2}(\rho^2 - y^2)^{3/2} H_n^{-3/2}(y;\rho)\,dy$$

$$- \frac{n\rho^{2n-1}}{2} \int_0^1 y^{(n-5)/2}(\rho^2 - y^2)^{3/2} H_n^{-3/2}(y;\rho)(1-y)\,dx.$$

Q.E.D.

PROOF OF THEOREM 4:

(i) *Case* $|\rho| < 1$. Using $F_n = 1 - \rho^2 x + (1-x)x^{2n-1}\rho^{2n} = 1 - \rho^2 x + O(\rho^{2n})$ and $n\rho^n = o(n^{-2})$, we have, for $|\rho| < 1$ from (A.17) and Lemma 3,

$$b_n(\rho) = \rho - \frac{3\rho}{2} \int_0^1 x^{(n-1)/2}(1 - \rho^2 x^2)^{1/2} F_n^{-1/2}\,dx$$

$$+ \frac{\rho}{2} \int_0^1 x^{(n-3)/2}(1 - \rho^2 x^2)^{3/2} F_n^{-3/2} + o(n^{-1})$$

$$= \rho - \frac{3\rho}{2} \int_0^1 x^{(n-1)/2}(1 - \rho^2 x^2)^{1/2}(1 - \rho^2 x)^{-1/2}\,dx$$

$$+ \frac{\rho}{2} \int_0^1 x^{(n-3)/2}(1 - \rho^2 x^2)^{3/2}(1 - \rho^2 x)^{-3/2}\,dx + o(n^{-2})$$

$$= \rho - \frac{2\rho}{n} + O(n^{-2}),$$

giving the well known asymptotic bias formula for $\hat{\rho}_n$ in the stationary case.

(ii) *Case* $\rho = \pm 1$. When $\rho = 1$, we have, $F_n(x;1) = 1 - x + (1-x)x^{2n-1} = (1-x)(1+x^{2n-1})$ and so, from (A.17),

$$b_n(1) = 1 - \frac{3}{2} \int_0^1 x^{(n-1)/2} \frac{(1+x)^{1/2}}{(1+x^{2n-1})^{1/2}}\,dx$$

$$+ \frac{1}{2} \int_0^1 x^{(n-3)/2} \frac{(1+x)^{3/2}}{(1+x^{2n-1})^{3/2}}\,dx$$

$$- \frac{n}{2} \int_0^1 x^{(5n-7)/2} \frac{(1+x)^{3/2}}{(1+x^{2n-1})^{3/2}}(1-x)\,dx.$$

Using (A.2) and (A.3), we have

$$\int_0^1 x^{(n-1)/2} \frac{(1+x)^{1/2}}{(1+x^{2n-1})^{1/2}}\,dx = \frac{2^{1/2}}{2n}\int_0^1 y^{-3/4}(1+y)^{-1/2}\,dy + O(n^{-2}),$$

$$\int_0^1 x^{(n-3)/2} \frac{(1+x)^{3/2}}{(1+x^{2n-1})^{3/2}}\,dx = \frac{2^{3/2}}{2n}\int_0^1 y^{-3/4}(1+y)^{-3/2}\,dy + O(n^{-2}),$$

$$n\int_0^1 x^{(n-1)/2} \frac{(1+x)^{3/2}}{(1+x^{2n-1})^{3/2}}(1-x)x^{2n-3}\,dx$$
$$= -\frac{2^{3/2}}{4n}\int_0^1 y^{1/4}(1+y)^{-3/2}\log y\,dy + O(n^{-2}).$$

It follows that

$$\text{(A.21)} \quad b_n(1) = 1 - 3\frac{2^{1/2}}{4n}\int_0^1 y^{-3/4}(1+y)^{-1/2}\,dy + \frac{2^{3/2}}{4n}\int_0^1 y^{-3/4}(1+y)^{-3/2}\,dy$$
$$- \frac{1}{2}\left\{-\frac{2^{3/2}}{4n}\int_0^1 y^{1/4}(1+y)^{-3/2}\log y\,dy\right\} + O(n^{-2}),$$

and numerical evaluation of the integrals gives

$$\text{(A.22)} \quad b_n(1) = 1 - \frac{3}{4n}2^{0.5}(3.7081) + \frac{2^{1.5}}{4n}(3.2683) - \frac{2^{1/2}}{4n}(0.45077) + O(n^{-2})$$
$$= 1 - \frac{1.7814}{n} + O(n^{-2}),$$

corresponding to the result found by SJ for the unit root case using different methods. The numerical value -1.7814 is the mean of the limit distribution of $n(\hat{\rho}_n - 1)$ when $\rho = 1$.

Similar calculations apply when $\rho = -1$, in which case we have

$$\text{(A.23)} \quad b_n(-1) = -1 + \frac{3}{4n}2^{0.5}(3.7081) - \frac{2^{1.5}}{4n}(3.2683)$$
$$+ \frac{2^{1/2}}{4n}(0.45077) + O(n^{-2})$$
$$= -1 + \frac{1.7814}{n} + O(n^{-2}),$$

giving the mirror image of (A.22).

(iii) *Case* $\rho = 1 + \frac{c}{n}$, $c < 0$. We consider the local to unity case with $\rho = 1 + \frac{c}{n}$, $c < 0$, and $|\rho| < 1$. The following arguments allow c to be fixed and $c \to -\infty$

as $n \to \infty$ with $c = o(n)$. The relevant expression for the bias when $|\rho| < 1$ is

$$b_n(\rho) = \rho - \frac{3\rho}{2}\int_0^1 x^{(n-1)/2}(1-\rho^2x^2)^{1/2}F_n^{-1/2}\,dx + \frac{\rho}{2}\int_0^1 x^{(n-3)/2}(1-\rho^2x^2)^{3/2}F_n^{-3/2}\,dx - \frac{n\rho^{2n-1}}{2}\int_0^1 x^{(5n-7)/2}(1-\rho^2x^2)^{3/2}F_n^{-3/2}(1-x)\,dx. \tag{A.24}$$

As before, set $y = x^{2n-1}$ so that $dy = (2n-1)x^{2n-2}\,dx = (2n-1) \times y^{(2n-2)/(2n-1)}\,dx = (2n-1)y^{1-1/(2n-1)}\,dx$. Then, using $F_n = 1 - \rho^2 x + (1 - x)x^{2n-1}\rho^{2n}$, we have for the first integral in (A.24),

$$\begin{aligned}
&\int_0^1 x^{(n-1)/2}(1-\rho^2x^2)^{1/2}F_n^{-1/2}\,dx \\
&= \frac{1}{2n-1}\int_0^1 y^{(n+1)/(4n-4)-1}\bigl(1-\rho^2 y^{2/(2n-1)}\bigr)^{1/2} \\
&\quad\times \bigl\{\bigl(1-\rho^2 y^{1/(2n-1)}\bigr) + \bigl(1-y^{1/(2n-1)}\bigr)y\rho^{2n}\bigr\}^{-1/2}\,dy \\
&= \frac{1}{2n-1}\int_0^1 y^{-3/4+1/(2n-2)}\bigl(1-\rho^2 y^{2/(2n-1)}\bigr)^{1/2} \\
&\quad\times \bigl\{\bigl(1-\rho^2 y^{1/(2n-1)}\bigr) + \bigl(1-y^{1/(2n-1)}\bigr)y\rho^{2n}\bigr\}^{-1/2}\,dy.
\end{aligned}$$

Since $y^{b/(2n+a)} = 1 + \frac{b}{2n+a}\log y + O(n^{-2})$, $\rho^2 = 1 + \frac{2c}{n} + \frac{c^2}{n^2}$, $\rho^{2n} = (1+\frac{c}{n})^{2n} = e^{2c}\{1 + O(\frac{c^2}{n})\}$, and $c < 0$ with $\frac{c}{n} = o(1)$, it follows that

$$\begin{aligned}
&\int_0^1 x^{(n-1)/2}(1-\rho^2x^2)^{1/2}F_n^{-1/2}\,dx \\
&= \frac{1}{2n-1}\int_0^1 y^{-3/4}\bigl(1-\rho^2 y^{2/(2n-1)}\bigr)^{1/2} \\
&\quad\times \bigl\{\bigl(1-\rho^2 y^{1/(2n-1)}\bigr) + \bigl(1-y^{1/(2n-1)}\bigr)y\rho^{2n}\bigr\}^{-1/2}\,dy\{1+O(n^{-1})\} \\
&= \frac{1}{2n}\int_0^1 y^{-3/4}\left\{\frac{1-\rho^2 y^{1/(2n-1)}}{1-\rho^2 y^{2/(2n-1)}} + \frac{1-y^{1/(2n-1)}}{1-\rho^2 y^{2/(2n-1)}}y\rho^{2n}\right\}^{-1/2}\,dy \\
&\quad\times\{1+O(n^{-1})\}
\end{aligned} \tag{A.25}$$

