ECONOMETRIC ANALYSIS OF FISHER'S EQUATION

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Econometric Analysis of Fisher's Equation

By PETER C. B. PHILLIPS*

ABSTRACT. Fisher's equation for the determination of the real rate of interest is studied from a fresh econometric perspective. Some new methods of data description for nonstationary time series are introduced. The methods provide a nonparametric mechanism for modelling the spatial densities of a time series that displays random wandering characteristics, like interest rates and inflation. Hazard rate functionals are also constructed, an asymptotic theory is given, and the techniques are illustrated in some empirical applications to real interest rates for the United States. The paper ends by calculating semiparametric estimates of long-range dependence in U.S. real interest rates, using a new estimation procedure called modified log periodogram regression and new asymptotics that covers the nonstationary case. The empirical results indicate that the real rate of interest in the United States is (fractionally) nonstationary over 1934–1997 and over the more recent subperiods 1961–1985 and 1961–1997. Unit root nonstationarity and short memory stationarity are both strongly rejected for all these periods.

I

Introduction

SINCE IRVING FISHER (1896, 1930) formalized[1] the notion of a real rate of interest, the concept has played a significant role in the formulation of a wide range of economic models. These include individual agent decision making regarding investment, savings, and portfolio allocations, options pricing models in finance, and the modern theory of inflation targeting in macroeconomics, to name but a few. Naturally enough, in light of the role that the real rate plays in economic theory models, a good deal of attention has been devoted in the literature, especially in macroeconomics, to the measurement of the real

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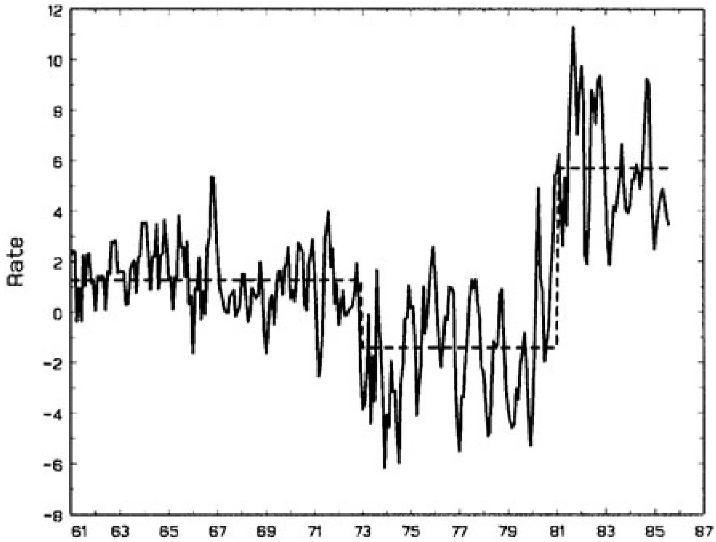
rate and to the characterization of its temporal dependence properties. *Prima facie*, this task seems like a simple exercise in time series econometrics. However, the empirical analysis is complicated by two factors: (1) the apparent nonstationary behavior of the series involved, particularly interest rates but also sometimes inflation; and (2) the fact that the *ex ante* real rate of interest depends on inflation expectations and is therefore not directly measured. Perhaps because of these complicating factors, no consensus seems to have emerged about the time series properties of the real rate of interest, in spite of intensive empirical study. In particular, while economic theory models routinely assume that the real rate of interest is a constant, or fluctuates in a stationary way about a constant mean, the empirical work indicates that this is not so or at best holds only over short regimes.

Figure 1 shows monthly data for the *ex post* real interest rate in the United States over the period 1961:1–1985:12. The series is calculated by taking the U.S. 90-day treasury bill rate for the nominal interest rate and by using the U.S. monthly CPI (all commodities, with no adjustment for housing costs) to compute three-month inflation rates. The figure also shows subgroup means calculated over the subperiods 1961:1–1973:1, 1973:2–1982:1, and 1982:2–1985:12. The data cover the same period as that studied recently by Garcia and Perron (1994), who used regime shift methods to estimate the *ex ante* real rate over approximately these subperiods. These authors concluded that the *ex ante* real rate of interest was effectively constant but subject to occasional mean shifts over 1961–1985. They found two mean shifts over this time period and gave results very similar to those obtained by subgroup means that are displayed in Figure 1. For these data, at least, the conclusion does not seem unreasonable.

Figures 2 and 3 graph the *ex post* real interest rate series calculated in the same way over the longer periods 1961–1997 and 1934–1997. Over the 1961–1997 period the graphs shows subsample means for the additional two subperiods 1990–1993 and 1994–1997. Apparently, there is a need to allow for continuing regime shifts in the mean level if this approach to modeling the real rate of interest is to give acceptable results. For the longer period, an even larger number of mean shifts is needed to accommodate this approach, and the results seem

Figure 1

Ex post real interest rate: 1961–1985.



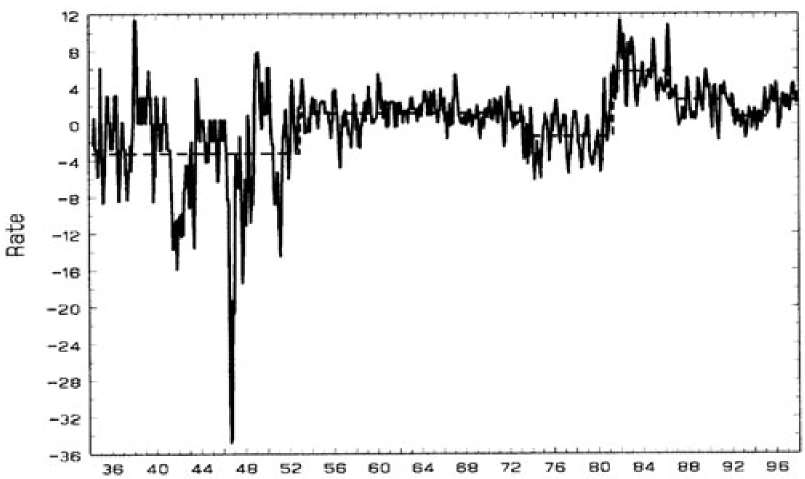
much less satisfying—we merely illustrate what is involved in Figure 3. It is hard to conclude that one is doing any more than simply curve fitting in such exercises and then possibly providing *ex post* rationalizations for the mean shifts. Note, however, that the data in Figure 3 clearly do support the conclusion reached in Fama's (1975) influential study that the real rate has a constant mean over the period 1953–1971. Notwithstanding the apparent constancy over 1953–1971, so many mean shifts are needed to model the data this way that the Garcia-Perron conclusion that the *ex ante* real rate of interest fluctuates about a constant mean, subject to occasional mean shifts, seems much less reasonable over the longer period 1934–1997 than it does for 1961–1986. Indeed, over longer periods such as this, most formal tests (e.g., Rose 1988; Walsh 1987) support a conclusion that is the opposite of stationarity and favor unit root nonstationary. One of the goals of the present paper is to determine whether there are other hypotheses that are more reasonable than these two alternatives.

Figure 2

Ex post real interest rate: 1961–1997.

Figure 3

Ex post real interest rate: 1934–1997.



Another goal of the paper is to contribute some new methods to assist in the econometric analysis of data of this type. The methods given here furnish a new way of describing and characterizing data like interest rates and inflation that appear to have nonstationary elements. More specifically, the paper proposes a nonparametric spatial density estimate as a new descriptive tool for nonstationary time series. Many series like the interest rates shown in Figures 1–3 behave as if they have no fixed mean. The random wandering characteristic of these series is hard to describe quantitatively. What a spatial density does is provide useful quantitative information about the spatial location characteristics of a time series, in just the same way as a probability density can be used to characterize stationary time series. As will be shown, in contrast to a probability density, the spatial density is a random process. However, it turns out that we can still obtain consistent estimates of spatial densities using nonparametric techniques, like those of kernel methods, which have proved useful in studying iid and strictly stationary time series. We outline these new procedures here and provide an asymptotic theory that characterizes their large sample properties and facilitates inference.

