SMOOTHING LOCAL-TO-MODERATE UNIT ROOT THEORY

BY

**PETER C. B. PHILLIPS, TASSOS MAGDALINOS,
and LIUDAS GIRAITIS**

COWLES FOUNDATION PAPER NO. 1309

**COWLES FOUNDATION FOR RESEARCH IN ECONOMICS
YALE UNIVERSITY
Box 208281
New Haven, Connecticut 06520-8281**

2010

http://cowles.econ.yale.edu/

Journal of Econometrics 158 (2010) 274–279

Contents lists available at ScienceDirect

Journal of Econometrics

journal homepage: www.elsevier.com/locate/jeconom

Smoothing local-to-moderate unit root theory[☆]

Peter C.B. Phillips[a,b,c,d,*], Tassos Magdalinos[e], Liudas Giraitis[f]

[a] Yale University, United States
[b] University of Auckland, New Zealand
[c] University of Southampton, United Kingdom
[d] Singapore Management University, Singapore
[e] Granger Centre for Time Series Econometrics, University of Nottingham, United Kingdom
[f] Queen Mary, University of London, United Kingdom

ARTICLE INFO

Article history:
Available online 1 February 2010

JEL classification:
C22

Keywords:
Edgeworth expansion
Local to unity
Moderate deviations
Unit root distribution

ABSTRACT

A limit theory is established for autoregressive time series that smooths the transition between local and moderate deviations from unity and provides a transitional form that links conventional unit root distributions and the standard normal. Edgeworth expansions of the limit theory are given. These expansions show that the limit theory that holds for values of the autoregressive coefficient that are closer to stationarity than local (i.e. deviations of the form $\rho = 1 + \frac{c}{n}$, where n is the sample size and $c < 0$) holds up to the second order. Similar expansions around the limiting Cauchy density are provided for the mildly explosive case.

1. Introduction

Earlier work by the authors (Phillips and Magdalinos, 2007, hereafter PM; Giraitis and Phillips, 2006, hereafter GP) provided a limit theory for autoregressive time series that allows for moderate deviations from unity in the autoregressive coefficient. This theory includes autoregressive roots of the form $\rho_n = 1+c/n^{\alpha}$, where the exponent α lies in the interval (0, 1). Such roots belong to larger neighborhoods of unity than conventional local to unity roots ($\rho_n = 1 + c/n$), the radial width of the neighborhood measured by the parameter α. The boundary value as $\alpha \to 1$ includes the conventional local to unity case, whereas the boundary value as $\alpha \to 0$ includes the stationary or explosive AR(1) process, depending on the value of c.

The limit theory developed in PM and GP was successful in establishing a continuous bridge for the rate of convergence between stationary, unit root and explosive asymptotics, as well as a continuous transition of the asymptotic distribution between moderately integrated time series and stationary or explosive AR(1) processes. However, the bridge provided in those papers is incomplete because there is still discontinuity in the form of the limit distributions between moderately integrated and local to unity processes.

The present paper contributes to this literature by showing how the local to unity limit distribution may be smoothly transitioned into a normal distribution on the stationary side of unity and a Cauchy distribution (corresponding to the invariance principle established in PM) on the explosive side of unity. By partitioning the sample size $n = mK$ into m blocks containing K observations, we consider roots representing "local-to-moderate deviations" from unity of the form $\rho_{n,m} = 1 + \frac{cm}{n}$, which approximate local to unity roots as $n \to \infty$ and m is kept fixed. This procedure yields the well-known local to unity limit distribution of Phillips (1987) (see (5) below). The innovation of this paper consists of deriving a second-order expansion of the above local to unity distribution as $m \to \infty$. The results reveal that the continuous bridging between moderately integrated and stationary/explosive AR(1) processes continues to hold for a second-order expansion of the limiting distribution function. More importantly, the asymptotic expansions of Theorems 1 and 2 provide insight into the transition of the local to unity limiting distribution to a Gaussian (Cauchy) variate as the autoregressive root approaches the boundary with the stationary (explosive) region. Further illustration of this transition in finite samples is given by means of Monte Carlo experiments.

☆ The authors thank the Co-Editor and referees for helpful comments on the original version, which was circulated in November 2006. Phillips thanks the NSF for partial research support under Grant Nos. SES 042254 and 06-47086. Giraitis gratefully acknowledges the financial support from ESRC grant RES062230790.

* Corresponding address: Yale University, 06520 New Haven, United States. Tel.: +1 203 432 3695.
E-mail address: peter.phillips@yale.edu (P.C.B. Phillips).

0304-4076/$ – see front matter © 2010 Elsevier B.V. All rights reserved.
doi:10.1016/j.jeconom.2010.01.009

2. A limit theory for local-to-moderate deviations from unity

Consider an autoregressive process with local-to-moderate deviations from unity root of the form $\rho_n = 1 + \frac{c}{K}$, where K passes to infinity with the sample size n and ρ_n approaches unity from the stationary or the explosive side according to the sign of c. It is convenient to think of such a time series as constituting m blocks of K observations with total sample size $n = mK$. Partitioning the chronological sequence $\{t = 1, \ldots, n\}$ by setting $t = \lfloor Kj \rfloor + k$ for $k \in \{1, \ldots, K\}$ and $j \in \{0, \ldots, m-1\}$, it is possible to study the asymptotic behavior of the time series $\{X_t : t = 1, \ldots, n\}$ via the asymptotic properties of the time series $\{X_{\lfloor Kj \rfloor + k} : j = 0, \ldots, m-1, \ k = 1, \ldots, K\}$. The latter representation is particularly useful for revealing the transition from non-stationary to stationary autoregression as the number of blocks m increases.