$$= \frac{1}{2n}\int_0^1 y^{-3/4}\left\{\frac{1-\rho^2\left(1+\frac{1}{2n}\log y\right)}{1-\rho^2\left(1+\frac{1}{n}\log y\right)}\right.$$

$$\left. - \frac{\frac{1}{2n}\log y}{1-\rho^2\left(1+\frac{1}{n}\log y\right)} y\rho^{2n}\right\}^{-1/2} dy$$

$$\times \{1+O(n^{-1})\}$$

$$= \frac{1}{2n}\int_0^1 y^{-3/4}\left\{\frac{-\frac{2c}{n}-\frac{1}{2n}\log y-\left(\frac{c}{n}\right)^2}{-\frac{2c}{n}-\frac{1}{n}\log y-\left(\frac{c}{n}\right)^2}\right.$$

$$\left. - \frac{\frac{1}{2n}\log y}{-\frac{2c}{n}-\frac{1}{n}\log y-\left(\frac{c}{n}\right)^2} ye^{2c}\left[1+O\left(\frac{c^2}{n}\right)\right]\right\}^{-1/2} dy$$

$$\times \{1+O(n^{-1})\}$$

$$= \frac{1}{2n}\int_0^1 y^{-3/4}\left\{\frac{(4c+\log y)\left(1+\frac{2c^2}{n(4c+\log y)}\right)}{(4c+2\log y)\left(1+\frac{2c^2}{n(4c+2\log y)}\right)}\right.$$

$$\left. + \frac{\log y}{(4c+2\log y)\left(1+\frac{2c^2}{n(4c+2\log y)}\right)} ye^{2c}\right\}^{-1/2} dy$$

$$\times \{1+O(n^{-1})\}$$

$$= \frac{1}{2n}\int_0^1 y^{-3/4}\left\{\frac{(4c+\log y)}{(4c+2\log y)}+\frac{\log y}{(4c+2\log y)}ye^{2c}\right\}^{-1/2} dy$$

$$\times \left\{1+O\left(\frac{c}{n}\right)\right\}.$$

The second integral in (A.24) can be reduced in the same way. Again, setting $y = x^{2n-1}$ with $dy = (2n-1)y^{1-1/(2n-1)}\,dx$ and $y^{b/(2n+a)} = 1+\frac{b}{2n+a}\log y + O(n^{-2})$

and using $c = o(n)$, we have

$$\begin{aligned}
\text{(A.26)}\quad & \int_0^1 x^{(n-3)/2}(1-\rho^2 x^2)^{3/2} F_n^{-3/2}\,dx \\
&= \frac{1}{2n-1}\int_0^1 y^{-3/4-(1/4)/(2n-2)}\big(1-\rho^2 y^{2/(2n-1)}\big)^{3/2} \\
&\quad \times \big\{\big(1-\rho^2 y^{1/(2n-1)}\big) + \big(1-y^{1/(2n-1)}\big) y\rho^{2n}\big\}^{-3/2}\,dy \\
&= \frac{1}{2n}\int_0^1 y^{-3/4}\left\{\frac{1-\rho^2 y^{1/(2n-1)}}{1-\rho^2 y^{2/(2n-1)}} + \frac{1-y^{1/(2n-1)}}{1-\rho^2 y^{2/(2n-1)}} y\rho^{2n}\right\}^{-3/2} dy \\
&\quad \times \{1+O(n^{-1})\} \\
&= \frac{1}{2n}\int_0^1 y^{-3/4}\left\{\frac{1-\rho^2\left(1+\frac{1}{2n}\log y\right)}{1-\rho^2\left(1+\frac{1}{n}\log y\right)}\right. \\
&\quad \left. - \frac{\frac{1}{2n}\log y}{1-\rho^2\left(1+\frac{1}{n}\log y\right)} y\rho^{2n}\right\}^{-3/2} dy\{1+O(n^{-1})\} \\
&= \frac{1}{2n}\int_0^1 y^{-3/4}\left\{\frac{-\frac{2c}{n}-\frac{1}{2n}\log y}{-\frac{2c}{n}-\frac{1}{n}\log y} - \frac{\frac{1}{2n}\log y}{-\frac{2c}{n}-\frac{1}{n}\log y} y e^{2c}\right\}^{-3/2} dy \\
&\quad \times \left\{1+O\left(\frac{c}{n}\right)\right\} \\
&= \frac{1}{2n}\int_0^1 y^{-3/4}\left\{\frac{4c+\log y}{4c+2\log y} + \frac{\log y}{4c+2\log y} y e^{2c}\right\}^{-3/2} dy \\
&\quad \times \left\{1+O\left(\frac{c}{n}\right)\right\}.
\end{aligned}$$

Finally, for the third integral in (A.24), we have in the same fashion,

$$\begin{aligned}
\text{(A.27)}\quad & \int_0^1 x^{(5n-7)/2}(1-\rho^2 x^2)^{3/2} F_n^{-3/2}(1-x)\,dx \\
&= \frac{1}{(2n-1)^2}\int_0^1 y^{(5n-7)/(4n-2)-1+1/(2n-1)}\big(1-\rho^2 y^{2/(2n-1)}\big)^{3/2}
\end{aligned}$$

$$\times \left\{\left(1-\rho^2 y^{1/(2n-1)}\right)+\left(1-y^{1/(2n-1)}\right)y\rho^{2n}\right\}^{-3/2}\log y\,dy\{1+O(n^{-1})\}$$

$$= \frac{1}{4n^2}\int_0^1 y^{(n-3)/(4n-2)}\left(1-\rho^2 y^{2/(2n-1)}\right)^{3/2}$$

$$\times \left\{\left(1-\rho^2 y^{1/(2n-1)}\right)+\left(1-y^{1/(2n-1)}\right)y\rho^{2n}\right\}^{-3/2}\log y\,dy\{1+O(n^{-1})\}$$

$$= \frac{1}{4n^2}\int_0^1 y^{1/4}\left\{\frac{1-\rho^2 y^{1/(2n-1)}}{1-\rho^2 y^{2/(2n-1)}}+\frac{1-y^{1/(2n-1)}}{1-\rho^2 y^{2/(2n-1)}}y\rho^{2n}\right\}^{-3/2}\log y\,dy$$

$$\times \{1+O(n^{-1})\}$$

$$= \frac{1}{4n^2}\int_0^1 y^{1/4}\left\{\frac{1-\left(1+\frac{2c}{n}\right)\left(1+\frac{1}{2n}\log y\right)}{1-\left(1+\frac{2c}{n}\right)\left(1+\frac{1}{n}\log y\right)}\right.$$

$$\left.-\frac{\frac{1}{2n}\log y}{1-\left(1+\frac{2c}{n}\right)\left(1+\frac{1}{n}\log y\right)}y\rho^{2n}\right\}^{-3/2}\log y\,dy\left\{1+O\left(\frac{c}{n}\right)\right\}$$

$$= \frac{1}{4n^2}\int_0^1 y^{1/4}\left\{\frac{-\frac{2c}{n}-\frac{1}{2n}\log y}{-\frac{2c}{n}-\frac{1}{n}\log y}-\frac{\frac{1}{2n}\log y}{-\frac{2c}{n}-\frac{1}{n}\log y}ye^{2c}\right\}^{-3/2}$$

$$\times \log y\,dy\left\{1+O\left(\frac{c}{n}\right)\right\}$$

$$= \frac{1}{4n^2}\int_0^1 y^{1/4}\left\{\frac{4c+\log y}{4c+2\log y}+\frac{\log y}{4c+2\log y}ye^{2c}\right\}^{-3/2}$$

$$\times \log y\,dy\left\{1+O\left(\frac{c}{n}\right)\right\}.$$

Combining results (A.25)–(A.27) gives the following approximation to the binding function for $\rho = 1+\frac{c}{n}$ with $c<0$, $c=o(n)$, and $|\rho|<1$:

(A.28)
$$b_n(\rho) = \rho - \frac{3\rho}{2}\int_0^1 x^{(n-1)/2}(1-\rho^2x^2)^{1/2}F_n^{-1/2}\,dx$$

$$+\frac{\rho}{2}\int_0^1 x^{(n-3)/2}(1-\rho^2x^2)^{3/2}F_n^{-3/2}\,dx$$

$$-\frac{n\rho^{2n-1}}{2}\int_0^1 x^{(5n-7)/2}(1-\rho^2x^2)^{3/2}F_n^{-3/2}(1-x)\,dx$$

$$
\begin{aligned}
&= \rho - \frac{3\rho}{4n} \int_0^1 y^{-3/4} \left\{ \frac{4c + \log y}{4c + 2\log y} + \frac{\log y}{4c + 2\log y} y e^{2c} \right\}^{-1/2} dy \\
&\quad \times \left\{ 1 + O\left(\frac{c}{n}\right) \right\} \\
&\quad + \frac{\rho}{4n} \int_0^1 y^{-3/4} \left\{ \frac{4c + \log y}{4c + 2\log y} + \frac{\log y}{4c + 2\log y} y e^{2c} \right\}^{-3/2} dy \\
&\quad \times \left\{ 1 + O\left(\frac{c}{n}\right) \right\} \\
&\quad + \frac{\rho^{2n-1}}{8n} \int_0^1 y^{1/4} \left\{ \frac{4c + \log y}{4c + 2\log y} + \frac{\log y}{4c + 2\log y} y e^{2c} \right\}^{-3/2} \\
&\quad \times \log y \, dy \left\{ 1 + O\left(\frac{c}{n}\right) \right\}.
\end{aligned}
$$

Observe the sign change in the last term because of the transformation in the integrand that involves $\log y$, which is negative for $y \in (0, 1)$, so the whole expression is negative.