Once nonparametric spatial density estimates have been obtained, we can use them in similar ways to that of a probability density—to study and quantify the locational characteristics of the time series. We illustrate these ideas by looking at nonparametric estimates of the spatial location of the *ex post* real rate of interest in the United States. This exercise provides some nonparametric evidence on recent empirical findings about real interest rates and the Fisher effect. In addition, we show how to construct a new type of hazard function for nonstationary time series. Hazard rates are particularly helpful in studying financial series, as they can be used to quantify the hazards of certain interest rate and inflation rate levels, for example. The methods can also be used to study how empirical hazard rates evolve over time. Again, we illustrate the techniques in some empirical applications to the *ex post* real rate of interest series for the U.S. economy shown in Figures 1–3.

Finally, we attempt to model the real rate of interest directly as a potentially nonstationary long memory process. This involves the econometric estimation of the long memory parameter (d), without making any delimiting assumptions about the short memory components in the data generating process. The procedure we use is a new semiparametric estimator developed by the author (1999a) in recent work on fractional processes. This new estimator is called modified log periodogram (LP) regression. Kim and Phillips (1999a) have shown that, in contrast to log periodogram regression, this method has good asymptotic properties for values of d over the full region $0 < d < 2$, so that it is well suited for use with time series, like interest rates, whose memory parameters may extend into the nonstationary region where

$$d \geq \frac{1}{2}.$$

We will briefly describe the new approach in Section VI and use the method to estimate d and to provide confidence intervals, thereby enabling us to answer the question of whether the preferred model for data on the real rate of interest is stationary or nonstationary.

The paper is organized as follows. The Fisher effect and associated regression equations are discussed in Section II. The econometric ideas that underlie the spatial modeling apparatus that we introduce are laid out in Sections III and IV. Section V provides an empirical application of the methods to U.S. data over the period 1934–1997. Section VI reports our new asymptotics and empirical estimates of the long memory parameter. Section VII concludes the paper and discusses some related issues. Proofs are collected together in Section VIII with some of the new asymptotic theory that is introduced in the paper.

II

The Fisher Effect

IRVING FISHER FORMULATED THE CONCEPT of the (*ex ante*) real rate of interest (r_t^e) to provide a rate of interest that accounted for the value of loan repayments in real dollar terms. That is, a nominal interest rate of i_t will assure a real rate of r_t^e when the anticipated price change is π_t^e provided $i_t = r_t^e + \pi_t^e + r_t^e \pi_t^e$, thereby adjusting the compensation to the lender for the anticipated losses in purchasing power in the principal and the interest. The term $r_t^e \pi_t^e$ is usually ignored because it is of smaller order, so that the Fisher equation is commonly written as

$$i_t = r_t^e + \pi_t^e. \qquad (1)$$

Equation (1) is sometimes made more specific by indexing the interest rate by the time period (m) of the bond to maturity and by using m-period ahead inflation expectations, leading to the alternate formulation

$$i_t^m = r_t^{e,m} + \pi_t^{e,m}. \qquad (2)$$

The empirical literature surrounding Equation (1) is vast and we will not attempt to provide a review here. Fisher himself began the empirical work and considered the problems surrounding the relationship between prices and interest rates to be "of such vital importance that I have gone to much trouble and expense to have such data as could be found compiled, compared, and analyzed" (1930: p.399).

The main object of Fisher's work was "to ascertain to what extent, if at all, a change in the general price level actually affects the market rates of interest" (1930: p.399). He conducted correlational studies between inflation and interest rates with annual data primarily for the United States and the United Kingdom, concluding as follows:

> Our first correlations seemed to indicate that the relationship between *P'* (inflation) and *i* (interest rate) is either very slight or obscured by other factors. But when we make the much more reasonable supposition that price changes do not exhaust their effects in a single year but manifest their influence with diminishing intensity over long periods which vary in length with the conditions, we find a very significant relationship, especially in the period which includes the World War, when prices were subject to violent fluctuations. (1930: 423)

Since this initial work by Fisher, a substantial amount of empirical work has been conducted with data from many countries, covering different periods of time and maturities. However, little consensus seems to have emerged from these studies about the nature of the Fisher effect. In particular, there seems to be little agreement about the statistical properties of the real rate r_t^e.

Two recent studies that make significant methodological departures in studying this problem are by Mishkin (1992) and the already cited paper by Garcia and Perron (1994). These papers bring some modern nonstationary time series and regime shift methods to bear in analyzing the Fisher equation. Using residual-based co-integration tests, Mishkin finds support in the data for a "long-run" Fisher effect in which inflation and interest rates share a common stochastic trend. Observe that if π_t^m is the *m*-period inflation rate, then Equation (2) gives the following relation between the *ex post* real rate of interest ($r_t^m = i_t^m - \pi_t^m$) and the *ex ante* rate of interest $r_t^{e,m}$:

$$r_t^m = r_t^{e,m} + \left(\pi_t^{e,m} - \pi_t^m\right) = r_t^{e,m} + \varepsilon_t. \qquad (3)$$

Under rational expectations, where agents or the market use all information efficiently in forecasting inflation, the forecast error $\varepsilon_t = \pi_t^{e,m} - \pi_t^m$ will be a martingale difference and can be assumed to be stationary, or integrated of order zero ($I(0)$). Under this hypothesis, the *ex post* and *ex ante* real rates differ by a stationary component and therefore have the same long-run time series properties. Thus, following Mishkin (1992), the *ex post* real rate can only be $I(1)$ if $r_t^{e,m}$ is $I(1)$. Hence, a test for a unit root in r_t^m against stationary alternatives can be interpreted as a test for a unit root in $r_t^{e,m}$ against a stationary *ex ante* real rate. Put another way, co-integration between i_t^m and π_t^m is the alternative hypothesis in a test for a unit root in the *ex post* real rate of interest. The Fisher effect then corresponds to the hypothesis that the *ex ante* real rate of interest is stationary, so that, under rational expectations and stationary forecast errors for inflation, the Fisher effect implies a stationary *ex post* real rate of interest.

In spite of their apparent simplicity, Equations (1) and (3) present a host of econometric difficulties arising from the fact that the variables r_t^e and π_t^e are unmeasured latent variables, and from the time series difficulties in modeling apparently nonstationary series like interest rates and inflation.

While Fisher (1930) analyzed the relationship between interest rates and inflation using correlational techniques, modern approaches rely on regression methods—sometimes, as in Summers (1983), in the frequency domain where low frequencies can be used to emphasize long-run properties. Depending on the properties of the real rate r_t^e, Equation (3) suggests a regression link between i_t and π_t^e. In particular, the Fisher effect asserts that the coefficient b should be unity in a regression of the form

$$i_t = c + b\pi_t^e + u_t$$

and the residuals u_t should be stationary. Under this hypothesis, we can write the *ex post* real rate of interest as

$$r_t = i_t - \pi_t = c + b(\pi_t^e - \pi_t) + u_t = c + w_t, \tag{4}$$

which implies stationary fluctuations about a constant level c.