Formally, a process with the above characteristics may be written in the form

$$X_t = \rho_{n,m} X_{t-1} + u_t, \quad u_t \sim iid\left(0, \sigma^2\right), \tag{1}$$

$$\rho_{n,m} = 1 + \frac{c}{K} = 1 + \frac{cm}{n}. \tag{2}$$

The usual local to unity model (Phillips, 1987) applies when $m = 1$ and the moderate deviation theory of PM and GP holds when $m \to \infty$.

Let W be a standard Brownian motion and $J_c(t) = \int_0^t e^{c(t-s)} \mathrm{d}W(s)$ be a corresponding Ornstein–Uhlenbeck process. For each m, letting

$$\tilde{W}(t) = \sqrt{m} W\left(\frac{t}{m}\right),$$

we observe that $\tilde{W}$ is also a standard Brownian motion and we denote by $\tilde{J}_c(t) = \int_0^t e^{c(t-s)} \mathrm{d}\tilde{W}(s)$ the associated Ornstein–Uhlenbeck process. For given $m \geq 1$, we may derive a limit theory for the least squares estimate $\hat{\rho}_{n,m}$ of $\rho_{n,m}$ in (1) as $n \to \infty$ using earlier results from standard local to unity asymptotics. Using the identities (see the Appendix for a proof)

$$\int_0^1 J_{cm}(s) \, \mathrm{d}W(s) = \frac{1}{m} \int_0^m \tilde{J}_c(s) \, \mathrm{d}\tilde{W}(s), \tag{3}$$

$$\int_0^1 J_{cm}(s)^2 \, \mathrm{d}s = \frac{1}{m^2} \int_0^m \tilde{J}_c(s)^2 \, \mathrm{d}s, \tag{4}$$

the results in Phillips (1987) imply that, for fixed m and $n \to \infty$, the asymptotic distribution of the least squares estimator takes the form

$$n\left(\hat{\rho}_n - \rho_{n,m}\right) \Rightarrow \frac{\int_0^1 J_{cm}(s) \, \mathrm{d}W(s)}{\int_0^1 J_{cm}^2(s) \, \mathrm{d}s} = m \frac{\int_0^m \tilde{J}_c(s) \, \mathrm{d}\tilde{W}(s)}{\int_0^m \tilde{J}_c(s)^2 \, \mathrm{d}s}. \tag{5}$$

When $c < 0$, sequential limits with $n \to \infty$ followed by $m \to \infty$ lead to the normal asymptotic theory given in PM and GP:

$$\frac{n}{\sqrt{m}}\left(\hat{\rho}_n - \rho_{n,m}\right) \Rightarrow \frac{\frac{1}{\sqrt{m}} \int_0^m \tilde{J}_c(s) \, \mathrm{d}\tilde{W}(s)}{\frac{1}{m} \int_0^m \tilde{J}_c(s)^2 \, \mathrm{d}s} \quad \text{for fixed } m \tag{6}$$

$$= \frac{\frac{1}{\sqrt{m}} \sum_{j=1}^{m} \int_{j-1}^{j} \tilde{J}_c(s) \, \mathrm{d}\tilde{W}(s)}{\frac{1}{m} \sum_{j=1}^{m} \int_{j-1}^{j} \tilde{J}_c(s)^2 \, \mathrm{d}s}$$

$$\Rightarrow \frac{N\left(0, \frac{1}{-2c}\right)}{\frac{1}{-2c}} \equiv N(0, -2c) \quad \text{as } m \to \infty \tag{7}$$

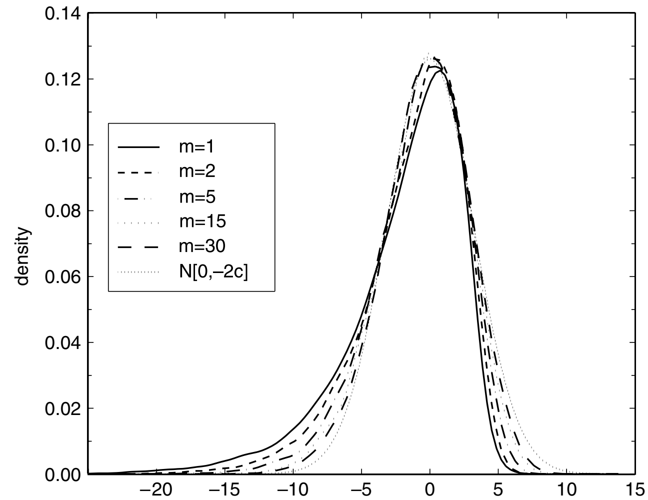

Fig. 1. Local limit densities of $\sqrt{\frac{m}{-2c}} \frac{\int_0^m J_c \mathrm{d}V}{\int_0^m J_c^2 \mathrm{d}s}$ for various m.

by a standard martingale CLT (e.g. Pollard, 1984, Theorem VIII.1) on the numerator and an ergodic theorem on the denominator ($\tilde{J}_c$ is a Gaussian diffusion with a stationary version for all $c < 0$).

Fig. 1 shows the limit distribution (6) for various values of m and $c = -5$. The graphs reveal a smooth transition from the local to unity limit distribution of $\sqrt{m/(-2c)} \int_0^m \tilde{J}_c \mathrm{d}\tilde{W} / \int_0^m \tilde{J}_c^2 \mathrm{d}s$ through to the standard normal.