The approximation (A.28) holds with an error of $O(n^{-2})$ uniformly for all c in compact sets of $\mathbb{R}_- = (-\infty, 0)$ when $n \to \infty$. It also holds with a relative error of $O(\frac{c}{n})$ when $c \to -\infty$ provided $c = o(n)$. In this case, terms involving e^{2c} become exponentially small. The approximation to the binding function when $n \to \infty$ and $c \to -\infty$ while $|\rho| = |1 + \frac{c}{n}| < 1$ and $c = o(n)$ as $n \to \infty$ has the form

$$
\begin{aligned}
\text{(A.29)} \quad b_n(\rho) &= \rho - \frac{3\rho}{4n} \int_0^1 y^{-3/4} \, dy + \frac{\rho}{4n} \int_0^1 y^{-3/4} \, dy + O\left(\frac{c}{n^2}\right) \\
&= \rho - \frac{\rho}{2n} \left[\frac{y^{1/4}}{1/4} \right]_0^1 + O\left(\frac{c}{n^2}\right) = \rho - \frac{2\rho}{n} + O\left(\frac{c}{n^2}\right),
\end{aligned}
$$

so that the leading term in the approximation corresponds to the case of fixed ρ with $|\rho| < 1$. From (A.29), the binding function is linear in ρ as $n \to \infty$ and $c \to -\infty$ with $|\rho| < 1$.

Next, when $c = 0$, we have

$$
\begin{aligned}
b_n(1) &= 1 - \frac{3}{4n} \int_0^1 y^{-3/4} \left\{ \frac{1}{2} + \frac{1}{2} y \right\}^{-1/2} dy \\
&\quad + \frac{1}{4n} \int_0^1 y^{-3/4} \left\{ \frac{1}{2} + \frac{1}{2} y \right\}^{-3/2} dy
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{8n}\int_0^1 y^{1/4}\left\{\frac{1}{2}+\frac{1}{2}y\right\}^{-3/2}\log y\,dy+O(n^{-2})\\
&=1-\frac{3}{4n}2^{1/2}\int_0^1 y^{-3/4}\{1+y\}^{-1/2}\,dy\\
&\quad+\frac{2^{3/2}}{4n}\int_0^1 y^{-3/4}\{1+y\}^{-3/2}\,dy\\
&\quad+\frac{2^{1/2}}{4n}\int_0^1 y^{1/4}\{1+y\}^{-3/2}\log y\,dy+O(n^{-2})\\
&=1-\frac{1.7814}{n}+O(n^{-2}),
\end{aligned}
$$

corresponding to (A.21). Thus, (A.28) encompasses both the stationary and unit root cases at the limits of the domain of definition for $c<0$.

(iv) *Case* $\rho=1+\frac{c}{n}>1$, $c>0$. We start with the local to unity case $\rho=1+c/n$ with $c>0$ and later consider fixed $\rho>1$. We allow for the case where $c\to\infty$ under the condition $c=o(n)$. The binding function formula for $\rho>1$ is

$$
\begin{aligned}
\text{(A.30)}\quad b_n(\rho)=\rho&+\frac{3\rho}{2}\int_1^\infty x^{(n-1)/2}(\rho^2x^2-1)^{1/2}G_n^{-1/2}\,dx\\
&-\frac{\rho}{2}\int_1^\infty x^{(n-3)/2}(\rho^2x^2-1)^{3/2}G_n^{-3/2}\,dx\\
&-\frac{n\rho^{2n-1}}{2}\int_1^\infty x^{(n-5)/2}(\rho^2x^2-1)^{3/2}G_n^{-3/2}(x-1)x^{2n-1}\,dx.
\end{aligned}
$$

We proceed by taking each term in turn. As before, set $y=x^{2n-1}$ so that

$$
\begin{aligned}
dy&=(2n-1)x^{2n-2}\,dx=(2n-1)y^{(2n-2)/(2n-1)}\,dx\\
&=(2n-1)y^{1-1/(2n-1)}\,dx,
\end{aligned}
$$

and use the expansion $y^{1/(2n-1)}=1+\frac{1}{2n-1}\log y+O(n^{-2})$. In $G_n=\rho^2x-1+(x-1)x^{2n-1}\rho^{2n}$, we have $\rho=1+\frac{c}{n}$ with $c>0$ and in what follows, we allow for $c\to\infty$ such that $c=o(n)$. As before, $\rho^2=1+\frac{2c}{n}+\frac{c^2}{n^2}$ and $\rho^{2n}=(1+\frac{c}{n})^{2n}=e^{2c}\{1+O(\frac{c^2}{n})\}$. The integral in the second term of (A.30) is then

$$
\begin{aligned}
\text{(A.31)}\quad &\int_1^\infty x^{(n-1)/2}(\rho^2x^2-1)^{1/2}G_n^{-1/2}\,dx\\
&=\int_1^\infty x^{(n-1)/2}\left\{\frac{\rho^2x^2-1}{\rho^2x-1+(x-1)x^{2n-1}\rho^{2n}}\right\}^{1/2}dx
\end{aligned}
$$

$$= \int_1^\infty y^{((n-1)/2)/(2n-1)} \left\{ \frac{\rho^2 y^{2/(2n-1)} - 1}{\rho^2 y^{1/(2n-1)} - 1 + (y^{1/(2n-1)} - 1) y \rho^{2n}} \right\}^{1/2}$$

$$\times \frac{dy}{(2n-1) y^{(2n-2)/(2n-1)}}$$

$$= \frac{1}{2n} \int_1^\infty y^{-3/4}$$

$$\times \left\{ \frac{\left(1 + \frac{2c}{n} + \frac{c^2}{n^2}\right)\left(1 + \frac{2}{2n}\log y\right) - 1}{\left(1 + \frac{2c}{n} + \frac{c^2}{n^2}\right)\left(1 + \frac{1}{2n}\log y\right) - 1 + \frac{\rho^{2n}}{2n} y \log y} \right\}^{1/2} dy$$

$$\times \{1 + O(n^{-1})\}$$

$$= \frac{1}{2n} \int_1^\infty y^{-3/4}$$

$$\times \left\{ \frac{(4c + 2\log y)\left(1 + \frac{2c^2}{n(4c + 2\log y)}\{1 + o(1)\}\right)}{(4c + \log y + e^{2c} y \log y)\left(1 + O\left(\frac{e^{-2c}c^2}{n}\right)\right)} \right\}^{1/2} dy$$

$$\times \{1 + O(n^{-1})\}$$

$$= \frac{1}{2n} \int_1^\infty y^{-3/4} \left\{ \frac{(4c + 2\log y)}{4c + \log y + e^{2c} y \log y} \right\}^{1/2} dy \left\{ 1 + O\left(\frac{c}{n}\right) \right\}.$$

Use the transformation $w = \log y$ so that $w \in [0, \infty)$ and $dy = e^w \, dw$, giving for the leading term of (A.31),

$$\text{(A.32)} \quad \frac{1}{2n} \int_1^\infty y^{-3/4} \left\{ \frac{4c + 2\log y}{4c + \log y + e^{2c} y \log y} \right\}^{1/2} dy$$

$$= \frac{1}{2n} \int_0^\infty e^{(1/4)w} \left\{ \frac{4c + 2w}{4c + w + e^{2c} w e^w} \right\}^{1/2} dw.$$

