BOOTSTRAPPING I(1) DATA

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Dedicated to Phoebus Dhrymes whose advanced textbooks in econometrics have trained and educated generations of econometricians and applied researchers.

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ABSTRACT

A functional law is given for an I(1) sample data version of the continuous-path block bootstrap of Paparoditis and Politis (2001a). The results provide an alternative demonstration that continuous-path block bootstrap unit root tests are consistent under the null.

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1. Introduction

Of the many professional contributions Phoebus Dhrymes has made to econometrics his impact on the teaching of econometrics has been particularly notable. Commencing with his advanced textbook (Dhrymes, 1970) and his treatise on distributed lags (Dhrymes, 1971), there have been a steady flow of texts and monographs covering topics like advanced probability and statistical methods (Dhrymes, 1989), mathematical methods (Dhrymes, 1978), simultaneous equations (Dhrymes, 1994), and unit roots and cointegration (Dhrymes, 1998). Through these texts Dhrymes has helped to transport probability foundations, advanced statistics, standard econometric methods, and new developments in econometrics to students, professionals, and practitioners. A defining goal in all of his monographs is to present advanced methods in a readable form that does not compromise rigor. Given the increasing technical demands placed on students and professionals seeking to keep abreast of developments in econometrics, this goal is especially admirable and it motivates the present work, which seeks to provide an alternative analysis of a fundamental result on bootstrapping I(1) series.

In time series applications the bootstrap must accurately capture the temporal dependence properties of the original time series if it is to be a useful aid to inference. Two approaches are now in common use to deal with temporal dependence: the sieve bootstrap (Kreiss, 1992; Bühlmann, 1997, 1998, among others), where a sequence of finite dimensional parametric models (like autoregressions) is used to remove temporal dependence; and block bootstrap methods (Carlstein, 1986; Künsch, 1989), where blocking techniques are used to deal with dependence. If temporal dependence is poorly captured or ignored, then the bootstrap will perform badly (Horowitz, 2001) and may well produce inconsistent estimates (Basawa et al., 1991). A dramatic example of inconsistency is that raw bootstrapping converts spurious regressions into cointegrating regressions (Phillips, 2001). Block bootstrapping also fails in this context and alters rates of convergence, again leading to inconsistent tests.

For unit root testing, the sieve and block bootstraps have been extensively analyzed in a series of papers that include Park (2002, 2003), Chang and Park (2002, 2003), Paparoditis and Politis (2003, 2005) and Parker et al. (2006). A detailed comparison of these

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various bootstrap unit root tests has recently been conducted by Palm et al. (2007), giving some recommendations for practical work. While differences occur in the construction of the tests in terms of the use of residuals or differences in parameter estimation, all of these bootstrap tests have in common the fact that they use a partial sum process to generate the bootstrap data under the null, thereby ensuring that the test conforms to a unit root limit distribution. This is appropriate when replicating properties of statistical distributions under the null of a unit root.

Paparoditis and Politis (2001a) developed a new bootstrap procedure for reproducing unit root data using a continuous-path block bootstrap. The continuous path block bootstrap works by adjusting the beginning of each new block to match up with the end of the last block in the random sequence of blocks. In subsequent work, the same authors (2001b; 2003) showed how to construct unit root data in a bootstrap algorithm based on partial sums of residuals from a first order autoregression of the sample data. Their approach produces consistent bootstrap estimates of the null unit root distribution under both the null and the alternative. This feature of the algorithm is important as it enables the generation of unit root pseudo-data with the correct residual dependence structure (and hence the correct unit root limit distribution) even when the original series are stationary.

The present note provides an alternative proof of the asymptotic behavior of the continuous path block bootstrap when the algorithm is applied to the original sample data rather than partial sums of residuals. This form of directly bootstrapping a nonstationary series is valid even when the initial condition is itself nonstationary. Further, unit root tests bootstrapped in this way are consistent under the null hypothesis of a unit root, but are not consistent for unit root distributions under the alternative, in contrast to the partial sum approach of Paparoditis and Politis (2001b, 2003). The results are proved with some of the embedding methods used in Phillips (2001).