In summary, the sequential limit theory as $(n, m) \to \infty$ on the stationary side is given by

$$Z_{n,m} = \frac{\sqrt{n}\left(\hat{\rho}_n - \rho_{n,m}\right)}{\sqrt{2\left(1 - \rho_{n,m}\right)}} = \frac{n}{\sqrt{m}} \frac{\left(\hat{\rho}_n - \rho_{n,m}\right)}{\sqrt{-2c}} \Rightarrow N(0, 1). \tag{8}$$

On the explosive side, $c > 0$, the martingale convergence theorem ensures that $\int_0^\infty e^{-cs} \mathrm{d}\tilde{W}(s)$ and $\tilde{J}_{-c}(\infty) := \lim_{m \to \infty} \tilde{J}_{-c}(m)$ both exist almost surely and consequently follow a $N\left(0, \frac{1}{2c}\right)$ distribution. Moreover, $\int_0^\infty e^{-cs} \mathrm{d}\tilde{W}(s)$ and $\tilde{J}_{-c}(\infty)$ can be seen to be independent by an elementary property of the stochastic integrals. Thus, in view of (5), taking sequential limits with $n \to \infty$ followed by $m \to \infty$ yields

$$\frac{1}{2c} \frac{n}{m} e^{cm}\left(\hat{\rho}_n - \rho_{n,m}\right) \Rightarrow_{n \to \infty} \frac{e^{-cm} \int_0^m \tilde{J}_c(s) \, \mathrm{d}\tilde{W}(s)}{2c e^{-2cm} \int_0^m \tilde{J}_c(s)^2 \, \mathrm{d}s} \quad \text{for fixed } m$$

$$= \frac{\tilde{J}_{-c}(m) \int_0^m e^{-cs} \mathrm{d}\tilde{W}(s)}{\left(\int_0^m e^{-cs} \mathrm{d}\tilde{W}(s)\right)^2} + O_p\left(m^{1/2} e^{-cm}\right)$$

$$\Rightarrow \frac{\tilde{J}_{-c}(\infty) \int_0^\infty e^{-cs} \mathrm{d}\tilde{W}(s)}{\left(\int_0^\infty e^{-cs} \mathrm{d}\tilde{W}(s)\right)^2} \quad \text{as } m \to \infty$$

$$= \frac{\tilde{J}_{-c}(\infty)}{\int_0^\infty e^{-cs} \mathrm{d}\tilde{W}(s)} =_d \mathcal{C}$$

where $\mathcal{C}$ denotes a standard Cauchy variate.

3. Edgeworth expansion on the stationary side

Recall that on the stationary side $c < 0$. The limit (7) may be derived by direct means as follows. We proceed using the joint moment-generating function (m.g.f.) of

$$\left(\frac{1}{\sqrt{m}} \int_0^m \tilde{J}_c \mathrm{d}\tilde{W}, \frac{1}{m} \int_0^m \tilde{J}_c^2 \mathrm{d}s\right)$$

$$= \left(\sqrt{m}\int_0^1 J_{cm}\mathrm{d}W,\, m\int_0^1 J_{cm}^2\mathrm{d}s\right) \tag{9}$$

which, from Phillips (1987, equation (A1)), is

$$L_m(w,z) = M_{cm}\left(\sqrt{m}w, mz\right),$$

where M_{cm} is defined in Proposition A.1. Setting

$$\tau_m = \left(c^2m^2 + 2\,cm^{3/2}w - 2mz\right)^{1/2},$$

the m.g.f. of the random vector in (9) can be written as

$$\begin{aligned} L_m(w,z) &= \left\{\frac{\mathrm{e}^{cm+\sqrt{m}w}}{2\tau_m}\left[\left(\tau_m - cm - \sqrt{m}w\right)\mathrm{e}^{\tau_m} + \left(\tau_m + cm + \sqrt{m}w\right)\mathrm{e}^{-\tau_m}\right]\right\}^{-1/2} \\ &= \mathrm{e}^{-\frac{1}{2}\left(\tau_m+cm+\sqrt{m}w\right)} \times \left[\frac{\tau_m - cm - \sqrt{m}w}{2\tau_m} + \frac{\left(\tau_m + cm + \sqrt{m}w\right)\mathrm{e}^{-2\tau_m}}{2\tau_m}\right]^{-1/2} \\ &= \mathrm{e}^{-\frac{1}{2}\left(\tau_m+cm+\sqrt{m}w\right)}\left[\frac{\tau_m - cm - \sqrt{m}w}{2\tau_m} + O\left(\mathrm{e}^{-dm}\right)\right]^{-1/2} \\ &= \mathrm{e}^{-\frac{1}{2}\left(\tau_m+cm+\sqrt{m}w\right)}\left[\frac{\tau_m - cm - \sqrt{m}w}{2\tau_m}\right]^{-1/2} + O\left(\mathrm{e}^{-dm}\right), \end{aligned} \tag{10}$$

for some $d > 0$, since $\tau_m \sim -cm$ and $\tau_m + cm + \sqrt{m}w \to \left(w^2 + 2z\right)/2c$ as $m \to \infty$ by (22) and (23) in the Appendix. Next we proceed to expand $L_m(w,z)$ as $m \to \infty$. Observe that the asymptotic expansions (22) and (23) for τ_m and τ_m^{-1} respectively give

$$\begin{aligned} \frac{1}{2\tau_m}\left(\tau_m - cm - \sqrt{m}w\right) &= \frac{1}{2}\left[2 + \frac{2w}{cm^{1/2}} - \frac{2w}{cm^{1/2}} + O\left(m^{-1}\right)\right] \\ &= 1 + O\left(m^{-1}\right), \end{aligned}$$

and

$$\begin{aligned} &\mathrm{e}^{-\frac{1}{2}\left(\tau_m+cm+\sqrt{m}w\right)} \\ &= \exp\left\{-\frac{1}{2}\left[\frac{w^2+2z}{2c} - \frac{w^3+2wz}{2c^2}m^{-1/2} + O\left(m^{-1}\right)\right]\right\} \\ &= \exp\left\{-\frac{w^2+2z}{4c}\right\}\exp\left\{\frac{w^3+2wz}{4c^2}m^{-1/2} + O\left(m^{-1}\right)\right\} \\ &= \exp\left\{-\frac{w^2+2z}{4c}\right\}\left[1 + \frac{w^3+2wz}{4c^2}m^{-1/2} + O\left(m^{-1}\right)\right]. \end{aligned}$$

Thus, (10) becomes, as $m \to \infty$,

$$L_m(w,z) = \exp\left\{-\frac{w^2+2z}{4c}\right\} \times \left[1 + \frac{w^3+2wz}{4c^2}m^{-1/2} + O\left(m^{-1}\right)\right] \tag{11}$$

with leading term given by

$$L_m(w,z) = \mathrm{e}^{-\frac{z}{2c} - \frac{1}{4c}w^2}\left[1 + O\left(\frac{1}{\sqrt{m}}\right)\right].$$

Therefore, the numerator of (6) has an $N\left(0, \frac{1}{-2c}\right)$ limit distribution and the denominator has constant probability limit $\frac{1}{-2c}$, as in (7) above. This establishes an alternative proof of (8).