In a celebrated study mentioned earlier, Fama (1975) found empirical evidence of a constant real rate of interest over the period 1953–1971. Mishkin (1981) subsequently rejected constancy in the real rate in a more extensive study covering the longer periods 1953–1979 and 1931–1952. Later investigations by Rose (1988) and Walsh (1987) found evidence in support of unit root nonstationarity in *ex post* real rates. Most recently, Garcia and Perron (1996) reanalyzed data over the period 1961–1986 using regime shift techniques and found support for a constant real rate of interest, subject to infrequent changes in the constant c. In short, the empirical evidence gives a mixed picture about the statistical properties of the real rate of interest, and it is probably fair to say that the generating mechanism for the real rate is very imperfectly understood. Although the bulk of the econometric evidence now points against the hypothesis of a pure Fisher effect in which the real rate is stationary about a constant mean, this does not rule out modified Fisher effects such as those supported in the Garcia-Perron study.

We now propose to take a very different approach to studying these data. The essence of the new approach is descriptive, but the asymptotic theory that we have developed for the quantities involved enables us to use them in an inferential framework as well. This helps us to corroborate and assess earlier empirical findings.

III

Spatial Densities for Nonstationary Series

OUR PROPOSAL IS TO STUDY the spatial characteristics of a time series rather than focus on their temporal dependence properties, as one would normally do in a stationary time series analysis. The starting point is to assume that, when appropriately normalized and transformed into a random function on the interval [0, 1], the time series trajectories converge weakly to those of a continuous stochastic process on the same interval. Such an assumption applies under a very wide variety of possible conditions, and it seems a very weak requirement if one is to make any headway in the development of asymptotic methods. Accordingly, we may suppose that the limit process, $M(r)$, say, of the normalized series is a continuous semi-martingale (see, e.g., Protter 1990). This requirement would then include the huge class of time series for which functional central limit theorems are known to apply, leading, for example, to Brownian motions and diffusion processes as special cases (see, e.g., Phillips and Solo 1992). While this class of time series is substantial, one case that is excluded by the semi-martingale requirement is time series that upon normalization tend to fractional processes like fractional Brownian motion (e.g., Mandlebrot and van Ness 1968; Taqqu 1975). This case does seem to be important in applications of our ideas to nonstationary series, because there is increasing evidence that economic time series are well modeled by long memory processes with fractional Brownian motion limits; for some recent macroeconomic time series evidence, see Gil-Alana and Robinson (1997) and Kim and Phillips (1999a). Fortunately, it seems that it will be possible to include fractional processes within our theory and, although this is not done here, some discussion and references are provided later (see the paragraph following Equation (12)).

For a semi-martingale $M(r)$ it is known that there exists an increasing stochastic process[2] (increasing in the argument r, that is) called the local time of M at s and denoted $L_M(r, s)$ that represents the amount of time that the limit process spends in the spatial vicinity of the point s. The local time process is defined as

$$L_M(r, s) = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_0^r 1(|M(t) - s| < \varepsilon) d[M]_t, \tag{5}$$

where $[M]_t$ is the quadratic variation process of M. Notice that this definition of "time spent in the vicinity of s" is expressed in units of variation, as measured by $d[M]_t$. In effect, $L_M(r, s)$ measures the contribution to the quadratic variation $[M]_r$ over the interval $[0, r]$ that comes from variation in $M(t)$ around the level s.

It turns out that we can also write down an inverse relation of the form

$$[M]_t = \int_{-\infty}^{\infty} L_M(t, s)ds, \tag{6}$$

which gives a decomposition of the quadratic variation into contributions to the conditional variance that come from fluctuations in the process that occur in the neighborhood of different spatial points $s \in [-\infty, \infty]$. In a sense, we can think of Equation (6) as the spatial equivalent for a continuous nonstationary random process of the decomposition of the variation of a stationary time series into contributions from different frequencies, that is,

$$\sigma^2 = \int_{-\pi}^{\pi} f_{xx}(\lambda)d\lambda,$$

where $f_x(\lambda)$ is the spectral density of X_t and $\sigma^2 = \text{var } X_t$. In this formula, λ is a continuous variable representing the frequency of the oscillations into which the time series variation is being decomposed. However, in Equation (6), the variable s represents spatial points in the real line, where the process spends some time. Fluctuations in the process around the point s, which can occur at various times in the time interval $[0, t]$, then contribute to the density $L_M(t, s)$ of the process at this spatial point.

Equation (6) is particularly interesting in the special case where $M(r)$ is the Brownian motion $B(r)$ with variance ω^2. Here, the parameter ω^2 arises as the long-run variance of the shocks which drive the unit root process that converges weakly to $B(r)$. In this case, we have

$$\omega^2 = \int_{-\infty}^{\infty} L_B(1, s)ds,$$

giving a decomposition of the long-run variance ω^2 into components that reflect the density of the fluctuations in the process around all spatial points $s \in (-\infty, \infty)$.

The local time process $L_M(r, s)$ is known to satisfy the following equation

$$|M(r) - s| = |M(0) - s| + \int_0^r \text{sgn}(M(t) - s)dM(t) + L_M(r, s), \tag{7}$$

where sgn(x) = 1, −1 for $x > 0$, $x \leq 0$. In fact, the process $L_M(r, s)$ is sometimes defined in this manner; for example, see Revuz and Yor (1994). Equation (7), gives a development of the function $|M(r) - s|$ is about its value at $r = 0$ and thereby provides a mechanism of generalizing the Ito stochastic calculus of functions that are continuously differentiable to the second order ($\mathbb{C}_2$ functions) to functions that are not everywhere differentiable and smooth, and, in particular, to convex functions, which we will show below. These extensions look to be particularly valuable in theoretical models where convex functions play a big role, but have not to my knowledge been used yet in economic analysis.

As is well known (e.g., Protter 1990: 74), if $f \in \mathbb{C}_2$ we have the Ito formula for continuous semi-martingales given by

$$f(M(r)) = f(M(0)) + \int_0^r f'(M(t))dM(t) + \frac{1}{2}\int_0^r f''(M(t))d[M]_t. \quad (8)$$

On the other hand, when f is convex we have, based on Equation (7),

$$f(M(r)) = f(M(0)) + \int_0^r f'_-(M(t))dM(t) + \frac{1}{2}\int_{-\infty}^{\infty} f''_g(p)L_M(r, p)dp, \quad (9)$$

where f'_- is the left derivative of f and f''_g is the second derivative of f in the generalized function sense. When $f \in \mathbb{C}_2$ we have $f'_-(M(t)) = f'(M(t))$, and $f''_g(p) = f''(p)$, the second derivative in the usual sense, and the occupation time formula (e.g., Revuz and Yor 1991: 209) gives

$$\frac{1}{2}\int_{-\infty}^{\infty} f''(p)L_M(r, p)dp = \frac{1}{2}\int_0^r f''(M(t))d[M]_t, \quad (10)$$

so that Equation (9) reduces to Equation (8) in this special case.

Observe that Equation (10) shows the sense in which $L_M(r, p)$ is a spatial density for the process $M(r)$, recording the amount of time (measured in units of the quadratic variation $[M]_t$) that the process has spent in the immediate vicinity of p over the time interval $t \in [0, r]$.