Proceeding in the same way with the integral in the third term of (A.30), we have

$$\text{(A.33)} \quad \int_1^\infty x^{(n-3)/2} (\rho^2 x^2 - 1)^{3/2} G_n^{-3/2} \, dx$$

$$= \int_1^\infty x^{(n-3)/2} \left\{ \frac{\rho^2 x^2 - 1}{\rho^2 x - 1 + (x-1) x^{2n-1} \rho^{2n}} \right\}^{3/2} dx$$

$$= \int_1^\infty y^{((n-3)/2)/(2n-1)} \left\{ \frac{\rho^2 y^{2/(2n-1)} - 1}{\rho^2 y^{1/(2n-1)} - 1 + (y^{1/(2n-1)} - 1) y \rho^{2n}} \right\}^{3/2}$$
$$\times \frac{dy}{(2n-1) y^{(2n-2)/(2n-1)}}$$
$$= \frac{1}{2n} \int_1^\infty y^{-3/4}$$
$$\times \left\{ \frac{\left(1 + \frac{2c}{n} + \frac{c^2}{n^2}\right)\left(1 + \frac{2}{2n} \log y\right) - 1}{\left(1 + \frac{2c}{n} + \frac{c^2}{n^2}\right)\left(1 + \frac{1}{2n} \log y\right) - 1 + \frac{\rho^{2n}}{2n} y \log y} \right\}^{3/2} dy$$
$$\times \{1 + O(n^{-1})\}$$
$$= \frac{1}{2n} \int_1^\infty y^{-3/4} \left\{ \frac{4c + 2 \log y}{4c + \log y + e^{2c} y \log y} \right\}^{3/2} dy \left\{ 1 + O\left(\frac{c}{n}\right) \right\}$$
$$= \frac{1}{2n} \int_0^\infty e^{1/4w} \left\{ \frac{4c + 2w}{4c + w + e^{2c} w e^w} \right\}^{3/2} dw \left\{ 1 + O\left(\frac{c}{n}\right) \right\}.$$

Finally, the integral in the fourth term of (A.30) is

$$\text{(A.34)} \quad \int_1^\infty x^{(n-5)/2} (\rho^2 x^2 - 1)^{3/2} G_n^{-3/2} (x-1) x^{2n-1}\, dx$$
$$= \int_1^\infty x^{(n-5)/2} \left\{ \frac{\rho^2 x^2 - 1}{\rho^2 x - 1 + (x-1) x^{2n-1} \rho^{2n}} \right\}^{3/2} (x-1) x^{2n-1}\, dx$$
$$= \int_1^\infty y^{((n-5)/2)/(2n-1)} \left\{ \frac{\rho^2 y^{2/(2n-1)} - 1}{\rho^2 y^{1/(2n-1)} - 1 + (y^{1/(2n-1)} - 1) y \rho^{2n}} \right\}^{3/2}$$
$$\times \frac{(y^{1/(2n-1)} - 1) y\, dy}{(2n-1) y^{(2n-2)/(2n-1)}}$$
$$= \frac{1}{2n} \int_1^\infty y^{1/4}$$
$$\times \left\{ \frac{\left(1 + \frac{2c}{n} + \frac{c^2}{n^2}\right)\left(1 + \frac{2}{2n} \log y\right) - 1}{\left(1 + \frac{2c}{n} + \frac{c^2}{n^2}\right)\left(1 + \frac{1}{2n} \log y\right) - 1 + \frac{\rho^{2n}}{2n} y \log y} \right\}^{3/2}$$
$$\times \left(\frac{1}{2n} \log y \right) dy \{1 + O(n^{-1})\}$$

$$= \left(\frac{1}{2n}\right)^2 \int_1^\infty y^{1/4} \left\{\frac{4c + 2\log y}{4c + \log y + e^{2c} y \log y}\right\}^{3/2} \log y \, dy \left\{1 + O\left(\frac{c}{n}\right)\right\}$$
$$= \left(\frac{1}{2n}\right)^2 \int_0^\infty e^{(5/4)w} \left\{\frac{4c + 2w}{4c + w + e^{2c} w e^w}\right\}^{3/2} w \, dw \left\{1 + O\left(\frac{c}{n}\right)\right\}.$$

Combining (A.32)–(A.34) in (A.30), we get for $\rho = 1 + \frac{c}{n}$ with $c > 0$,

$$b_n(\rho) = \rho + \frac{3\rho}{4n} \int_0^\infty e^{(1/4)w} \left\{\frac{4c + 2w}{4c + w + e^{2c} w e^w}\right\}^{1/2} dw \left\{1 + O\left(\frac{c}{n}\right)\right\}$$
$$- \frac{\rho}{4n} \int_0^\infty e^{(1/4)w} \left\{\frac{4c + 2w}{4c + w + e^{2c} w e^w}\right\}^{3/2} dw \left\{1 + O\left(\frac{c}{n}\right)\right\}$$
$$- \frac{\rho^{2n-1}}{8n} \int_0^\infty e^{(5/4)w} \left\{\frac{4c + 2w}{4c + w + e^{2c} w e^w}\right\}^{3/2} w \, dw \left\{1 + O\left(\frac{c}{n}\right)\right\}.$$

Hence the bias function to $O(n^{-1})$ in this case of local to unity on the explosive side of unity is

$$\text{(A.35)} \quad b_n\left(1 + \frac{c}{n}\right)$$
$$= 1 + \frac{c}{n} + \frac{3}{4n} \int_0^\infty e^{(1/4)w} \left\{\frac{4c + 2w}{4c + w + e^{2c} w e^w}\right\}^{1/2} dw \left\{1 + O\left(\frac{c}{n}\right)\right\}$$
$$- \frac{1}{4n} \int_0^\infty e^{(1/4)w} \left\{\frac{4c + 2w}{4c + w + e^{2c} w e^w}\right\}^{3/2} dw \left\{1 + O\left(\frac{c}{n}\right)\right\}$$
$$- \frac{\rho^{2n}}{8n} \int_0^\infty e^{(5/4)w} \left\{\frac{4c + 2w}{4c + w + e^{2c} w e^w}\right\}^{3/2} w \, dw \left\{1 + O\left(\frac{c}{n}\right)\right\}.$$

The approximation (A.35) holds with an error of $O(n^{-2})$ uniformly for all c in compact sets of $\mathbb{R}_+ = (0, \infty)$ when $n \to \infty$. The formula also produces a valid approximation to the binding function for $\rho = 1 + \frac{c}{n} > 1$ when $c \to \infty$ and $c = o(n)$ as $n \to \infty$. In this case, noting the relative error order in (A.35) and the behavior of $k^+(w; c)$ as $c \to \infty$, the approximation to the binding function has the form

$$\text{(A.36)} \quad b_n\left(1 + \frac{c}{n}\right) = 1 + \frac{c}{n} + O\left(\frac{1}{ne^c}\right),$$

so that for large c the binding function is approximately linear.

Combining (A.29) and (A.36), we deduce that the binding function $b_n(1 + \frac{c}{n})$ is approximately linear as $n \to \infty$ when $|c| \to \infty$ with $c = o(n)$.

(v) *Case* $|\rho| > 1$. We now turn to the case of fixed $\rho > 1$. The relevant bias expression is from (A.19):

$$
\text{(A.37)} \quad \begin{aligned} b_n(\rho) = \rho &+ \frac{3\rho}{2}\int_1^\infty x^{(n-1)/2}(\rho^2x^2-1)^{1/2}G_n^{-1/2}\,dx \\ &- \frac{\rho}{2}\int_1^\infty x^{(n-3)/2}(\rho^2x^2-1)^{3/2}G_n^{-3/2}\,dx \\ &- \frac{n\rho^{2n-1}}{2}\int_1^\infty x^{(n-5)/2}(\rho^2x^2-1)^{3/2}G_n^{-3/2}(x-1)x^{2n-1}\,dx. \end{aligned}
$$

We examine the order of magnitude of each term in turn as $n \to \infty$. For the first term,

$$
\begin{aligned}
&\int_1^\infty x^{(n-1)/2}(\rho^2x^2-1)^{1/2}G_n^{-1/2}\,dx \\
&= \int_1^\infty x^{(n-1)/2}\left\{\frac{\rho^2x^2-1}{\rho^2x-1+(x-1)x^{2n-1}\rho^{2n}}\right\}^{1/2}dx \\
&= \frac{1}{\rho^n}\int_1^\infty \frac{x^{(n-1)/2}}{x^{n-1/2}}\left\{\frac{\rho^2x^2-1}{(x-1)+\dfrac{\rho^2x-1}{x^{2n-1}\rho^{2n}}}\right\}^{1/2}dx \\
&\le \frac{B}{\rho^n}\int_1^\infty \frac{1}{x^{n/2}}\,dx = O(n^{-1}\rho^{-n}).
\end{aligned}
$$

In a similar way, the second term is

$$
\begin{aligned}
&\int_1^\infty x^{(n-3)/2}(\rho^2x^2-1)^{3/2}G_n^{-3/2}\,dx \\
&= \int_1^\infty x^{(n-3)/2}\left\{\frac{\rho^2x^2-1}{\rho^2x-1+(x-1)x^{2n-1}\rho^{2n}}\right\}^{3/2}dx \\
&= O(n^{-1}\rho^{-3n}).
\end{aligned}
$$