The derivation of an Edgeworth expansion requires considering the next term in the expansion of the joint moment-generating function, i.e. including powers of $m^{-1/2}$ in (11). We provide an expansion of the distribution of the statistic

$$Q_m = \frac{\sqrt{-2c}\left(\frac{1}{\sqrt{m}}\int_0^m \tilde{J}_c\mathrm{d}\tilde{W}\right)}{(-2c)\left(\frac{1}{m}\int_0^m \tilde{J}_c^2\mathrm{d}s\right)} = \frac{A_m}{B_m},$$

which is the limit of $Z_{n,m}$ in (8) as $n \to \infty$, for fixed m. Towards this end, define $D_m = A_m - xB_m$ and note that

$$P\left(Q_m < x\right) = P\left(A_m - xB_m < 0\right) = P\left(D_m < 0\right).$$

An expansion for the moment-generating function of D_m can be obtained from (11) as follows:

$$\begin{aligned} E\left(\mathrm{e}^{D_m s}\right) &= E\left(\mathrm{e}^{sA_m - sxB_m}\right) = L_m\left(s\sqrt{-2c}, -sx(-2c)\right) \\ &= \mathrm{e}^{-sx+\frac{1}{2}s^2}\left[1 + \frac{-2s^2x(-2c)^{3/2} + s^3(-2c)^{3/2}}{4c^2\sqrt{m}} + O\left(\frac{1}{m}\right)\right] \\ &= \mathrm{e}^{-sx+\frac{1}{2}s^2}\left[1 + \frac{s^3 - 2s^2x}{\sqrt{-2c}\sqrt{m}} + O\left(\frac{1}{m}\right)\right]. \end{aligned} \tag{12}$$

Next, $\mathrm{e}^{-sx+\frac{1}{2}s^2}$ is the m.g.f. of the $N(-x, 1)$ distribution so that, as in Satchell (1984), we have

$$P\left(D_m < 0\right) = \int_{-\infty}^0 \frac{\mathrm{e}^{-\frac{1}{2}(t+x)^2}}{\sqrt{2\pi}}\mathrm{d}t = \int_{-\infty}^x \frac{\mathrm{e}^{-\frac{u^2}{2}}}{\sqrt{2\pi}}\mathrm{d}u = \Phi(x), \tag{13}$$

the standard normal c.d.f. Again, as in Satchell (1984), terms such as $s^p\mathrm{e}^{-sx+\frac{1}{2}s^2}$ in the expansion of the m.g.f. (12) correspond to terms of the form $-H_{p-1}(x)\varphi(x)$ in the distributional expansion, where $\varphi(x)$ is the standard normal density and $H_p(x)$ is the Hermite polynomial of order p. From this correspondence and (13), we find that the m.g.f. expansion leads to

$$\begin{aligned} P\left(Q_m < x\right) &= \Phi(x) + \frac{1}{\sqrt{m}}\varphi(x)\frac{-H_2(x) + 2H_1(x)x}{\sqrt{-2c}} + O\left(\frac{1}{m}\right) \\ &= \Phi(x) + \frac{1}{\sqrt{m}}\varphi(x)\frac{1 - x^2 + 2x^2}{\sqrt{-2c}} + O\left(\frac{1}{m}\right) \\ &= \Phi(x) + \frac{1}{\sqrt{m}}\varphi(x)\frac{1 + x^2}{\sqrt{-2c}} + O\left(\frac{1}{m}\right). \end{aligned} \tag{14}$$

Theorem 1. *For each $c < 0$, the distribution of*

$$Q_m = \frac{\sqrt{-2c}\left(\frac{1}{\sqrt{m}}\int_0^m \tilde{J}_c\mathrm{d}\tilde{W}\right)}{(-2c)\left(\frac{1}{m}\int_0^m \tilde{J}_c^2\mathrm{d}s\right)}$$

admits the following Edgeworth expansion:

$$F(x) = \Phi(x) + \frac{1}{\sqrt{m}}\varphi(x)\frac{1 + x^2}{\sqrt{-2c}} + O\left(\frac{1}{m}\right).$$

Remark 1. We may compare (14) with the corresponding expansion for the stationary (fixed ρ with $|\rho| < 1$) case where

$$X_t = \rho X_{t-1} + u_t, \quad u_t \sim iid\left(0, \sigma^2\right). \tag{15}$$

In this case, the Edgeworth expansion of the distribution function F_n of the standardized and centred estimator

$$\frac{\sqrt{n}\left(\hat{\rho}_n - \rho\right)}{\sqrt{1-\rho^2}}$$

was shown in Phillips (1977) to have the form

$$F_n(x) = \Phi(x) + \frac{\rho}{\sqrt{1-\rho^2}} \frac{1+x^2}{\sqrt{n}} \varphi(x) + O\left(\frac{1}{n}\right). \quad (16)$$

The two results may be related by setting $\rho = 1 + \frac{c}{K}$ in (16), leading to

$$F_n(x) = \Phi(x) + \frac{1+O\left(K^{-1}\right)}{\sqrt{-2c/K}} \frac{1+x^2}{\sqrt{n}} \varphi(x) + O\left(\frac{1}{n}\right)$$
$$= \Phi(x) + \frac{1}{\sqrt{-2c}} \frac{1+x^2}{\sqrt{m}} \varphi(x) + O\left(\frac{1}{m} + \frac{1}{K^{3/2}m^{1/2}}\right),$$

which is the same as (14). Therefore, the moderate deviation limit theory is uniform to the second order in the sense that the Edgeworth expansions of the distributions are the same to the first correction term.