2. Preliminaries

Suppose x_t is a unit root process generated by

$$x_t = x_{t-1} + u_t, \quad t = 1, \dots, n \tag{1}$$

where u_t a linear process satisfying condition **LP** below (see Phillips and Solo, 1992). We allow for random initializations in which $x_0 = O_{a.s}(1)$ or where there may be a distant past initialization in which $x_0 = \sum_{j=1}^{[n\kappa]} u_{-j}$ for some finite[1] $\kappa > 0$ (so that x_0 and x_t have similar stochastic properties) and where u_{-j} also satisfies assumption **LP**.

Assumption LP. u_t has the Wold representation

$$u_t = C(L)\varepsilon_t = \sum_{j=0}^{\infty} c_j \varepsilon_{t-j}, \qquad \sum_{j=0}^{\infty} j^s |c_j| < \infty,$$
$$s \geq 1,\ C(1) \neq 0, \tag{2}$$

with $\varepsilon_t = iid\left(0, \sigma_\varepsilon^2\right)$ and with $E(|\varepsilon_t|^q) < \infty$, for some $q > 4$.

The summability condition in (2) is satisfied by a wide class of parametric and nonparametric models for u_t and, in conjunction with the moment condition, enables the use of almost sure invariance principles (IP) for the partial sums of u_t. To perform the latter, which are especially useful here, we expand the probability space as needed so that the partial sum process $S_k = \sum_{t=1}^{k} u_t$ of u_t can be represented up to a negligible error in terms of a Brownian motion defined on the same space. An IP of this type for partial sums of linear processes was given in Phillips (2007, Lemma 3.1) and a version of that lemma is repeated here for convenience.

[1] Phillips and Magdalinos (2009) explore the effects of distant past initializations on unit root limit theory when $\kappa = \kappa_n$ may pass to infinity as $n \to \infty$.

Lemma 2.1. *Let $S_k = \sum_{j=1}^{k} u_j$ for $k \geq 1$, and $S_0 = 0$, for $k = 0$, where u_j satisfies Assumption LP. Then, the probability space on which the u_j and S_k are defined can be expanded in such a way that there is a process distributionally equivalent to S_k and a Brownian motion $B(\cdot)$ with variance $\omega^2 = \sigma_\varepsilon^2 C(1)^2$ on the new space for which*

$$\sup_{0 \leq k \leq n} \left\| \frac{S_k}{\sqrt{n}} - B\left(\frac{k}{n}\right) \right\| = o_{a.s}\left(\frac{1}{n^{\frac{1}{2}-\frac{1}{p}}}\right) \tag{3}$$

provided $E\|u_t\|^q < \infty$ for some $q > 2p > 4$.

When $x_0 = O_{a.s}(1)$ we have $x_t = S_t + O_{a.s.}(1)$, and it follows from (3) that, after changing the probability space as required,

$$\sup_{0 \leq t \leq n} \left\| \frac{x_t}{\sqrt{n}} - B\left(\frac{t}{n}\right) \right\| = o_{a.s}\left(\frac{1}{n^{\frac{1}{2}-\frac{1}{p}}}\right). \tag{4}$$

Similarly, when there is a distant past initialization, an almost sure IP applies to $n^{-\frac{1}{2}} x_0 = \sum_{j=1}^{[n\kappa]} u_{-j}$ and we have, after appropriate enrichment of the probability space,

$$\left| \frac{x_0}{\sqrt{n}} - B_0\left(\frac{[n\kappa]}{n}\right) \right| = o_{a.s}\left(\frac{1}{n^{\frac{1}{2}-\frac{1}{p}}}\right). \tag{5}$$

We shall proceed as if these changes have been made without adding the qualification and noting that in the original space these convergences translate into weak convergence of measures.