We now propose to use these concepts in studying the spatial properties of time series like interest rates and inflation. Our first task is to estimate the local time process $L_M(r, p)$ for a particular time series. Just as the spectral density of a stationary time series is estimated by nonparametric methods, the local time $L_M(r, p)$ can be estimated in a nonparametric manner as follows. Let X_t be a time series that satisfies a functional law of the form

$$\frac{1}{n^{\alpha}} X_{[nr]} \Rightarrow M(r), \tag{11}$$

where $M(r)$ is a continuous semi-martingale for $r \in [0, 1]$. For instance, when X_t is an integrated process of order 1, standard functional central limit theorems (e.g., Phillips and Solo 1992) lead to Equation (11) with

$$\alpha = \frac{1}{2} \text{ and } M(r) = B(r),$$

a Brownian motion with variance

$$\omega^2 = 2\pi f_{\Delta x}(0).$$

When X_t is near integrated, so that the quasi differences

$$\Delta_c X_t = X_t - \left(1 + \frac{c}{n}\right) X_{t-1}$$

are stationary with positive spectral density at the origin, then

$$\alpha = \frac{1}{2} \text{ and } M(r) = J_c(r),$$

a linear diffusion process (e.g., Phillips 1987). On the other hand, when X_t is a fractionally integrated process whose memory parameter

$$d \in \left(\frac{1}{2}, 1\right],$$

whose initialization is at $t = 0$, and for which the differenced process $(1 - L)^d X_t = u_t$ is stationary with positive spectrum $\omega^2 > 0$ at the origin, then we have the following functional law (cf. Akonom and Gourieroux 1987)

$$\frac{1}{n^{d-\frac{1}{2}}} X_{[nr]} \Rightarrow B_{d-1}(r) = \frac{\omega}{\Gamma(d)} \int_0^r (r-s)^{d-1} dW(s),$$

where $B_{d-1}(r)$ is a fractional Brownian motion. In this case, the limit process $M(r) = B_{d-1}(r)$ is not a semi-martingale and, indeed, the process B_{d-1} actually has infinite quadratic variation. Hence, in place of Equation (5), where quadratic variation is being distributed spatially, it is more helpful to define local time according to

$$\bar{L}_M(r, s) = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_0^r 1(|M(t) - s| < \varepsilon)dt, \tag{12}$$

which measures the amount of time in chronological units that M_t spends in the vicinity of s over the interval [0, r]. Accordingly, $\bar{L}_M(r, s)$ is called the chronological local time of M at s over [0, r] (cf Phillips and Park 1998). Local time for the fractional process B_{d-1} can be defined using Equation (12), and Tyurin and Phillips (1999) give a development of the theory of local time using this approach and discuss its empirical estimation.

These examples seem to cover most cases of empirical interest that arise in the present literature on nonstationary economic time series. In the development that follows we will assume we are working with an integrated process X_t for which

$$n^{-\frac{1}{2}} X_{[nr]} \Rightarrow B(r).$$

Or, if we want to allow for distant initial conditions (at time $-[n\kappa]$ for some $\kappa \geq 0$) in the origination of X_t then we can use the limit

$$n^{-\frac{1}{2}} X_{[nr]} \Rightarrow B(r) + B_0(\kappa),$$

where B and B_0 are independent Brownian motions, with the process B_0 extending in a reverse (negative) direction reflecting the effect of distant initial conditions at some fraction κ of the sample size n. This will cover a wide class of interesting practical cases. Extensions of the theory to the more general case of Equation (11) are also possible but are more difficult and will call upon strong approximation versions of Equation (11) that, at least to the author's present knowledge, do not seem to be available in the probability literature as yet for a general class of processes (although a strong approximation for diffusion processes is given in Phillips 1998). A complete development of the asymptotic theory in our case to cover situations as general as Equation (11) is beyond the scope of the present paper. Extensions to the important fractional Brownian motion case are considered by Tyurin and Phillips (1999).

A natural candidate for estimating the local time of the limit process of

$$n^{-\frac{1}{2}} X_{[nr]}$$

at s is the scaled kernel estimate

$$\frac{\hat{\omega}^2}{nh_n}\sum_{t=1}^{n} K\left(\frac{s-X_t}{h_n}\right) = \frac{\hat{\omega}^2}{nh_n}\sum_{t=1}^{n} K\left(c_n\left\{\frac{s}{\sqrt{n}} - \frac{X_t}{\sqrt{n}}\right\}\right), \quad c_n = \frac{\sqrt{n}}{h_n}, \quad (13)$$

where h_n is a bandwidth parameter, $K(\cdot)$ is a symmetric kernel function, and $\hat{\omega}^2$ is a consistent estimate of $\omega^2 = 2\pi f_{\Delta x}(0)$. Upon restandardization of this estimate we have the following extension of a result obtained recently in Phillips and Park (1998).

A. Theorem

Suppose $\hat{\omega}^2 \to_p \omega^2 = 2\pi f_{\Delta x}(0)$. *If* $s = s_0 + \sqrt{n}\,a$, *with* s_0 *and* a *fixed, and if assumptions VIII.A–VIII.D in the Appendix hold, then as* $n \to \infty$

$$\hat{L}_B\left(r, \frac{s}{\sqrt{n}}\right) = \frac{\hat{\omega}^2}{\sqrt{n}h_n}\sum_{t=1}^{[nr]} K\left(\frac{s-X_t}{h_n}\right) \to_p L_B(r, a - B_0(\kappa)). \quad (14)$$

The definition of

$$\hat{L}_B\left(r, n^{-\frac{1}{2}}s\right)$$

involves scaling the conventional kernel estimator given in Equation (13) by $\sqrt{n}$. The reason for this restandardization is that in the nonstationary case the process X_t wanders away from the location s at the rate $\sqrt{n}$ and, for such departures from s, $K(h_n^{-1}(s - X_t))$ is negligibly small. In effect, the stochastic trend property of X_t reduces the order of magnitude of the kernel estimate compared with the stationary case.

Note that the estimate

$$\hat{L}_B\left(r, n^{-\frac{1}{2}}s\right)$$

evaluates the local time at a spatial point

$$n^{-\frac{1}{2}}s$$

that is affected by the rate of convergence of

$$n^{-\frac{1}{2}}X_{[nr]}$$

to its Brownian motion limit process. Hence, if s is fixed and initial conditions are $O_p(1)$, i.e., $\kappa = 0$ as $n \to \infty$, then the estimate

$$\hat{L}_B\left(r, n^{-\frac{1}{2}}s\right)$$

is consistent for the local time of the limit process at the origin, that is, $L_B(r, 0)$. When $\kappa > 0$, the random initialization shifts the spatial point of evaluation of the local time estimate so that it is centered around $B_0(\kappa)$. So, initial conditions generally play a role in the spatial density of the process, as we might well expect for a nonstationary series.

Our next step is to construct confidence regions for the density estimate

$$\hat{L}_B\left(r, n^{-\frac{1}{2}}s\right).$$

This can be done using the limiting distribution theory for

$$\hat{L}_B\left(r, n^{-\frac{1}{2}}s\right)$$

given in Theorem III.B below. The limit distribution turns out to be mixed normal (denoted *MN* in what follows) with a mixing variate that is proportional to the spatial density itself. This limit theory justifies the construction of confidence intervals in the usual manner.