4. An expansion on the explosive side

In a related way, we may develop an expansion on the explosive side of unity with $c > 0$. Again, the m.g.f. method initiated by White (1958) is employed and a second-order expansion is obtained as in Satchell (1984). However, the point of expansion in our approach is now the Cauchy distribution delivered by the invariance principle in PM.

We are interested in expanding the moment-generating function of

$$\left(\frac{2c}{e^{cm}} \int_0^m \tilde{J}_c d\tilde{W}, \frac{4c^2}{e^{2cm}} \int_0^m \tilde{J}_c^2 ds\right)$$
$$= \left(\frac{2cm}{e^{cm}} \int_0^1 J_{cm} dW, \frac{4c^2m^2}{e^{2cm}} \int_0^1 J_c^2 ds\right), \quad (17)$$

which in view of Proposition A.1 takes the following form:

$$\Psi_m(u, v) = M_{cm}\left(2cme^{-cm}u, 4c^2m^2e^{-2cm}v\right)$$
$$= \left\{\frac{1}{2\lambda_m} e^{cm+2cme^{-cm}u} \Lambda_m(u, v)\right\}^{-1/2}, \quad (18)$$

where

$$\lambda_m = \left(c^2m^2 + 4c^2m^2e^{-cm}u - 8c^2m^2e^{-2cm}v\right)^{1/2},$$
$$\Lambda_m(u, v) = \left(\lambda_m - cm - 2cme^{-cm}u\right)e^{\lambda_m} + \left(\lambda_m + cm + 2cme^{-cm}u\right)e^{-\lambda_m}.$$

Using the asymptotic expansions (26) and (27) for Λ_m and $\lambda_m^{-1}e^{cm+2cme^{-cm}u}$, respectively, we obtain, as $m \to \infty$,

$$\frac{1}{\lambda_m} e^{cm+2cme^{-cm}u} \Lambda_m(u, v) = 2\left(1 - u^2 - 2v\right) - 8cu\left(u^2+2v\right)me^{-cm} + 8u\left(u^2+2v\right)e^{-cm} + O\left(m^2e^{-2cm}\right).$$

Hence, by (18), the m.g.f. of the random vector in (17) admits the following asymptotic expansion as $m \to \infty$:

$$\Psi_m(u, v) = \left(1-u^2-2v\right)^{-1/2} \times \left\{1 - \frac{4u\left(u^2+2v\right)}{1-u^2-2v}(cm-1)e^{-cm} + O\left(m^2e^{-2cm}\right)\right\}^{-1/2}$$
$$= \left(1-u^2-2v\right)^{-1/2} \times \left\{1 + \frac{2u\left(u^2+2v\right)}{1-u^2-2v}(cm-1)e^{-cm} + O\left(m^2e^{-2cm}\right)\right\}.$$

Thus, the approximate m.g.f. of $(2ce^{-cm}\int_0^m \tilde{J}_c d\tilde{W}, 4c^2e^{-2cm}\int_0^m \tilde{J}_c^2 ds)$ is given by

$$\Psi_m(u, v) = \frac{1}{\left(1-u^2-2v\right)^{1/2}} + \frac{2u\left(u^2+2v\right)}{\left(1-u^2-2v\right)^{3/2}}(cm-1)e^{-cm}. \quad (19)$$

Having obtained an asymptotic approximation for $\Psi_m(u, v)$, we employ a method similar to Satchell (1984) in order to derive an approximation for the density function of $\left(2ce^{-cm}\int_0^m \tilde{J}_c d\tilde{W}\right)\left(4c^2e^{-2cm}\int_0^m \tilde{J}_c^2 ds\right)^{-1}$. Using a standard result for the density of ratios of random variables, the density function of $(2ce^{-cm}\int_0^m \tilde{J}_c d\tilde{W})(4c^2e^{-2cm}\int_0^m \tilde{J}_c^2 ds)^{-1}$ is given by

$$f(r) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \left.\frac{\partial\Psi_m(u, v-ru)}{\partial v}\right|_{v=0} du. \quad (20)$$

Differentiating the expression

$$\Psi_m(u, v-ru) = \frac{1}{\left(1-u^2-2v+2ru\right)^{1/2}} + \frac{2u\left(u^2+2v-2ru\right)(cm-1)e^{-cm}}{\left(1-u^2-2v+2ru\right)^{3/2}}$$

and using the identity

$$\frac{u^3-2ru^2}{\left(1-u^2+2ru\right)^{\frac{5}{2}}} = \frac{-u}{\left(1-u^2+2ru\right)^{\frac{3}{2}}} + \frac{u}{\left(1-u^2+2ru\right)^{\frac{5}{2}}}$$

yields

$$\left.\frac{\partial\Psi_m(u, v-ru)}{\partial v}\right|_{v=0} = \frac{1}{\left(1-u^2+2ru\right)^{\frac{3}{2}}} + 4(cm-1)e^{-cm}\frac{u}{\left(1-u^2+2ru\right)^{\frac{3}{2}}} + 6(cm-1)e^{-cm}\frac{u^3-2ru^2}{\left(1-u^2+2ru\right)^{\frac{5}{2}}}$$
$$= \frac{1}{\left(1-u^2+2ru\right)^{\frac{3}{2}}} - 2(cm-1)e^{-cm}\frac{u}{\left(1-u^2+2ru\right)^{\frac{3}{2}}} + 6(cm-1)e^{-cm}\frac{u}{\left(1-u^2+2ru\right)^{\frac{5}{2}}}.$$