3. The continuous moving block bootstrap

Let the bootstrap block size be given by the positive integer m, and let $N_1, \dots, N_M$ be *iid* uniform draws from $\{0, 1, \dots, n-m\}$ with $M = [n/m]$, where $[\cdot]$ denotes integer part. A typical moving block (Künsch, 1989) bootstrap observation is $x^{mb*}_{(j-1)m+k} = x_{N_j+k}$ for $1 \leq j \leq M$ and $1 \leq k \leq m$. Here, N_j are the block division points and M is the number of blocks. Using the construction of Paparoditis and Politis (2001a), we can create continuous path moving block bootstrap observations by re-initializing at the division points as follows:

CB1. First Block ($s = 0$, say): $x_k^{cb*} = x_1 + (x_{N_1+k} - x_{N_1})$ for $k = 1, \dots, m$

CB2. Succeeding Blocks ($s = 1, \dots, M-1$, say):

$$x^{cb*}_{sm+k} = x^{cb*}_{sm} + (x_{N_s+k} - x_{N_s})$$
$$= x_1 + \sum_{a=1}^{s} (x_{N_{a-1}+m} - x_{N_{a-1}}) + (x_{N_s+k} - x_{N_s})$$

for $k = 1, \dots, m$.

This algorithm produces bootstrap data directly from the sample observations x_t and does not use residuals from a preliminary regression or partial sum processes to construct the series. Note that there are m observations in each block. The first block ends at the point $x_m^{cb*} = x_1 + (x_{N_1+m} - x_{N_1})$ and the second block starts at $x^{cb*}_{m+1} = x^{cb*}_m + (x_{N_2+1} - x_{N_2})$, thereby differing from the end of the first block by the $O_{p^*}(1)$ random element $x_{N_2+1} - x_{N_2}$ after re-initializing by removing x_{N_2}. There are $\ell = mM$ observations in total and $n = m[M + O(1)]$. In what follows, it is assumed that $\frac{1}{M} + \frac{M}{n^{1/2-1/p}} \to 0$ as $n \to \infty$. The bootstrapped series is initialized at the first sample observation x_1 plus the $O_{p^*}(1)$ random element $x_{N_1+1} - x_{N_1}$, and so is asymptotically equivalent to the sample initialization x_1. (The algorithm of Paparoditis and Politis (2001a), initializes directly on

x_1). This common initialization is important, particularly if x_1 is random and has an implicit distant past initialization which figures in the limit theory. Thus, if $x_1 = \sum_{j=0}^{[n\kappa]} u_{-j}$ for some finite $\kappa > 0$ and where the u_{-j} satisfy **LP** then from (5) $n^{-\frac{1}{2}}x_1 \to_d B_0(\kappa)$, where B_0 is a Brownian motion with variance ω^2, independent of B. In this event, the standardized sample observations $n^{-\frac{1}{2}}x_{[nr]} \to_d B_0(\kappa) + B(r)$, and $B_0(\kappa)$ figures in the limit theory. Step CB1 of the bootstrap construction above ensures that this feature is reproduced in the bootstrap limit theory. Now consider the limit of a standardized bootstrap sample constructed in this manner. As earlier, there exists a probability space in which we can write for $t = (j-1)m+k$

$$\begin{aligned}\frac{x_t^{cb*}}{\sqrt{n}} &= \frac{x_{(j-1)m+k}^{cb*}}{\sqrt{n}} \\ &= \frac{x_1}{\sqrt{n}} + \sum_{a=1}^{j-1}\left(\frac{x_{N_a-1+m}}{\sqrt{n}} - \frac{x_{N_{a-1}}}{\sqrt{n}}\right) + \left(\frac{x_{N_{j-1}+k}}{\sqrt{n}} - \frac{x_{N_{j-1}}}{\sqrt{n}}\right) \\ &= \frac{x_1}{\sqrt{n}} + \sum_{a=1}^{j-1}\left(B\left(R_{N_{a-1}} + \frac{m}{n}\right) - B\left(R_{N_{a-1}}\right)\right) \\ &\quad + \left(B\left(R_{N_{j-1}} + \frac{k}{n}\right) - B\left(R_{N_{j-1}}\right)\right) + o_{a.s.}\left(\frac{M}{n^{\frac{1}{2}-\frac{1}{p}}}\right), \qquad (6)\end{aligned}$$