The third term is

$$
\begin{aligned}
&\frac{n\rho^{2n-1}}{2}\int_1^\infty x^{(n-5)/2}(\rho^2x^2-1)^{3/2}G_n^{-3/2}(x-1)x^{2n-1}\,dx \\
&= \frac{n\rho^{2n-1}}{2}\int_1^\infty x^{(n-5)/2}\left\{\frac{\rho^2x^2-1}{\rho^2x-1+(x-1)x^{2n-1}\rho^{2n}}\right\}^{3/2}(x-1)x^{2n-1}\,dx
\end{aligned}
$$

$$= \frac{n\rho^{-n-1}}{2}\int_1^\infty \frac{x^{(n-5)/2+2n-1}}{x^{3n-3/2}}\left\{\frac{\rho^2x^2-1}{(x-1)+\dfrac{\rho^2x-1}{x^{2n-1}\rho^{2n}}}\right\}^{3/2}(x-1)\,dx$$

$$= \frac{n\rho^{-n-1}}{2}\int_1^\infty \frac{1}{x^{n/2+2}}\left\{\frac{\rho^2x^2-1}{(x-1)+\dfrac{\rho^2x-1}{x^{2n-1}\rho^{2n}}}\right\}^{3/2}(x-1)\,dx$$

$$= O(\rho^{-n}),$$

and therefore dominates (A.37). It follows that $b_n(\rho) = \rho + O(\rho^{-n})$, showing that the bias is exponentially small (and negative, in view of the sign of the final term of (A.37)) for $\rho > 1$. A similar result holds when $\rho < -1$, in which case the bias is exponentially small and positive. *Q.E.D.*

PROOF OF THE ALTERNATE FORM OF $g^-(c)$: We need to show that the leading term $g^-(c)$ in the binding function when $\rho = 1 + c/n$ and $c < 0$ has the alternate form

$$g^-(c) = -\frac{3}{4}\int_0^\infty e^{-(1/4)v}k^-(v;c)^{1/2}\,dv + \frac{1}{4n}\int_0^\infty e^{-(1/4)v}k^-(v;c)^{3/2}\,dv - \frac{e^{2c}}{8}\int_0^\infty e^{-(5/4)v}k^-(v;c)^{3/2}v\,dv, \tag{A.38}$$

where

$$k^-(v;c) := \frac{4c-2v}{4c-v-e^{2c}ve^{-v}}. \tag{A.39}$$

We start with the following expression for $g^-(c)$ which is established in the paper (see (A.28) above):

$$g^-(c) = -\frac{3}{4}\int_0^1 y^{-3/4}\left\{\frac{4c+\log y}{4c+2\log y} + \frac{\log y}{4c+2\log y}ye^{2c}\right\}^{-1/2}dy + \frac{1}{4}\int_0^1 y^{-3/4}\left\{\frac{4c+\log y}{4c+2\log y} + \frac{\log y}{4c+2\log y}ye^{2c}\right\}^{-3/2}dy + \frac{e^{2c}}{8}\int_0^1 y^{1/4}\left\{\frac{4c+\log y}{4c+2\log y} + \frac{\log y}{4c+2\log y}ye^{2c}\right\}^{-3/2}\log y\,dy.$$

Transform $w = \log y$ so that $w \in [0, \infty)$ and $dy = e^{w}\,dw$, with range $w \in (-\infty, 0]$ for $x \in (0, 1]$, giving for $c \leq 0$,

$$
\begin{aligned}
\text{(A.40)} \quad g^{-}(c) &= -\frac{3}{4}\int_0^1 y^{-3/4}\left\{\frac{4c+\log y + ye^{2c}\log y}{4c+2\log y}\right\}^{-1/2} dy \\
&\quad + \frac{1}{4}\int_0^1 y^{-3/4}\left\{\frac{4c+\log y + ye^{2c}\log y}{4c+2\log y}\right\}^{-3/2} dy \\
&\quad + \frac{e^{2c}}{8}\int_0^1 y^{1/4}\left\{\frac{4c+\log y + ye^{2c}\log y}{4c+2\log y}\right\}^{-3/2} \log y\, dy \\
&= -\frac{3}{4}\int_{-\infty}^0 e^{(1/4)w}\left\{\frac{4c+2w}{4c+w+e^{2c}we^{w}}\right\}^{1/2} dw \\
&\quad + \frac{1}{4}\int_{-\infty}^0 e^{(1/4)w}\left\{\frac{4c+2w}{4c+w+e^{2c}we^{w}}\right\}^{3/2} dw \\
&\quad + \frac{e^{2c}}{8}\int_{-\infty}^0 e^{(5/4)w}\left\{\frac{4c+2w}{4c+w+e^{2c}we^{w}}\right\}^{3/2} w\, dw \\
&= -\frac{3}{4}\int_0^{\infty} e^{-(1/4)v}\left\{\frac{4c-2v}{4c-v-e^{2c}ve^{-v}}\right\}^{1/2} dv \\
&\quad + \frac{1}{4}\int_0^{\infty} e^{-(1/4)v}\left\{\frac{4c-2v}{4c-v-e^{2c}ve^{-v}}\right\}^{3/2} dv \\
&\quad - \frac{e^{2c}}{8}\int_0^{\infty} e^{-(5/4)v}\left\{\frac{4c-2v}{4c-v-e^{2c}ve^{-v}}\right\}^{3/2} v\, dv,
\end{aligned}
$$

which gives the stated result (A.38). Observe that for $v \geq 0$, we have

$$
k^{-}(v; c) = \frac{4c-2v}{4c-v-e^{2c}ve^{-v}} 1\{c < 0\} > 0,
$$

so the expressions with fractional exponents in (A.40) are all nonnegative quantities. At $c = 0$, we have

$$
\begin{aligned}
g^{-}(0) &= -\frac{3}{4}\int_0^{\infty} e^{(1/4)v}\left\{\frac{2v}{v+ve^{v}}\right\}^{1/2} dv + \frac{1}{4}\int_0^{\infty} e^{(5/4)v}\left\{\frac{2v}{v+ve^{v}}\right\}^{3/2} dv \\
&\quad - \frac{1}{8}\int_0^{\infty} e^{(1/4)v}\left\{\frac{2v}{v+ve^{v}}\right\}^{3/2} v\, dv \\
&= -\frac{3}{4}5.2441 + \frac{9.2441}{4} - \frac{1.275}{8} = -1.7814
\end{aligned}
$$

upon direct evaluation, reproducing the earlier result at $c = 0$. Further note that, as $c \to -\infty$, we have

$$\text{(A.41)} \qquad \lim_{c \to -\infty} g^{-}(c) = -\frac{2}{4} \int_0^\infty e^{-(1/4)v}\, dv = -2. \qquad \textit{Q.E.D.}$$

PROPERTIES OF $g^{-}(c)$: We start with some properties of the functions $k^{-}(v; c)$ and $g^{-}(c)$ in (A.39) and (A.38), namely

$$k^{-}(v; c) := \frac{4c - 2v}{4c - v - e^{2c} v e^{-v}} = \frac{2v - 4c}{v + e^{2c} v e^{-v} - 4c},$$

$$g^{-}(c) = -\frac{3}{4} \int_0^\infty e^{-(1/4)v} k^{-}(v; c)^{1/2}\, dv + \frac{1}{4} \int_0^\infty e^{-(1/4)v} k^{-}(v; c)^{3/2}\, dv - \frac{e^{2c}}{8} \int_0^\infty e^{-(5/4)v} k^{-}(v; c)^{3/2} v\, dv.$$