B. Theorem

If $s = s_0 + \sqrt{n}a$, *with* s_0 *and a fixed, if assumptions VIII.A–VIII.D in the Appendix hold, and if* $\sqrt{c_n}(\hat{\omega}^2 - \omega^2) = o_p(1)$ *where* $c_n = h_n^{-1}\sqrt{n}$, *then as* $n \to \infty$

$$\begin{aligned} &\sqrt{c_n}\left[\hat{L}_B\left(r, n^{-\frac{1}{2}}s\right) - L_B(r, a - B_0(\kappa))\right] \\ &\Rightarrow 2\int_{-\infty}^{\infty} K(p)Q(L_B(r, a - B_0(\kappa)), p)dp \\ &\equiv MN(0, 8K_2L_B(r, a - B_0(\kappa))), \end{aligned}$$

where Q(a, b) is a standard Brownian sheet, W(p) is a standard Brownian motion, and

$$K_2 = \int_0^{\infty}\int_0^{\infty} K(p)(p \wedge t)K(t)dpdt.$$

This result enables us to construct confidence intervals for $L_B(r, a - B_0(\kappa))$ for each spatial point. Thus,

$$\hat{L}_B\left(r, n^{-\frac{1}{2}}s\right) \pm 1.96\left(\frac{8K_2\hat{L}_B\left(r, n^{-\frac{1}{2}}s\right)}{c_n}\right)^{\frac{1}{2}}$$

is a 95% confidence interval for $L_B(r, a - B_0(\kappa))$. When $K(\cdot)$ is a normal kernel, some calculations show that K_2 takes the value

$$K_2 = \pi^{-\frac{1}{2}}\left(2^{\frac{1}{2}} - 1\right).$$

Note that we are measuring spatial departures from the origin in units of $\sqrt{n}$ in the limit. So, if we want a confidence interval for the spatial density of the process at a point like $\sqrt{n}\,a^0$, we set $s = \sqrt{n}\,a^0 + X_0$ and compute

$$\hat{L}_B\left(r, a^0 + n^{-\frac{1}{2}}X_0\right) \pm 1.96\left(\frac{8K_2\hat{L}_B\left(r, a^0 + n^{-\frac{1}{2}}X_0\right)}{c_n}\right)^{\frac{1}{2}}. \tag{15}$$

It is interesting to compare local time confidence intervals like Equation (15) above with the confidence intervals for a probability density from kernel estimates of a probability density. By traditional theory here (e.g., Silverman 1986) we have the following 95% confidence interval for a density $f(x)$,

$$\hat{f}(x) \pm 1.96\left(\frac{k_2\hat{f}(x)}{nh}\right)^{\frac{1}{2}} \tag{16}$$

where

$$k_2 = \int_{-\infty}^{\infty} K(r)^2\,dr. \tag{17}$$

The differences between Equations (15) and (16) involve: (1) the scale factors k_2 and $8K_2$, where the difference is due to the temporal dependence in the trajectory of X_t (so that the covariance kernel enters the definition of K_2) and definitional differences between local time and a probability density; and (2) the rate of convergence $-\sqrt{c_n}$ in the case of the spatial density, compared with $\sqrt{nh}$ in the case of the probability density. In other respects, Equation (15) simply extends our existing theory of nonparametric density estimation to spatial density estimation for stochastic processes.

IV

Hazard Rates for Nonstationary Time Series

One advantage of a nonparametric treatment of spatial density is that we can define interesting functionals derived from the density that help us to shed light on the nature of variation in the data. While there are many obvious notions that arise in this way, one that we will pay attention to here is the idea of a spatial hazard function. We define the spatial hazard function $H_M(t, a)$ associated with a given spatial density $L_M(t, s)$ as follows:

$$H_M(t, a) = \frac{L_M(t, a)}{\int_a^\infty L_M(t, s)ds}. \tag{18}$$

The form of Equation (18) is analogous to that of the hazard rate $\theta(x)$ associated with a probability density $f(x)$ as

$$\theta(x) = \frac{f(x)}{\int_x^\infty f(s)ds} = \frac{f(x)}{1 - F(x)} = \frac{f(x)}{\overline{F}(x)},$$

where $F(x)$ is the *cdf* of the distribution and $\overline{F}(x) = 1 - F(x)$ is the survivor function. Such conventional hazard rates are now extensively used in empirical econometric work to help model and understand phenomena like unemployment duration and quits in the labor market and have widespread use in other fields, such as the statistical analysis of medical phenomena relating to the contraction of diseases.

How do we interpret such hazard rates in the case of nonstationary time series? Suppose the time series under study X_t is inflation and $M(r)$ is the weak limit process of $n^{-\alpha}X_{[nr]}$, with local time $L_M(r, s)$. The spatial hazard

$$H_M\left(t, a = \frac{s}{n^\alpha}\right)$$

measures the conditional risk over the period $[0, t]$ (which is expressed in standardized units of fractions of the overall sample) of an inflation rate of s, given that inflation is at least as great as s. We can then study the form of this hazard as a function of the inflation rate s, just as we might look at unemployment duration in the conventional hazard analysis of independent data. Thus, we can see whether the hazard declines, increases, or stays constant as we increase s. We can also look at the hazard as a function of the length of the time interval t and examine what happens to the hazard rate as the time period evolves and new data are introduced. So, we can see whether the hazard rate of a certain rate of inflation rises or falls over time. Of course, we might ultimately contemplate modeling such hazard rates through the use of covariates or policy interventions. These possibilities exploit the dual-argument property of the spatial density $L_M(r, s)$ and hazard $H_M(t, a)$ so that it becomes possible to analyze time series effects on hazard rates. Like $L_M(r, s)$, the function $H_M(t, a)$ is a random process in its two arguments, but unlike conventional hazard rates which depend on the unknown but estimable probability distribution of the data, the hazard $H_M(t, a)$ is an unknown but estimable path dependent stochastic process.

Now take the case where $M(r)$ is Brownian motion $B(r)$. The spatial hazard function $H_B(t, a)$ is empirically estimable using the nonparametric spatial density estimate

$$\hat{L}_B\left(t, \frac{s}{\sqrt{n}}\right)$$

discussed in the last section of the paper. For $s = s_0 + \sqrt{n}\dot{a}$, we construct the estimator

$$\hat{H}_B(t, a) = \frac{\hat{L}_B(t, a)}{\int_a^\infty \hat{L}_B\left(t, \frac{p}{\sqrt{n}}\right)\frac{dp}{\sqrt{n}}}.$$

This estimator is consistent for $H_M(t, a)$ as the following theorem shows, where the result is given in the case of $\sqrt{n}$ convergence to a Brownian motion limit process.

A. Theorem

Suppose $\hat{\omega}^2 \to_p \omega^2 = 2\pi f_{\Delta x}(0)$. *If* $s = s_0 + \sqrt{n}a$, *with* s_0 *and* a *fixed, and if Assumptions VIII.A–VIII.D in the Appendix hold, then as* $n \to \infty$

$$\hat{H}_B\left(t, \frac{s}{\sqrt{n}}\right) = \frac{\hat{L}_B\left(t, \frac{s}{\sqrt{n}}\right)}{\int_{\frac{s}{\sqrt{n}}}^\infty \hat{L}_B\left(t, \frac{p}{\sqrt{n}}\right)\frac{dp}{\sqrt{n}}} \to_p H_B(t, a - B_0(\kappa))$$

where

$$\hat{L}_B\left(r, \frac{s}{\sqrt{n}}\right) = \frac{\hat{\omega}^2}{\sqrt{n}h_n} \sum_{t=1}^{[nr]} K\left(\frac{s - X_t}{h_n}\right).$$

As in the case of the spatial density, when the limit process is reached at a $\sqrt{n}$ rate of convergence, we measure spatial departures from the origin in units of $\sqrt{n}$. So, if we want to estimate the hazard of the process at a point like a^0, we set $s = \sqrt{n}a^0 + X_0$ and compute

$$\hat{H}_B\left(t, a^0 + \frac{X_0}{\sqrt{n}}\right).$$

The function $F_M(t, a) = \int_a^\infty L_M(t, s)ds$ can be called the survivor function corresponding to $L_M(t, s)$, and is analogous to the survivor function $\bar{F}(a) = \int_a^\infty f(s)ds$ of a probability density f. Since we are using kernel estimates of $L_M(t, s)$ we can estimate $F_M(t, a)$ using

$$\begin{aligned} \hat{F}_M\left(t, \frac{s}{\sqrt{n}}\right) &= \int_{\frac{s}{\sqrt{n}}}^\infty \hat{L}_M\left(t, \frac{p}{\sqrt{n}}\right) \frac{dp}{\sqrt{n}} \\ &= \frac{\hat{\omega}^2}{\sqrt{n}} \sum_{t=1}^{[nr]} \overline{\mathbb{K}}\left(\frac{s - X_t}{h_n}\right) \end{aligned}$$

where

$$\mathbb{K}(b) = \int_{-\infty}^b \mathbb{K}(x)dx, \quad \overline{\mathbb{K}}(b) = 1 - \mathbb{K}(b).$$