where $R_{N_a} = \frac{N_a}{n}$ is uniformly distributed over $\{0, \frac{1}{n}, \ldots, \frac{n-m}{n}\}$ for each a (see Phillips, 2001, for more details on this construction). The error order given in (6) holds uniformly in $j \le M$. The sequence of Brownian increments

$$\begin{aligned}&\left\{B\left(R_{N_{a-1}} + \frac{m}{n}\right) - B\left(R_{N_{a-1}}\right) : a = 1, \ldots, j-1\right\} \\ &\quad \equiv iid\ N\left(0, \omega^2\frac{m}{n}\right),\end{aligned}$$

and these are independent of $B(R_{N_{j-1}} + \frac{k}{n}) - B(R_{N_{j-1}})$. There exists a new Brownian motion $V = BM\left(\omega^2\right)$ for which we can write

$$B\left(R_{N_{a-1}} + \frac{m}{n}\right) - B\left(R_{N_{a-1}}\right) =_d \int_{\frac{a-1}{M}}^{\frac{a-1}{M}+\frac{m}{n}} \mathrm{d}V,$$

and then using the $a.s.(P)$ continuity of the sample paths of V we have the representation

$$\begin{aligned}&\sum_{a=1}^{j-1}\left(B\left(R_{N_{a-1}} + \frac{m}{n}\right) - B\left(R_{N_{a-1}}\right)\right) =_d \sum_{a=1}^{j-1}\int_{\frac{a-1}{M}}^{\frac{a-1}{M}+\frac{m}{n}} \mathrm{d}V \\ &\quad = V\left(\frac{j-2}{M} + \frac{m}{n}\right) + o_{a.s.}(1). \qquad (7)\end{aligned}$$

Next, suppose that $t = (j-1)m + k = [nr]$ for some $r > 0$. Then,

$$\frac{j-1}{M} = \frac{[nr]-k}{mM} = r + O\left(\frac{1}{M}\right),$$

uniformly for $k = 1, \ldots, m$. Since V has continuous sample paths almost surely, it follows from (6) and (7) that we can write

$$\begin{aligned}\frac{x_t^{cb*}}{\sqrt{n}} - \frac{x_1}{\sqrt{n}} &=_d V\left(\frac{j-2}{M} + \frac{m}{n}\right) + \left(V\left(\frac{j-1}{M} + \frac{k}{n}\right)\right. \\ &\qquad \left. - V\left(\frac{j-1}{M}\right)\right) + o_{a.s.}\left(\frac{1}{n^{\frac{1}{2}-\frac{1}{p}}}\right) \\ &= V\left(\frac{j-1}{M} + \frac{k}{n}\right) + o_{a.s.}(1) \\ &= V(r) + o_{a.s.}(1), \qquad (8)\end{aligned}$$

uniformly in $t = [nr]$ with $r \in [0, 1]$. Noting the distributional equivalence $V(r) =_d B(r)$, we deduce that in the original probability space we have the weak convergence

$$\frac{x_{[nr]}^{cb*}}{\sqrt{n}} - \frac{x_1}{\sqrt{n}} \to_{d^*} B(r), \quad \text{a.s.}(P), \qquad (9)$$

and hence

$$\frac{x_{[nr]}^{cb*}}{\sqrt{n}} \to_{d^*} B_0(\kappa) + B(r), \quad \text{a.s.}(P),$$

where $\kappa > 0$ when there is distant past initialization. Thus, the CMB bootstrap provides a consistent mechanism for reproducing integrated time series from a random and possibly distant initialization.

4. Unit root testing

Now consider the limit distribution of serial correlation coefficients obtained from the bootstrap sample. Take the first order serial correlation $\widehat{\rho}^{cb*} = n^{-2}\sum_{t=1}^n x_t^{cb*}x_{t-1}^{cb*}/n^{-2}\sum_{t=1}^n x_{t-1}^{cb*2}$ and for simplicity let the initialization be such that $\kappa = 0$ in what follows. Using (9) and standard limit theory, the denominator of $\widehat{\rho}^{cb*}$ can be shown to have the following limit

$$\frac{1}{n^2}\sum_1^n x_{t-1}^{cb*2} \to_{d^*} \int_0^1 B(r)^2\mathrm{d}r \quad \text{a.s.}(P). \qquad (10)$$