First note the limits at the domain of definition of c,

$$\lim_{c \to 0} k^{-}(v; c) = \frac{2}{1 + e^{-v}} \le 2,$$

$$\lim_{c \to -\infty} k^{-}(v; c) = 1,$$

and so

$$\lim_{c \to -\infty} g^{-}(c) = -\frac{2}{4} \int_0^\infty e^{-(1/4)v}\, dv = -2$$

as in (A.41) above. Next note that $v + e^{2c} v e^{-v} - 4c \le 2v - 4c$ as $e^{2c} e^{-v} < 1$, so that

$$k^{-}(v; c) \ge \frac{2v - 4c}{v + v e^{-v} - 4c} \ge \frac{2v - 4c}{2v - 4c} = 1,$$

and since $e^{2c} v e^{-v} > 0$ and $-2c \ge 0$,

$$k^{-}(v; c) \le \frac{2v - 4c}{v - 4c} \le \frac{2v - 4c}{v - 2c} = 2,$$

we have

$$\text{(A.42)} \qquad k^{-}(v; c) \in [1, 2].$$

Next consider the derivative $k_c^-(v;c) = \frac{\partial}{\partial c} k^-(v;c)$, which has the form

$$
\begin{aligned}
\text{(A.43)} \quad k_c^-(v;c) &= \frac{(v + e^{2c} v e^{-v} - 4c)(-4) - (2v - 4c)(2e^{2c} v e^{-v} - 4)}{(v + e^{2c} v e^{-v} - 4c)^2} \\
&= \frac{(v + e^{2c} v e^{-v} - 4c)(-4) - 2(2v - 4c) e^{2c} v e^{-v} + 4(2v - 4c)}{(v + e^{2c} v e^{-v} - 4c)^2} \\
&= \frac{4v - 4e^{2c} v e^{-v} - 4e^{2c} v^2 e^{-v} + 8c e^{2c} v e^{-v}}{(v + e^{2c} v e^{-v} - 4c)^2} \\
&= 4 \frac{v - e^{2c} v(1+v) e^{-v} + 2c e^{2c} v e^{-v}}{(v + e^{2c} v e^{-v} - 4c)^2} \\
&= 4v \frac{1 - e^{2c}(1+v) e^{-v} + 2c e^{2c} e^{-v}}{(v + e^{2c} v e^{-v} - 4c)^2} \\
&= 4v \frac{1 - e^{2c} e^{-v}(1 + v - 2c)}{(v + e^{2c} v e^{-v} - 4c)^2} \\
&= 4v \frac{1 - e^{2c} e^{-v}(1 + v - 2c)}{((v - 2c) + v e^{-(v-2c)} - 2c)^2} \\
&= 4v \frac{1 - e^{2c} e^{-v}(1 + v - 2c)}{((v - 2c) + (v - 2c) e^{-(v-2c)} - 2c(1 - e^{-(v-2c)}))^2} \\
&= 4v \frac{1 - e^{2c} e^{-v}(1 + v - 2c)}{((v - 2c) + (v - 2c) e^{-(v-2c)} - 2c(1 - e^{-(v-2c)}))^2}.
\end{aligned}
$$

This expression is bounded above as

$$
\begin{aligned}
\text{(A.44)} \quad k_c^-(v;c) &\leq 4v \frac{1 - e^{-(v-2c)}(1 + v - 2c)}{(v - 2c)^2 (1 + e^{-(v-2c)})^2} \\
&= \frac{4v}{(v - 2c)(1 + e^{-(v-2c)})^2} \frac{1 - e^{-(v-2c)}(1 + v - 2c)}{(v - 2c)} \\
&\leq \frac{v}{v - 2c} \frac{1 - e^{-(v-2c)}(1 + v - 2c)}{(v - 2c)} \\
\text{(A.45)} \quad &\leq \frac{v}{v - 2c} \max_{x \geq 0} \frac{1 - e^{-x}(1 + x)}{x} < 0.3 \frac{v}{v - 2c} \leq 0.3,
\end{aligned}
$$

since

$$
\text{(A.46)} \quad \max_{x \geq 0} \frac{1 - e^{-x}(1 + x)}{x} = \max_{v \geq 0, c \leq 0} \frac{1 - e^{-(v-2c)}(1 + v - 2c)}{(v - 2c)} = 0.29843.
$$

A lower bound for $k_c^-(v;c)$ is obtained from (A.43) as

$$
\begin{aligned}
k_c^-(v;c) &= 4v\frac{1-e^{2c}e^{-v}(1+v-2c)}{((v-2c)+(v-2c)e^{-(v-2c)}-2c(1-e^{-(v-2c)}))^2} \\
&\geq 4v\frac{1-e^{2c}e^{-v}(1+v-2c)}{((v-2c)+(v-2c)e^{-(v-2c)}+v-2c)^2} \\
&= \frac{4v}{(v-2c)^2}\frac{1-e^{2c}e^{-v}(1+v-2c)}{(2+e^{-(v-2c)})^2} \\
&\geq \frac{4v}{9}\frac{1-e^{-(v-2c)}(1+v-2c)}{(v-2c)^2} \\
&= \frac{4}{9}\frac{v}{v-2c}\frac{1-e^{-(v-2c)}(1+v-2c)}{(v-2c)}.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\frac{4}{9}\frac{v}{v-2c}\frac{1-e^{-(v-2c)}(1+v-2c)}{(v-2c)} &\leq k_c^-(v;c) \\
&\leq \frac{v}{v-2c}\times\frac{1-e^{-(v-2c)}(1+v-2c)}{(v-2c)}
\end{aligned}
\tag{A.47}
$$

or

$$
\frac{4}{9}K(v;c) \leq k_c^-(v;c) \leq K(v;c), \tag{A.48}
$$

where

$$
K(v;c) = \frac{v}{v-2c}\frac{1-e^{-(v-2c)}(1+v-2c)}{(v-2c)}.
$$

Next note the following inequalities, which follow from (A.42), (A.47), and (A.46):

$$
\frac{1}{2} \leq \frac{1}{k^-(v;c)} \leq 1, \quad -1 \leq -\frac{1}{k^-(v;c)^{1/2}} \leq -\frac{1}{2^{1/2}}, \tag{A.49}
$$

$$
1 \leq k^-(v;c)^{1/2} \leq 2^{1/2}, \quad 1 \leq k^-(v;c)^{3/2} \leq 2^{3/2}, \tag{A.50}
$$

and

$$
0 \leq \frac{4}{9}K(v;c) \leq k_c^-(v;c) \leq K(v;c) \leq 0.3. \tag{A.51}
$$

Now the derivative of $g^{-}(c)$ is

$$g^{-\prime}(c) = -\frac{3}{8}\int_0^\infty e^{-(1/4)v} k_c^{-}(v;c) k^{-}(v;c)^{-1/2}\,dv + \frac{3}{8}\int_0^\infty e^{-(1/4)v} k_c^{-}(v;c) k^{-}(v;c)^{1/2}\,dv - \frac{e^{2c}}{4}\int_0^\infty e^{-(5/4)v} k^{-}(v;c)^{3/2} v\,dv - \frac{3e^{2c}}{16}\int_0^\infty e^{-(5/4)v} k_c^{-}(v;c) k^{-}(v;c)^{1/2} v\,dv. \tag{A.52}$$

We consider each term of (A.52) in turn. Using (A.49)–(A.51), we have

$$\begin{aligned}
&-\frac{3}{8}\int_0^\infty e^{-(1/4)v} k_c^{-}(v;c) k^{-}(v;c)^{-1/2}\,dv \\
&\quad \geq -\frac{3}{8}\int_0^\infty e^{-(1/4)v} K(v;c) k^{-}(v;c)^{-1/2}\,dv \\
&\quad \geq -\frac{3}{8}\int_0^\infty e^{-(1/4)v} K(v;c)\,dv, \\
&+\frac{3}{8}\int_0^\infty e^{-(1/4)v} k_c^{-}(v;c) k^{-}(v;c)^{1/2}\,dv \\
&\quad \geq \frac{4}{9}\frac{3}{8}\int_0^\infty e^{-(1/4)v} K(v;c) k^{-}(v;c)^{1/2}\,dv \\
&\quad \geq \frac{4}{9}\frac{3}{8}\int_0^\infty e^{-(1/4)v} K(v;c)\,dv, \\
&-\frac{e^{2c}}{4}\int_0^\infty e^{-(5/4)v} k^{-}(v;c)^{3/2} v\,dv \\
&\quad \geq -\frac{2^{3/2}}{4}\int_0^\infty e^{-(5/4)v} v\,dv,
\end{aligned}$$

and

$$\begin{aligned}
&-\frac{3e^{2c}}{16}\int_0^\infty e^{-(5/4)v} k_c^{-}(v;c) k^{-}(v;c)^{1/2} v\,dv \\
&\quad \geq -\frac{3e^{2c}}{16} 2^{1/2}\int_0^\infty e^{-(5/4)v} K(v;c) v\,dv.
\end{aligned}$$