Thus, using a closed-form expression for the contour integrals,

$$h(j, k) = \frac{1}{2i}\int_{-i\infty}^{i\infty} \frac{u^j}{\left(1-u^2+2ru\right)^{\frac{2k+1}{2}}} du, \quad 0 \le j \le k, \quad (21)$$

similar to Satchell (1984), (20) becomes

$$f(r) = \frac{1}{\pi}h(0,1) - \frac{2(cm-1)}{\pi}e^{-cm}h(1,1) + \frac{6(cm-1)}{\pi}e^{-cm}h(1,2)$$
$$= \frac{1}{\pi}\frac{1}{1+r^2} - \frac{2(cm-1)}{\pi}e^{-cm}\frac{r}{1+r^2}$$

$$+\frac{6\,(cm-1)}{\pi}\mathrm{e}^{-cm}\frac{2r}{3\left(r^4+2r^2+1\right)}$$
$$=\frac{1}{\pi\left(1+r^2\right)}+\frac{2r\left(1-r^2\right)}{\pi\left(1+r^2\right)^2}\,(cm-1)\,\mathrm{e}^{-cm}.$$

Theorem 2. *For each $c>0$, the approximate density of*

$$\left(2c\mathrm{e}^{-cm}\int_0^m \tilde{J}_c \mathrm{d}\tilde{W}\right)\left(4c^2\mathrm{e}^{-2cm}\int_0^m \tilde{J}_c^2 \mathrm{d}s\right)^{-1}$$

is given by

$$f(r)=\frac{1}{\pi\left(1+r^2\right)}+\frac{2r\left(1-r^2\right)}{\pi\left(1+r^2\right)^2}\,(cm-1)\,\mathrm{e}^{-cm},$$

with an approximation error of order $O\left(m^2\mathrm{e}^{-2cm}\right)$ as $m\to\infty$.

Remark 2. We may compare the approximate density of Theorem 2 with the approximate density of the normalized and centred least squares estimator in the purely explosive case. Satchell (1984) derives the approximate density of

$$\frac{\rho^n\left(\hat{\rho}_n-\rho\right)}{\rho^2-1}$$

generated by (15) with Gaussian innovations u_t, fixed $\rho\in(-1,1)$ and $X_0=0$. Using MAPLE to calculate the contour integrals in (21), we have obtained the following expression for this density[1]:

$$f_n(r)=\frac{1}{\pi\left(1+r^2\right)}-\frac{1}{\pi}\frac{\rho}{|\rho|^n}\left[2-\frac{\left(\rho^2-1\right)n}{\rho^2}\right]\frac{r-r^3}{\left(1+r^2\right)^2}.$$

Letting $\rho=1+cm/n$, we obtain $|\rho|^{-n}=\mathrm{e}^{-cm}[1+O(m^2/n)]$, giving

$$\frac{\rho}{|\rho|^n}\left[2-\frac{\left(\rho^2-1\right)n}{\rho^2}\right]=-2\,(cm-1)\,\mathrm{e}^{-cm}\left[1+O\left(\frac{m^2}{n}\right)\right].$$

Thus, $f_n(r)$ agrees with the approximate density derived in Theorem 2 as long as $m^2/n\to 0$. The latter condition ensures that $\rho_{n,m}^n=\left(1+\frac{cm}{n}\right)^n$ is approximated by e^{cm}: since $m/n\to 0$,

$$\rho_{n,m}^n=\exp\left\{n\log\left(1+\frac{cm}{n}\right)\right\}=\exp\left\{n\left[\frac{cm}{n}+O\left(\frac{m^2}{n^2}\right)\right]\right\}$$
$$=\mathrm{e}^{cm}\mathrm{e}^{O\left(\frac{m^2}{n}\right)}.$$

5. Discussion

The paper provides a second-order expansion of the local to unity distribution around the standard normal distribution for the stationary side of unity and around the Cauchy distribution on the explosive side of unity. Using the local-to-moderate parameterization for the autoregressive root $\rho_{n,m}=1+\frac{cm}{n}$, the results are obtained by employing sequential asymptotics. A (first-order) limit for the normalized and centred least squares estimator $\hat{\rho}_n$ is obtained in terms of the local to unity distribution as $n\to\infty$. A second-order expansion of the local to unity distribution is obtained as $m\to\infty$.

From an analytical point of view, this procedure is equivalent to a second-order approximation of the Phillips (1987) local to unity distribution for large values of the localizing coefficient. Setting $C=cm$, the local-to-moderate root of (2) becomes the standard local to unit root $\rho_n=1+\frac{C}{n}$, and a second-order limit theory for

$$Q_{1C}=\frac{(-2C)^{1/2}\int_0^1 J_C\mathrm{d}W}{-2C\int_0^1 J_C^2\mathrm{d}s}\quad\text{and}\quad Q_{2C}=\frac{2C\mathrm{e}^{-C}\int_0^1 J_C\mathrm{d}W}{4C^2\mathrm{e}^{-2C}\int_0^1 J_C^2\mathrm{d}s}$$

is given by Theorems 1 and 2 upon substituting $C=cm$:

$$F_{Q_{1C}}(x)=\Phi(x)+\frac{1+x^2}{\sqrt{-2C}}\varphi(x)+O\left(\frac{1}{C}\right)$$
$$f_{Q_{2C}}(r)=\frac{1}{\pi\left(1+r^2\right)}+\frac{2r\left(1-r^2\right)}{\pi\left(1+r^2\right)^2}\,(C-1)\,\mathrm{e}^{-C}.$$

The adequacy of the above approximation may be assessed using the Monte Carlo experiment of Fig. 1 with $C=cm$. However, we favour the use of the parameterization $\rho_{n,m}=1+\frac{cm}{n}$ as it provides more insight into the sample segmentation principle that drives the limit theory and also because it provides a more natural design in Monte Carlo experiments, as m is integer valued.