Using (8), the numerator can be written as

$$\begin{aligned}&\frac{1}{n}\sum_1^n\left(\frac{x_t^{cb*}}{\sqrt{n}}\right)\left(\frac{x_{t-1}^{cb*}}{\sqrt{n}}\right) \\ &\quad = \frac{1}{mM}\sum_{j=0}^{M-1}\sum_{k=1}^{m}\left[V\left(\frac{j-1}{M} + \frac{k}{n}\right) + o_{a.s.}(1)\right] \\ &\qquad \times\left[V\left(\frac{j-1}{M} + \frac{k-1}{n}\right) + o_{a.s.}(1)\right] \\ &\quad = \frac{1}{M}\sum_{j=0}^{M-1}\int_{\frac{j}{M}+\frac{1}{n}}^{\frac{j}{M}+\frac{m}{n}} V(r)^2\mathrm{d}r + o_{a.s.}(1) \\ &\quad = \int_{\frac{1}{n}}^{\frac{M-1}{M}+\frac{m}{n}} V(r)^2\mathrm{d}r + o_{a.s.}(1) \\ &\quad \to_{d^*} \int_0^1 B(r)^2\mathrm{d}r \quad \text{a.s.}(P). \qquad (11)\end{aligned}$$

It follows that $\widehat{\rho}^{cb*} \to_{p^*} 1$ a.s. (P). Next, for the limit distribution of $\widehat{\rho}^{cb*}$, write

$$n(\widehat{\rho}^{cb*} - 1) = \frac{1}{n}\sum_{t=1}^n \Delta x_t^{cb*}x_{t-1}^{*} \Big/ \frac{1}{n^2}\sum_{t=1}^n x_{t-1}^{cb*2}. \qquad (12)$$

The limit of the denominator is given in (10). For the numerator, first observe that for $t = sm+k$ we have $x_{sm+k}^{cb*} = x_{sm}^{cb*} + (x_{N_s+k} - x_{N_j})$ and so

$$\Delta x_t^{cb*} = \Delta x_{N_s+k} = u_{N_s+k}, \quad \text{for all } k \ge 1. \qquad (13)$$

The identity

$$\begin{aligned}\Delta\sum_{t=1}^n x_t^{cb*2} &= \sum_{t=1}^n \Delta x_t^{cb*}x_t^{cb*} + \sum_{t=1}^n x_{t-1}^{cb*}\Delta x_t^{cb*} \\ &= 2\sum_{t=1}^n x_{t-1}^{cb*}\Delta x_t^{cb*} + \sum_{t=1}^n\left(\Delta x_t^{cb*}\right)^2\end{aligned}$$

leads to

$$\begin{aligned}\frac{1}{n}\sum_{t=1}^{n}x_{t-1}^{cb*}\Delta x_t^{cb*} &= \frac{1}{2}\left\{\left(\frac{x_n^{cb*}}{\sqrt{n}}\right)^2-\frac{1}{n}\sum_{t=1}^{n}\left(\Delta x_t^{cb*}\right)^2\right\}\\ &= \frac{1}{2}\left\{\left(\frac{x_n^{cb*}}{\sqrt{n}}\right)^2-\frac{1}{n}\sum_{j=1}^{M}\sum_{k=1}^{m}\left(\Delta x_{N_{j-1}+k}\right)^2\right\}\\ &= \frac{1}{2}\left\{\left(\frac{x_n^{cb*}}{\sqrt{n}}\right)^2-\frac{1}{n}\sum_{j=1}^{M}\sum_{k=1}^{m}u_{N_{j-1}+k}^2\right\}.\end{aligned}$$