Combining these inequalities, we find that

$$\begin{aligned}
g^{-\prime}(c) &= -\frac{3}{8}\int_0^\infty e^{-(1/4)v} k_c^-(v;c) k^-(v;c)^{-1/2}\,dv \\
&\quad + \frac{3}{8}\int_0^\infty e^{-(1/4)v} k_c^-(v;c) k^-(v;c)^{1/2}\,dv \\
&\quad - \frac{e^{2c}}{4}\int_0^\infty e^{-(5/4)v} k^-(v;c)^{3/2}\,dv \\
&\quad - \frac{3e^{2c}}{16}\int_0^\infty e^{-(5/4)v} k_c^-(v;c) k^-(v;c)^{1/2} v\,dv \\
&\geq -\frac{3}{8}\int_0^\infty e^{-(1/4)v} K(v;c)\,dv + \frac{4}{9}\frac{3}{8}\int_0^\infty e^{-(1/4)v} K(v;c)\,dv \\
&\quad - \frac{2^{3/2}}{4}\int_0^\infty e^{-(5/4)v} v\,dv \\
&\quad - \frac{3e^{2c}}{16} 2^{1/2}\int_0^\infty e^{-(5/4)v} K(v;c) v\,dv \\
&\geq -\frac{5}{24}\int_0^\infty e^{-(1/4)v} K(v;c)\,dv + 0 - \frac{2^{3/2}}{4}\frac{16}{25} \\
&\quad - \frac{3}{16} 2^{1/2}\int_0^\infty e^{-(5/4)v} K(v;c) v\,dv \\
&\geq -\frac{5}{6}0.3 - \frac{2^{3/2}4}{25} - 0.3\frac{3}{16}2^{1/2}\int_0^\infty e^{-(5/4)v} v\,dv \\
&= -\frac{5}{6}0.3 - \frac{2^{3/2}4}{25} - 0.3\frac{3}{16}\frac{16}{25}2^{1/2} \\
&= -0.75346.
\end{aligned}$$

It follows that the limit function $h^-(c) = c + g^-(c)$ has derivative

$$h^{-\prime}(c) = 1 + g^{-\prime}(c) \geq 1 - 0.75346 = 0.24654 > 0,$$

so that $h^-(c)$ is an increasing function of c for all $c \in (-\infty, 0)$. This is a lower bound. Direct computation of $h^{-\prime}(c)$ shows that $h^{-\prime}(c) \geq 1$, as is evident in Figure A.1. Note also that

$$\lim_{c\to-\infty} k_c^-(v;c) = \lim_{c\to-\infty} 4v\frac{1 - e^{2c}e^{-v}(1 + v - 2c)}{(v + e^{2c}ve^{-v} - 4c)^2} = 0,$$

so that $\lim_{c\to-\infty} g^{-\prime}(c) = 0$ and $\lim_{c\to-\infty} h^{-\prime}(c) = 1$.

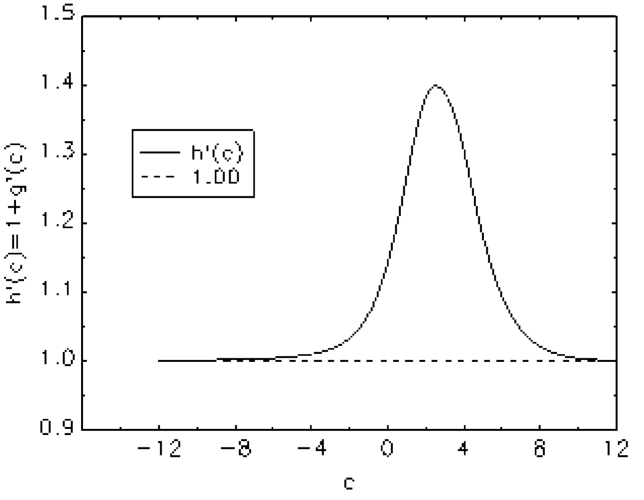

FIGURE A.1.—Derivative $h'(c)$ of the limit function $h(c)$.

PROPERTIES OF $g^+(c)$: We start with some properties of the functions $k^+(v; c)$ and $g^+(c)$, which we restate here for convenience:

$$k^+(w; c) := \frac{4c + 2w}{4c + w + e^{2c}we^{w}},$$

$$g^+(c) = \frac{3}{4}\int_0^\infty e^{(1/4)w}k^+(w; c)^{1/2}\,dw - \frac{1}{4}\int_0^\infty e^{(1/4)w}k^+(w; c)^{3/2}\,dw$$
$$- \frac{e^{2c}}{8}\int_0^\infty e^{(5/4)w}k^+(w; c)^{3/2}w\,dw.$$

First note that at the limits at the domain of definition of c, we have

$$\lim_{c\to 0} k^+(w; c) = \frac{2}{1+e^w} \leq 1, \qquad \lim_{c\to\infty} k^+(w; c) = 0,$$

and

$$\lim_{c\to\infty} g^+(c) = 0. \tag{A.53}$$

Next, as $e^{2c}e^w \geq 1$,

$$k^+(w; c) = \frac{4c + 2w}{4c + 2w + w(e^{w+2c} - 1)} \leq \frac{2w + 4c}{2w + 4c} = 1,$$

so we have

$$k^+(w; c) \in [0, 1].$$

Now consider the derivative $k_c^+(w;c) = \frac{\partial}{\partial c} k^+(w;c)$, which has the form

$$
\begin{aligned}
k_c^+(w;c) &= \frac{(4c+w+e^{2c}we^w)(4) - (2w+4c)(2e^{2c}we^w+4)}{(4c+w+e^{2c}we^w)^2} \\
&= \frac{(-w+e^{2c}we^w)(4) - (2w+4c)2e^{2c}we^w}{(4c+w+e^{2c}we^w)^2} \\
&= \frac{-4w+4e^{2c}we^w - 4e^{2c}w^2e^w - 8ce^{2c}we^w}{(4c+w+e^{2c}we^w)^2} \\
&= 4w\frac{-1+e^{2c}e^w - e^{2c}we^w - 2ce^{2c}e^w}{(4c+w+e^{2c}we^w)^2} \\
&= 4w\frac{-1+e^{2c}e^w(1-w-2c)}{(4c+w+e^{2c}we^w)^2} = -4w\frac{1+e^{w+2c}(w+2c-1)}{(4c+w+we^{w+2c})^2},
\end{aligned}
$$

which is negative since $1+e^x(x-1) \geq 0$ for $x = w+2c \geq 0$. The derivative of $g^+(c)$ is

$$
\begin{aligned}
g^{+\prime}(c) = {} & \frac{3}{8}\int_0^\infty e^{(1/4)w} k_c^+(w;c) k^+(w;c)^{-1/2}\, dw \\
& - \frac{3}{8}\int_0^\infty e^{(1/4)w} k_c^+(w;c) k^+(w;c)^{1/2}\, dw \\
& - \frac{e^{2c}}{4}\int_0^\infty e^{(5/4)w} k^+(w;c)^{3/2} w\, dw \\
& - \frac{3e^{2c}}{16}\int_0^\infty e^{(5/4)w} k_c^+(w;c) k^+(w;c)^{1/2} w\, dw.
\end{aligned} \tag{A.54}
$$

It follows that $h^+(c) = c + g^+(c)$ has derivative $h^{+\prime}(c) = 1 + g^{+\prime}(c)$, and, by direct computation, $h^{+\prime}(c) \geq 1$ and the limit function $h^+(c)$ is an increasing function of c, as is evident in Figure A.1. Since $\lim_{c\to\infty} k^+(w;c) = 0$ and

$$
\lim_{c\to\infty} k_c^+(w;c) = \lim_{c\to\infty}\left\{-4w\frac{1+e^{w+2c}(w+2c-1)}{(4c+w+we^{w+2c})^2}\right\} = 0,
$$

we deduce that $\lim_{c\to\infty} g^{+\prime}(c) = 0$ and $\lim_{c\to\infty} h^{+\prime}(c) = 1$.

PROOF OF CONTINUITY OF $g(c)$ AT $c = 0$: Observe that

$$
\begin{aligned}
g^-(0) = {} & -\frac{3}{4}\int_0^\infty e^{(1/4)v}\left\{\frac{2}{1+e^v}\right\}^{1/2} dv + \frac{1}{4}\int_0^\infty e^{(5/4)v}\left\{\frac{2}{1+e^v}\right\}^{3/2} dv \\
& - \frac{1}{8}\int_0^\infty e^{(1/4)v}\left\{\frac{2}{1+e^v}\right\}^{3/2} v\, dv
\end{aligned}
$$

and

$$g^+(0) = \frac{3}{4}\int_0^\infty e^{(1/4)w}\left(\frac{2}{1+e^w}\right)^{1/2} dw - \frac{1}{4}\int_0^\infty e^{(1/4)w}\left(\frac{2}{1+e^w}\right)^{3/2} dw - \frac{1}{8}\int_0^\infty e^{(5/4)w}\left(\frac{2}{1+e^w}\right)^{3/2} w\,dw,$$