For most kernels, although not the Gaussian, $\overline{\mathbb{K}}(b)$ is available in closed form, simplifying calculations for the hazard and survivor functions. For example, if K is the Epanechnikov kernel, we have

$$K(s) = \begin{cases} \frac{3}{4\sqrt{5}}\left(1 - \frac{s^2}{\sqrt{5}}\right) & -\sqrt{5} \le s \le \sqrt{5} \\ 0 & \text{otherwise} \end{cases},$$

and

$$\mathbb{K}(s) = \begin{cases} 0 & s < -\sqrt{5} \\ \frac{3}{4\sqrt{5}}(s + \sqrt{5}) - \frac{1}{45^{\frac{3}{2}}}\left(s^3 + 5^{\frac{3}{2}}\right) & -\sqrt{5} \le s \le \sqrt{5} \\ 1 & s > \sqrt{5} \end{cases}.$$

An asymptotic theory for the estimated hazard function

$$\hat{H}_B\left(t, \frac{s}{\sqrt{n}}\right)$$

is given in the following theorem.

B. Theorem

If $s = s_0 + \sqrt{n}\,a$, with s_0 and a fixed, if Assumptions VIII.A–VIII.D in the Appendix hold with $h_n \to 0$, and if $\sqrt{c_n}(\hat{\omega}^2 - \omega^2) = o_p(1)$ where $c_n = h_n^{-1}\sqrt{n}$, then as $n \to \infty$

$$\sqrt{c_n}\left[\hat{H}_B\left(r, n^{-\frac{1}{2}}s\right) - H_B(r, a - B_0(\kappa))\right]$$

$$\Rightarrow 2\int_{-\infty}^{\infty} K(p)Q\left(\frac{L_B(r, a - B_0(\kappa))}{\left(\int_{a-B_0(\kappa)}^{\infty} L_M(r, s)ds\right)^2}, p\right)dp$$

$$\equiv \; MN\left(0, 8K_2\frac{H_B(r, a - B_0(\kappa))^2}{L_B(r, a - B_0(\kappa))}\right),$$

where $Q(a, b)$ is a standard Brownian sheet.

This result delivers an asymptotic standard error for the hazard

$$\hat{H}_B\left(r, n^{-\frac{1}{2}}s\right),$$

viz.

$$\left(\frac{8K_2\hat{H}_B\left(r, n^{-\frac{1}{2}}s\right)^2}{c_n\hat{L}_B\left(r, n^{-\frac{1}{2}}s\right)}\right)^{\frac{1}{2}},$$

which can be used to assess significance of the hazard estimates in practical applications. Interestingly, this formula is analogous to that obtained in traditional hazard analysis for a probability density, which takes the well-known form (cf. Silverman 1986: 148)

$$\left(\frac{k_2\theta(x)^2}{nhf(s)}\right)^{\frac{1}{2}}.$$

The differences arise only from differences in the rate of convergence to the spatial density and the probability density and the scale constants $8K_2$ and k_2.

V

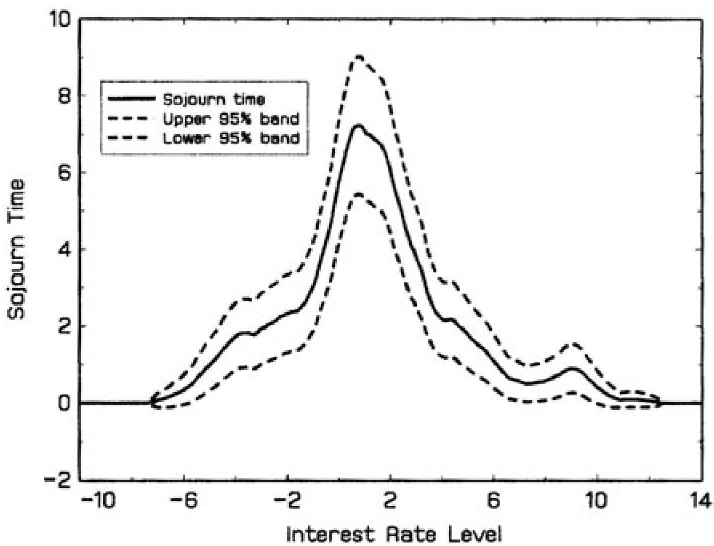
The Empirical Density of Real Interest Rates

We now illustrate the use of these concepts in analyzing the empirical spatial density of the real rate of interest. Hopefully, this will help to shed some new light on the nature of the Fisher effect and provide corroborative evidence for other studies that use conventional econometric methods.

We start with the period 1961–1985 studied by Garcia and Perron (1994). Figure 4 gives the spatial density estimate[3] for the *ex post* real rate of interest shown in Figure 1. The estimated spatial density shows a dominant mode around the level 1.5% and evidence for four minor modes around –4%, –2%, 4%, and 9%. Only the modes at 1.5% and 9% appear to be statistically significant. These results provide partial

Figure 4

Spatial density of *ex post* real rate: 1961–1985.

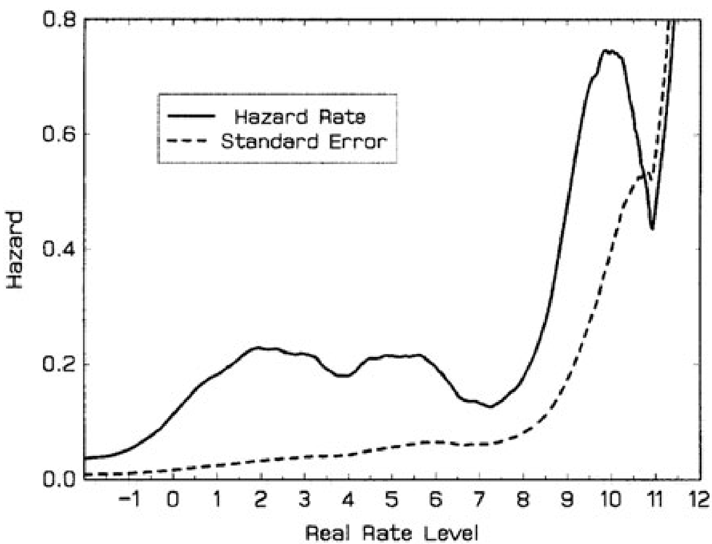


support for Garcia and Perron's conclusion in favor of the hypothesis that the real rate of interest over 1961–1985 fluctuated about three constant levels around –2%, 1.5%, and 4%. However, our nonparametric analysis indicates that there are also secondary modes around –4% and 9% and that these modes, especially the latter, appear to be more significant than those around –2% and 4%, which were the only secondary levels identified in the Garcia-Perron switching regimes approach. The significant mode around 9% seems to have been missed by Garcia and Perron.

The estimated hazard rates for the same period of data are shown in Figure 5. The graph shows that there are two peak levels of hazard for the real rate—levels around 1.5% to 3% and 4.5% to 6%—and a further peak around 9%. The sharply rising hazard at higher levels is not significant and can be ignored. Roughly speaking, these results tell us that, if we are to have high real rates of interest, the risk rises with the real rate to a plateau when rates are around 1.5% to 3%, tapers off, and then rises again to a plateau in the 4.5% to 6% region before falling off again and rising to a further peak around 9%. Note

Figure 5

Hazard function for *ex post* real rate: 1961–1985.



that the very high peak at 9% is not significant, as the standard error is rising quickly at this point. Moreover, the peak at 9% indicates that if the real rate is to be as high as this, then it is very likely that it will be in the region of 9%, given the observed data over the period 1961–1985.