Now

$$\frac{x_n^{cb*}}{\sqrt{n}}\to_{d^*}B(1)\quad \text{a.s.}(P),$$

and for all j

$$\frac{1}{m}\sum_{k=1}^{m}u_{N_{j-1}+k}^2\to_{p^*}\sigma_u^2\quad \text{a.s.}(P),$$

where $\sigma_u^2=E(u_t^2)$, and then

$$\begin{aligned}&\frac{1}{n}\sum_{j=1}^{M}\sum_{k=1}^{m}u_{N_{j-1}+k}^2\\ &\quad=\frac{1}{M+O(1)}\sum_{j=1}^{M}\frac{1}{m}\sum_{k=1}^{m}u_{N_{j-1}+k}^2\to_{p^*}\sigma_u^2\quad \text{a.s.}(P).\end{aligned}$$

We deduce that

$$\frac{1}{n}\sum_{t=1}^{n}x_{t-1}^{cb*}\Delta x_t^{cb*}\to_{d^*}\frac{1}{2}\left[B(1)^2-\sigma_u^2\right]=\int_0^1 BdB+\lambda,$$

with $\lambda=\omega^2-\sigma_u^2=\sum_{k=1}^{\infty}E(u_0u_k)$. Thus,

$$n(\widehat{\rho}^{cb*}-1)\to_{d^*}\frac{\int_0^1 BdB+\lambda}{\int_0^1 B^2dr}\quad \text{a.s.}(P), \tag{14}$$

and the bootstrap distribution is consistent for that of $n(\widehat{\rho}-1)$. Adjustments to remove the serial dependence manifested in λ in (14) can be made in the usual way, by semiparametric corrections as in Phillips (1987) or augmented regression (Said and Dickey, 1984; Xiao and Phillips, 1998). If such corrected bootstrap statistics are to have the same limit theory as the statistics in the sample data (and therefore be pivotal) we need the corresponding estimate λ^{cb*} of λ to be consistent, so that $\lambda^{cb*}\to_{p^*}\lambda$ a.s. (P), or the effects of λ to be consistently removed through augmented regression. In what follows, we concentrate on the semiparametric correction approach to testing. In that case, we use the regression

$$x_t^{cb*}=\widehat{\rho}^{cb*}x_{t-1}^{cb*}+u_t^{**},$$

to form the kernel estimate λ^{cb*} from the residuals u_t^{**}

$$\lambda^{cb*}=\sum_{h=1}^{H}k\left(\frac{h}{H}\right)\frac{1}{n}\sum_{t=h+1}^{n}u_t^{**}u_{t-h}^{**}, \tag{15}$$

using some suitable lag kernel function $k(\cdot)$ and truncation parameter H. Since

$$u_t^{**}=\Delta x_t^{cb*}-\left(\widehat{\rho}^{cb*}-1\right)x_{t-1}^{cb*},$$

and $\widehat{\rho}^{cb*}\to_{p^*}1$ a.s. (P) and $\widehat{\rho}^{cb*}-1=O_{p^*}(\frac{1}{n})$ a.s. (P), we obtain

$$\begin{aligned}\frac{1}{n}\sum_{t=h+1}^{n}u_t^{**}u_{t-h}^{**} &= \frac{1}{n}\sum_{t=h+1}^{n}\left[\Delta x_t^{cb*}-\left(\widehat{\rho}^{cb*}-1\right)x_{t-1}^{cb*}\right]\\ &\quad\times\left[\Delta x_{t-h}^{cb*}-\left(\widehat{\rho}^{cb*}-1\right)x_{t-1-h}^{cb*}\right]\\ &= \frac{1}{n}\sum_{t=h+1}^{n}\Delta x_t^{cb*}\Delta x_{t-h}^{cb*}-\left(\widehat{\rho}^{cb*}-1\right)\frac{1}{n}\sum_{t=h+1}^{n}x_{t-1}^{cb*}\Delta x_{t-h}^{cb*}\\ &\quad-\left(\widehat{\rho}^{cb*}-1\right)\frac{1}{n}\sum_{t=h+1}^{n}x_{t-1-h}^{cb*}\Delta x_t^{cb*}\\ &\quad+n\left(\widehat{\rho}^{cb*}-1\right)^2\frac{1}{n^2}\sum_{t=h+1}^{n}x_{t-1}^{cb*}x_{t-1-h}^{cb*}\\ &= \frac{1}{n}\sum_{t=h+1}^{n}\Delta x_t^{cb*}\Delta x_{t-h}^{cb*}+O_{p^*}\left(\frac{1}{n}\right)\quad \text{a.s.}(P).\end{aligned}$$