so that

$$\begin{aligned}
\text{(A.55)}\quad & g^+(0) - g^-(0) \\
&= \frac{3}{2}\int_0^\infty e^{(1/4)w}\left(\frac{2}{1+e^w}\right)^{1/2} dw \\
&\quad - \frac{1}{4}\int_0^\infty e^{(1/4)w}(1+e^w)\left(\frac{2}{1+e^w}\right)^{3/2} dw \\
&\quad - \frac{1}{8}\int_0^\infty e^{(5/4)w}\left(\frac{2}{1+e^w}\right)^{3/2} w\,dw \\
&\quad + \frac{1}{8}\int_0^\infty e^{(1/4)v}\left\{\frac{2}{1+e^v}\right\}^{3/2} v\,dv \\
&= \int_0^\infty e^{(1/4)w}\left(\frac{2}{1+e^w}\right)^{1/2} dw \\
&\quad - \frac{1}{8}\int_0^\infty e^{(1/4)w}(e^w-1)\left(\frac{2}{1+e^w}\right)^{3/2} w\,dw \\
&= \int_0^\infty e^{(1/4)w}\left(\frac{2}{1+e^w}\right)^{1/2}\left(1 - \frac{1}{4}\frac{e^w-1}{1+e^w}w\right) dw \\
&= \frac{1}{4}\int_0^\infty e^{(1/4)w}\left(\frac{2}{1+e^w}\right)^{1/2}\frac{4+4e^w-we^w+w}{1+e^w}\,dw \\
&= \frac{1}{8}\int_0^\infty e^{(1/4)w}\left(\frac{2}{1+e^w}\right)^{3/2}(4+w+e^w(4-w))\,dw \\
&= \frac{2^{3/2}}{8}\int_0^\infty e^{-(5/4)w}(1+e^{-w})^{-3/2}(4+w+e^w(4-w))\,dw.
\end{aligned}$$

Now upon expansion and integration term by term, which is valid by majorization of the series, (A.55) can be written as

$$
\begin{aligned}
&\int_0^\infty e^{-(5/4)w}(1+e^{-w})^{-3/2}(4+w+e^w(4-w))\,dw \\
&= \int_0^\infty e^{-(5/4)w} \\
&\quad\times \sum_{j=0}^\infty \frac{\left(\frac{3}{2}\right)_j}{j!}(-1)^j e^{-jw}(4+w+e^w(4-w))\,dw \\
&= \sum_{j=0}^\infty \frac{\left(\frac{3}{2}\right)_j}{j!}(-1)^j \\
&\quad\times \int_0^\infty e^{-((5/4)+j)w}(4+w+e^w(4-w))\,dw \\
&= \sum_{j=0}^\infty \frac{\left(\frac{3}{2}\right)_j}{j!}(-1)^j \\
&\quad\times \left\{\int_0^\infty e^{-((5/4)+j)w}(4+w)\,dw + \int_0^\infty e^{-((1/4)+j)w}(4-w)\,dw\right\} \\
&= \sum_{j=0}^\infty \frac{\left(\frac{3}{2}\right)_j}{j!}(-1)^j \left\{\frac{4(16j+24)}{(4j+5)^2} + \frac{4}{(4j+1)^2}16j\right\} \\
&= \sum_{j=0}^\infty \frac{\left(\frac{3}{2}\right)_j}{j!}(-1)^j \frac{4(16j+24)}{(4j+5)^2} + \sum_{j=0}^\infty \frac{\left(\frac{3}{2}\right)_j}{j!}(-1)^j \frac{4}{(4j+1)^2}16j \\
&= 32\sum_{j=0}^\infty \frac{\left(\frac{3}{2}\right)_j}{j!}(-1)^j \frac{(2j+3)}{(4j+5)^2} + 32\sum_{j=0}^\infty \frac{\left(\frac{3}{2}\right)_j}{j!}(-1)^j \frac{2j}{(4j+1)^2} \\
&= 32\sum_{j=0}^\infty \frac{\left(\frac{3}{2}\right)_j}{j!}(-1)^j \frac{(2j+3)}{(4j+5)^2} + 32\sum_{j=1}^\infty \frac{\left(\frac{3}{2}\right)_j}{j!}(-1)^j \frac{2j}{(4j+1)^2}
\end{aligned}
$$

$$
\begin{aligned}
&= 32\sum_{j=0}^{\infty}\frac{\left(\frac{3}{2}\right)_j}{j!}(-1)^j\frac{(2j+3)}{(4j+5)^2}+32\sum_{j=1}^{\infty}\frac{\left(\frac{3}{2}\right)_j}{j!}(-1)^{(j-1)+1}\\
&\quad\times\frac{2(j-1)+2}{(4(j-1)+5)^2}\\
&= 32\sum_{j=0}^{\infty}\frac{\left(\frac{3}{2}\right)_j}{j!}(-1)^j\frac{(2j+3)}{(4j+5)^2}-32\sum_{k=0}^{\infty}\frac{\left(\frac{3}{2}\right)_{k+1}}{(k+1)!}(-1)^k\frac{2k+2}{(4k+5)^2}\\
&= 32\sum_{j=0}^{\infty}\frac{(-1)^j}{(4j+5)^2}\left\{\frac{\left(\frac{3}{2}\right)_j(2j+3)}{j!}-\frac{\left(\frac{3}{2}\right)_{j+1}(2j+2)}{(j+1)!}\right\}\\
&= 32\sum_{j=0}^{\infty}\frac{(-1)^j}{(4j+5)^2}\left\{\frac{\left(\frac{3}{2}\right)_j(2j+3)}{j!}-\frac{2\left(\frac{3}{2}\right)_{j+1}}{j!}\right\}\\
&= 32\sum_{j=0}^{\infty}\frac{(-1)^j}{(4j+5)^2j!}\left\{\left(\frac{3}{2}\right)_j(2j+3)-2\left(\frac{3}{2}\right)_{j+1}\right\}\\
&= \frac{32}{\Gamma\left(\frac{3}{2}\right)}\sum_{j=0}^{\infty}\frac{(-1)^j}{(4j+5)^2j!}\left\{\Gamma\left(\frac{3}{2}+j\right)(2j+3)-2\Gamma\left(\frac{3}{2}+j+1\right)\right\}\\
&= \frac{32}{\Gamma\left(\frac{3}{2}\right)}\sum_{j=0}^{\infty}\frac{(-1)^j}{(4j+5)^2j!}\left\{2\Gamma\left(\frac{3}{2}+j+1\right)-2\Gamma\left(\frac{3}{2}+j+1\right)\right\}=0,
\end{aligned}
$$

which proves that $g^{+}(0)-g^{-}(0)=0$, as required. *Q.E.D.*

PROPERTIES OF THE LIMIT FUNCTION $h(c)$: Combining (A.54) with (A.52), we have

$$g'(c)=g^{-\prime}(c)1_{\{c\le 0\}}+g^{+}(c)1_{\{c>0\}}.$$

Write $h(c)=c+g(c)=h^{-}(c)1_{\{c\le 0\}}+h^{+}(c)1_{\{c>0\}}$. The derivative is then

$$
\begin{aligned}
h'(c)&=1+g'(c)=h^{-\prime}(c)1_{\{c\le 0\}}+h^{+\prime}(c)1_{\{c>0\}}\\
&=1+g^{-\prime}(c)1_{\{c\le 0\}}+g^{+}(c)1_{\{c>0\}}
\end{aligned}
$$

and tends to unity as $|c| \to \infty$, as seen in Figure A.1 The limit function $h(c)$ is monotonically increasing over $c \in (-\infty, \infty)$. Moreover, in view of (A.41) and (A.53), we have

$$h(c) \sim \begin{cases} c-2, & \text{as } c \to -\infty, \\ c, & \text{as } c \to \infty, \end{cases} \tag{A.56}$$

so that $h(c)$ is linear in c for large $|c|$.

PROPERTIES OF THE DERIVATIVES $b_n^{(j)}(\rho)$: For $\rho = 1 + \frac{c}{n}$ and for large n, we have

$$\frac{\partial}{\partial c} b_n\left(1 + \frac{c}{n}\right) = \frac{1}{n} b_n^{(1)}\left(1 + \frac{c}{n}\right) = \frac{1}{n}(1 + g'(c)) + O(n^{-2}) > 0,$$

and so

$$\frac{\partial}{\partial \rho} b_n(\rho) = 1 + g'(n(\rho - 1)) + O(n^{-1}) > 0. \tag{A.57}$$

Hence, the binding function $b_n(\rho)$ is monotonic and increasing for large enough n. Furthermore,

$$\frac{\partial^j}{\partial c^j} b_n\left(1 + \frac{c}{n}\right) = \frac{1}{n^j} b_n^{(j)}\left(1 + \frac{c}{n}\right) = \frac{1}{n}\big(1 + g^{(j)}(c)\big)\{1 + o(1)\}$$

and then

$$\begin{aligned} \frac{\partial^j}{\partial \rho^j} b_n(\rho) &= n^{j-1}\big(1 + g^{(j)}(c)\big)\{1 + o(1)\} \\ &= n^{j-1}\big(1 + g^{(j)}(n(\rho - 1))\big)\{1 + o(1)\} \\ &= O(n^{j-1}). \end{aligned}$$

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