The spatial density of the real rate changes as we add data for the period through to 1997 and back to 1934. First, consider the effect of the data through to 1997. As is apparent in Figure 6, there are now only three modes: around –4%, 1.5%, and 9% in the spatial density. Only the modes around 1.5% and 9% appear significant, as before. The data over 1985–1997 has smoothed out the spatial density in the region between 2% to 6% and there is no evidence of modes around 4% and −2%, at least for the bandwidth choice

$$h_n = n^{-\frac{1}{5}}$$

that we are using here. Next, as we add data for the period back to 1934, we see from Figure 8 that modes around −5% and −2% have

Figure 6

Spatial density of *ex post* real rate: 1961–1997.

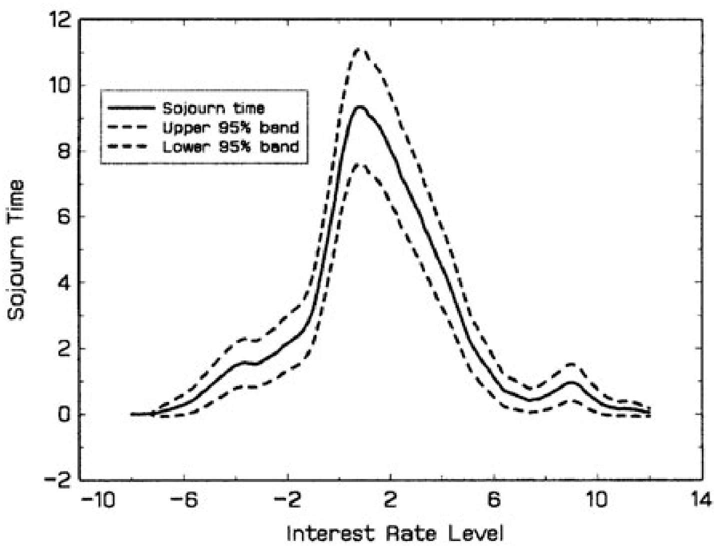
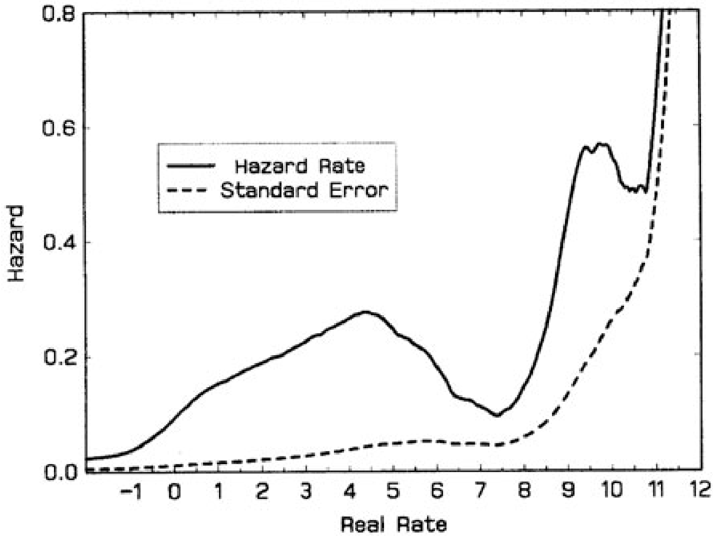


Figure 7

Hazard function for *ex post* real rate: 1961–1997.



reappeared, although there is no evidence of a mode around 4%. At least for this data set, the spatial density computed over the full period 1934–1997 indicates that the real interest rate has regions of sojourn time around −5%, −2%, and 9% and the dominant mode is around 1.5%.

Hazard rates over the period 1961–1997 are shown in Figure 7 and in Figure 9 for 1934–1997. In both cases, there are now only two peaks in the hazard function, around 3% to 4% and 9%. Both of these appear to be more significant than they are for the shorter data set. Looking at Figure 9, it seems clear that the substantial region of hazard for positive real rates is in the 1% to 6% zone, and for high rates, a zone around 9%.

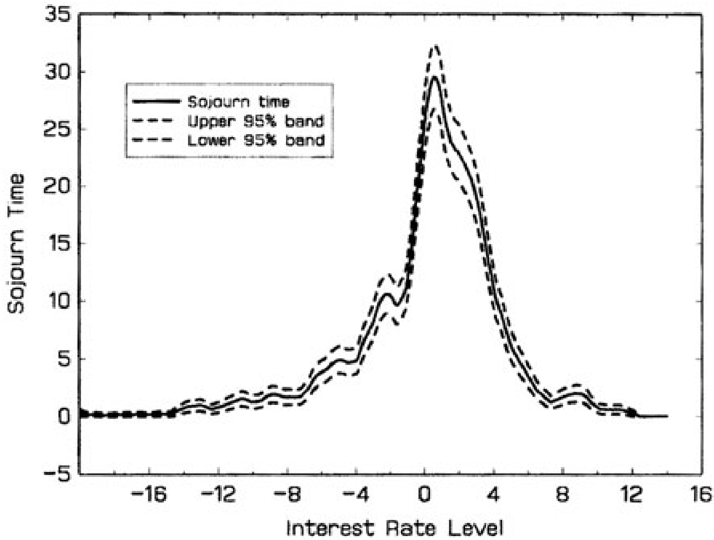
VI

Is the Real Rate Nonstationary?

Tests for nonstationarity in the real rate of interest have in the past focused on comparing unit root nonstationarity with stationary

Figure 8

Spatial density of real rate: 1934–1997.



alternatives. We now consider a broader range of alternatives accommodated by allowing for fractional integration.

Our approach is semi-parametric, so that we can retain as much generality as possible regarding the generating mechanism for the real rate of interest r_t. In particular, we consider a model for the *ex post* real rate of the form

$$(1 - L)^d r_t = u_t, \tag{19}$$

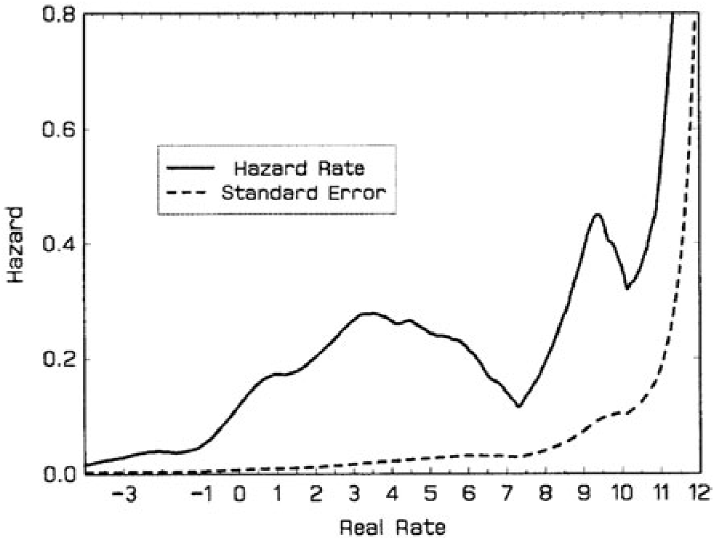
where u_t is a zero mean stationary process with spectral density $f_{uu}(\lambda)$ and is assumed to satisfy Assumption VIII.A in the Appendix. By virtue of Equation (19), the spectrum of r_t has the following asymptotic form in the vicinity of the origin

$$f_{rr}(\lambda) \sim \frac{f_{uu}(0)}{\lambda^{2d}}, \lambda \sim 0. \tag{20}$$

Several semi-parametric estimation procedures for d are available, and a brief review of some of the procedures is given in Robinson

Figure 9

Hazard function for real rate: 1934–1997.