Appendix

Proposition A.1. *The moment-generating function of $\left(\int_0^1 J_{cm}\mathrm{d}W,\int_0^1 J_c^2\mathrm{d}s\right)$ is given by*

$$M_{cm}(u,v)=\left\{\frac{1}{2\kappa_m}\mathrm{e}^{cm+u}\left[\left(\kappa_m-cm-u\right)\mathrm{e}^{\kappa_m}+\left(\kappa_m+cm+u\right)\mathrm{e}^{-\kappa_m}\right]\right\}^{-1/2}$$

where $\kappa_m=\left(c^2m^2+2cmu-2v\right)^{1/2}$.

A.1. Proof of (3) and (4)

Recalling that $\tilde{W}(t)=\sqrt{m}W(t/m)$ is a standard BM, we obtain

$$\int_0^1 J_{cm}(s)\,\mathrm{d}W(s)=\int_0^1\int_0^s \mathrm{e}^{cm(s-r)}\mathrm{d}W(r)\,\mathrm{d}W(s)$$
$$=\int_0^1 \mathrm{e}^{cms}\int_0^{ms}\mathrm{e}^{-cu}\mathrm{d}W\left(\frac{u}{m}\right)\mathrm{d}W(s)$$
$$=\int_0^m \mathrm{e}^{cv}\int_0^v \mathrm{e}^{-cu}\mathrm{d}W\left(\frac{u}{m}\right)\mathrm{d}W\left(\frac{v}{m}\right)$$
$$=\frac{1}{m}\int_0^m\int_0^v \mathrm{e}^{c(v-u)}\mathrm{d}\left(\sqrt{m}W\left(\frac{u}{m}\right)\right)\times\mathrm{d}\left(\sqrt{m}W\left(\frac{v}{m}\right)\right)$$
$$=\frac{1}{m}\int_0^m\int_0^v \mathrm{e}^{c(v-u)}\mathrm{d}\tilde{W}(u)\,\mathrm{d}\tilde{W}(v)$$
$$=\frac{1}{m}\int_0^m \tilde{J}_c(s)\,\mathrm{d}\tilde{W}(s),$$

where we have used the substitution $v=ms$. Similarly,

$$\int_0^1 J_{cm}(s)^2\,\mathrm{d}s=\int_0^1\left(\int_0^s \mathrm{e}^{cm(s-r)}\mathrm{d}W(r)\right)^2\mathrm{d}s$$
$$=\int_0^1 \mathrm{e}^{2cms}\left(\int_0^{ms}\mathrm{e}^{-cu}\mathrm{d}W\left(\frac{u}{m}\right)\right)^2\mathrm{d}s$$
$$=\frac{1}{m}\int_0^1 \mathrm{e}^{2cms}\left(\int_0^{ms}\mathrm{e}^{-cu}\mathrm{d}\tilde{W}(u)\right)^2\mathrm{d}s$$

[1] There seems to be an omission of a factor of 2 in the second term of Satchell's formula (4.6) for the approximate density.

$$= \frac{1}{m^2}\int_0^m e^{2cv}\left(\int_0^v e^{-cu}\mathrm{d}\tilde{W}(u)\right)^2 \mathrm{d}v$$
$$= \frac{1}{m^2}\int_0^m \left(\int_0^v e^{c(v-u)}\mathrm{d}\tilde{W}(u)\right)^2 \mathrm{d}v$$
$$= \frac{1}{m^2}\int_0^m \tilde{J}_c(s)^2\,\mathrm{d}s. \quad \square$$

A.2. Expansion for τ_m, τ_m^{-1}

For each $c < 0$, we have, as $m \to \infty$,

$$\tau_m = -cm - wm^{1/2} + \frac{w^2+2z}{2c} - \frac{w^3+2wz}{2c^2}m^{-1/2} + O\left(m^{-1}\right), \tag{22}$$

$$\tau_m^{-1} = \frac{1}{-cm} + \frac{w}{c^2m^{3/2}} + O\left(m^{-2}\right). \tag{23}$$

Proof. Using the Taylor expansion for $(1+x)^{1/2}$, we can write

$$\tau_m = |c|\,m\left\{1 + \frac{2}{cm^{1/2}}w - \frac{2z}{mc^2}\right\}^{1/2}$$
$$= -cm\left\{1 + \frac{w}{cm^{1/2}} - \frac{z}{mc^2} - \frac{1}{8}\left(\frac{2}{cm^{1/2}}w - \frac{2z}{mc^2}\right)^2 + \frac{w^3}{2c^3m^{3/2}} + O\left(\frac{1}{m^2}\right)\right\}$$
$$= -cm\left\{1 + \frac{w}{cm^{1/2}} - \frac{w^2+2z}{2c^2m} + \frac{w^3+2wz}{2c^3m^{3/2}} + O\left(\frac{1}{m^2}\right)\right\}.$$

The expansion for τ_m^{-1} is obtained by an identical argument. $\square$

A.3. Expansion for λ_m, λ_m^{-1}

For each $c > 0$, we have, as $m \to \infty$,

$$\lambda_m = cm + 2cume^{-cm} - 2c\left(u^2+2v\right)me^{-2cm} + 4cu\left(u^2+2v\right)me^{-3cm} + O\left(me^{-4cm}\right), \tag{24}$$

$$\lambda_m^{-1} = \frac{1}{cm} - \frac{2ue^{-cm}}{cm} + \frac{\left(6u^2-4v\right)e^{-2cm}}{cm} + O\left(m^{-1}e^{-3cm}\right). \tag{25}$$