For $t=sm+k$ and $t-h=sm+k-h$ with $k>h$ we have

$$\Delta x_t^{cb*}\Delta x_{t-h}^{cb*}=u_{N_s+k}u_{N_s+k-h}.$$

When $k\leq h$ we have $t-h=(s-1)m+(m-h+k)$ and then

$$\Delta x_t^{cb*}\Delta x_{t-h}^{cb*}=u_{N_s+k}u_{N_{s-1}+m+k-h}.$$

The sample covariances are therefore

$$\begin{aligned}\frac{1}{n}\sum_{t=h+1}^{n}\Delta x_t^{cb*}\Delta x_{t-h}^{cb*} &= \frac{1}{n}\sum_{s=0}^{M-1}\left[\sum_{k=h+1}^{m}u_{N_s+k}u_{N_s+k-h}\right.\\ &\quad\left.+\sum_{k=1}^{h}u_{N_s+k}u_{N_{s-1}+m+k-h}\right]\\ &= \frac{1}{M+O(1)}\sum_{s=0}^{M-1}\left[\frac{1}{m}\sum_{k=h+1}^{m}u_{N_s+k}u_{N_s+k-h}\right.\\ &\quad\left.+\frac{1}{m}\sum_{k=1}^{h}u_{N_s+k}u_{N_{s-1}+m+k-h}\right].\end{aligned}$$

Now, as $m\to\infty$

$$\frac{1}{m}\sum_{k=h+1}^{m}u_{N_s+k}u_{N_s+k-h}\to_{p^*}E\left(u_tu_{t-h}\right)\quad \text{a.s.}(P),$$

whereas for all finite h

$$\frac{1}{m}\sum_{k=1}^{h}u_{N_s+k}u_{N_{s-1}+m+k-h}\to_{p^*}0\quad \text{a.s.}(P).$$

It follows by conventional arguments that the kernel estimate (15) has the following limit

$$\lambda^{cb*}\to_{p^*}\lambda=\sum_{h=1}^{\infty}E\left(u_tu_{t-h}\right)\quad \text{a.s.}(P).$$

We deduce that the unit root test statistic

$$Z\left(\widehat{\rho}^{cb*}\right)=n(\widehat{\rho}^{cb*}-1)-\frac{\lambda^{cb*}}{n^{-2}\sum_{t=1}^{n}x_{t-1}^{cb*2}}\to_{d^*}\frac{\int_0^1 BdB}{\int_0^1 B^2dr}\quad \text{a.s.}(P),$$

and so $Z(\widehat{\rho}^{cb*})$ is consistent for the limit distribution of the usual Z-test (Phillips, 1987),

$$Z\left(\widehat{\rho}\right)=n(\widehat{\rho}-1)-\frac{\widehat{\lambda}}{n^{-2}\sum_{t=1}^{n}x_{t-1}^2},$$

where $\widehat{\lambda}=\sum_{h=1}^{H}k\left(\frac{h}{H}\right)\frac{1}{n}\sum_{t=h+1}^{n}\hat{u}_t\hat{u}_{t-h}$, $\widehat{\rho}=n^{-2}\sum_{t=1}^{n}x_tx_{t-1}/n^{-2}\sum_{t=1}^{n}x_{t-1}^2$, $\hat{u}_t=x_t-\widehat{\rho}x_{t-1}$, and H is the lag truncation parameter. Similar results apply to the Z_t test, the ADF test and tests in models with fitted drift or in tests against trend breaks.

5. Conclusion

The continuous moving block bootstrap of Paparoditis and Politis (2001a) is consistent under the null hypothesis that the sample data is I(1). In particular, it can be used to construct estimates of the finite sample distribution of unit root tests that involve semiparametric corrections for serial correlation. However, this approach does not produce consistent estimates of the null unit root distribution under the alternative, as in the procedures of Paparoditis and Politis (2001b, 2003) or some other procedures that use residual-based partial summation methods in constructing the bootstrap sample.

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