Proof. Using the Taylor expansion for $(1+x)^{1/2}$, we can write

$$\frac{\lambda_m}{cm} = \left(1 + 4e^{-cm}u - 8e^{-2cm}v\right)^{1/2}$$
$$= 1 + \frac{1}{2}\left(4e^{-cm}u - 8e^{-2cm}v\right) - \frac{1}{8}\left(4e^{-cm}u - 8e^{-2cm}v\right)^2 + \frac{1}{16}\left(4e^{-cm}u - 8e^{-2cm}v\right)^3 + O\left(e^{-4cm}\right)$$
$$= 1 + 2e^{-cm}u - 4e^{-2cm}v - 2e^{-2cm}u^2 + 8uve^{-3cm} + 4e^{-3cm}u^3 + O\left(e^{-4cm}\right),$$

and (24) follows upon multiplication by cm. For (25), expanding the reciprocal of (24), we obtain

$$cm\lambda_m^{-1} = \left\{1 - \left[-2ue^{-cm} + 2\left(u^2+2v\right)e^{-2cm} + O\left(e^{-3cm}\right)\right]\right\}^{-1}$$
$$= 1 - 2ue^{-cm} + \left(6u^2-4v\right)e^{-2cm} + O\left(e^{-3cm}\right). \quad \square$$

A.4. Expansion for $\Lambda_m(u, v)$

For each $c > 0$, we have, as $m \to \infty$,

$$\Lambda_m(u,v) = 2c\left(1-u^2-2v\right)me^{-cm} - 4c^2u\left(1+u^2+2v\right)m^2e^{-2cm} + 4cu\left(1+u^2+2v\right)me^{-2cm} + O\left(m^3e^{-3cm}\right). \tag{26}$$

Proof. By (24) and the Taylor series for the exponential function, we obtain

$$e^{\lambda_m} = e^{cm}\exp\left\{2cume^{-cm} + O\left(me^{-2cm}\right)\right\}$$
$$= e^{cm}\left[1 + 2cume^{-cm} + O\left(m^2e^{-2cm}\right)\right]$$
$$= e^{cm} + 2cum + O\left(m^2e^{-cm}\right).$$

Similarly, (25) yields

$$e^{-\lambda_m} = e^{-cm}\exp\left\{-2cume^{-cm} + O\left(me^{-2cm}\right)\right\}$$
$$= e^{-cm}\left[1 - 2cume^{-cm} + O\left(m^2e^{-2cm}\right)\right]$$
$$= e^{-cm} - 2cume^{-2cm} + O\left(m^2e^{-3cm}\right).$$

Using the above expansions for e^{λ_m} and $e^{-\lambda_m}$ together with (24), we obtain

$$\Lambda_m(u,v) = \left[-2c\left(u^2+2v\right)me^{-2cm} + 4cu\left(u^2+2v\right)me^{-3cm} + O\left(me^{-4cm}\right)\right] \times \left[e^{cm} + 2cum + O\left(m^2e^{-cm}\right)\right] + \left[2\,cm + 4cume^{-cm} + O\left(me^{-2cm}\right)\right]\left[e^{-cm} - 2cume^{-2cm} + O\left(m^2e^{-3cm}\right)\right]$$
$$= -2c\left(u^2+2v\right)me^{-cm} + 4cu\left(u^2+2v\right)me^{-2cm} - 4c^2u\left(u^2+2v\right)m^2e^{-2cm} + 2cme^{-cm} + 4cume^{-2cm} - 4c^2um^2e^{-2cm} + O\left(m^3e^{-3cm}\right)$$
$$= 2c\left(1-u^2-2v\right)me^{-cm} - 4c^2u\left(1+u^2+2v\right)m^2e^{-2cm} + 4cu\left(1+u^2+2v\right)me^{-2cm} + O\left(m^3e^{-3cm}\right). \quad \square$$

A.5. Expansion for $\lambda_m^{-1}e^{cm+2cme^{-cm}u}$

For each $c > 0$, we have, as $m \to \infty$,

$$\lambda_m^{-1}e^{cm+2cme^{-cm}u} = \frac{1}{cm}e^{cm} + 2u - \frac{2u}{cm} + \frac{\left(6u^2-4v\right)e^{-cm}}{cm} - 4u^2e^{-cm} + 2cu^2me^{-cm} + O\left(m^2e^{-2cm}\right). \tag{27}$$

Proof. Writing

$$e^{cm+2cme^{-cm}u} = e^{cm}\exp\left\{2cme^{-cm}u\right\}$$
$$= e^{cm}\left[1 + 2cmue^{-cm} + 2c^2u^2m^2e^{-2cm} + O\left(m^3e^{-3cm}\right)\right]$$
$$= e^{cm} + 2cmu + 2c^2u^2m^2e^{-cm} + O\left(m^3e^{-2cm}\right),$$

we can obtain (27) by multiplying the above expression with the expansion for λ_m^{-1} in (25). $\square$

References

Giraitis, L., Phillips, P.C.B., 2006. Uniform limit theory for stationary autoregression. Journal of Time Series Analysis 27, 51–60.

Phillips, P.C.B., 1977. Approximations to some finite sample distributions associated with a first order stochastic difference equation. Econometrica 45 (2), 463–485.

Phillips, P.C.B., 1987. Towards a unified asymptotic theory for autoregression. Biometrika 74, 535–547.

Phillips, P.C.B., Magdalinos, T., 2007. Limit theory for moderate deviations from a unit root. Journal of Econometrics 136, 115–130.

Pollard, D., 1984. Convergence of Stochastic Processes. Springer-Verlag.

Satchell, S.E., 1984. Approximation to the finite sample distribution for nonstationary first order stochastic difference equations. Econometrica 52, 1271–1290.

White, J.S., 1958. The limiting distribution of the serial correlation coefficient in the explosive case. Annals of Mathematical Statistics 29, 1188–1197.