(1995). We propose to use the modified log periodogram estimator suggested recently in Phillips (1999a). This estimator is, like log periodogram regression, simple to use, as it employs only least squares techniques. Kim and Phillips (1999a) have shown that the estimator has good asymptotic properties that make it suitable for inference in possibly nonstationary regions of d and have validated the construction of confidence intervals that extend into the nonstationary region. It therefore seems well suited to the study of interest-rate data.

Log periodogram (LP) regression involves the least squares regression (over $s = 1, \ldots, m$ for some $m < n$ with

$$\frac{1}{m} + \frac{m}{n} \to 0$$

as $n \to \infty$)

$$\ln(I_r(\lambda_s)) = c^* - d^* \ln\left|1 - e^{i\lambda_s}\right|^2 + \text{error},$$

where $I_r(\lambda_s) = |w_r(\lambda_s)|^2$ is the periodogram and $w_r(\lambda_s)$ is the discrete Fourier transform of r_t, both evaluated at the fundamental frequencies

$$\lambda_s = \frac{2\pi s}{n} \text{ for } s = 1, \ldots, m.$$

Modified LP regression involves the similar linear regression

$$\ln(I_v(\lambda_s)) = \hat{c} - \hat{d}\ln\left|1 - e^{i\lambda_s}\right|^2 + \text{error}, \tag{21}$$

in which the periodogram ordinates, $I_r(\lambda_s)$, are replaced by the modified periodogram ordinates $I_v(\lambda_s) = |v_x(\lambda_s)|^2$, where

$$v_x(\lambda_s) = w_r(\lambda_s) + \frac{e^{i\lambda_s}}{1 - e^{i\lambda_s}} \frac{r_n}{\sqrt{2\pi n}}. \tag{22}$$

This procedure was suggested in Phillips (1999a), and Kim and Phillips (1999a) show that the modified LP estimator $\hat{d}$ is consistent for all $d \in (0, 2)$ and has the following limit theory

$$\sqrt{m}\left(\hat{d} - d\right) \xrightarrow{d} N\left(0, \frac{\pi^2}{24}\right) \tag{23}$$

for

$$d \in \left(\frac{1}{2}, 2\right).$$

Thus, the limit theory for $\hat{d}$ is the same as that of the conventional LP estimator d^* in the stationary case (Robinson 1995 and Hurvich; Deo, and Brodsky 1998). By contrast, the usual log periodogram estimator d^* has a mixed normal limit theory when $d = 1$, as shown in Phillips (1999b) and is inconsistent when $d > 1$ (Kim and Phillips 1999b). Thus, the modified regression in Equation (21) is especially useful in the nonstationary case when

$$d > \frac{1}{2}$$

and the limit theory makes possible statistical testing and the construction of confidence intervals for d that extend into the nonstationary case.

With this methodology and using

$$m = n^{\frac{3}{4}}$$

frequency ordinates, we found semi-parametric estimates of d in Equation (19) for the *ex post* real rate for the three time periods 1961–1985, 1961–1997, and 1934–1997. The results are given in Table 1.

For each period the estimates of d are greater than 0.5 and indi-

Table 1

Empirical Estimates of d: *Ex Post* Real Rate

Period	$\hat{d}$	$s_{\hat{d}}$	95% Confidence Interval
1961–1985	0.6353	0.0761	[0.486,0.784]
1961–1997	0.5538	0.0654	[0.426,0.682]
1934–1997	0.5159	0.0532	[0.412,0.620]

cate that the *ex post* real rate of interest is nonstationary. Moreover, the estimates of d are all quite close, which is interesting and perhaps surprising given that the sample path of r_t varies substantially over the full time period and that other approaches require multiple regime shifts to model this data even over shorter periods. The confidence intervals show that unit root nonstationarity and short memory are both clearly rejected. However, in every case the confidence intervals for d include some long memory stationary alternatives with d less than 0.5 but greater than 0.4.

Thus, our estimates of d provide empirical evidence that supports the conclusion of Rose (1988) and Walsh (1987) that the real rate is nonstationary. But we reject unit root nonstationarity and the estimates of d are in every case not significantly greater than 0.5, so there is also some support from our estimates for the hypothesis that the real rate is marginally stationary but with very long memory.

Figures 10–12 show the effect of d-differencing the real rate of interest using the relevant estimates $\hat{d}$ for each of the subperiods. The figures also display the mean value of the residual series $\hat{u}_t = (1 - L)^{\hat{d}} r_t$ for each period. As is apparent from the figures, in each case $\hat{u}_t$ appears to be stationary with short memory.

Finally, Figure 13 gives spatial density estimates for the differenced series $\hat{u}_t = (1 - L)^{\hat{d}} r_t$ calculated for each subperiod. The densities are calculated here by normalizing the spatial densities in the same way for each subperiod—that is, by

$$\frac{1}{nh}$$

in place of

$$\frac{\hat{\omega}^2}{\sqrt{n}h}$$

Figure 10

$(1 - L)^{\hat{d}} r_t$ for 1934–1997.

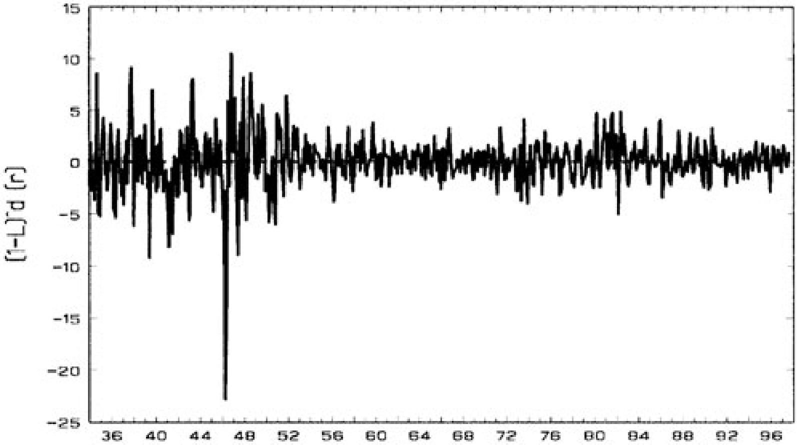


Figure 11

$(1 - L)^{\hat{d}} r_t$ for 1961–1985.

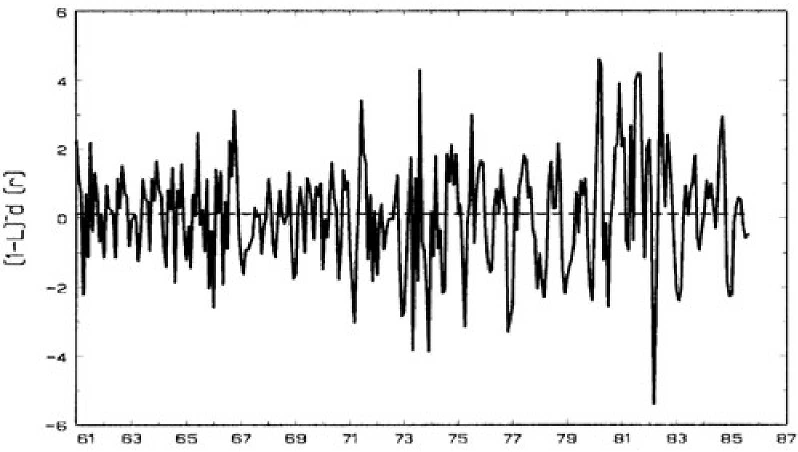


Figure 12

$(1 - L)^{\hat{d}} r_t$ for 1961–1997.

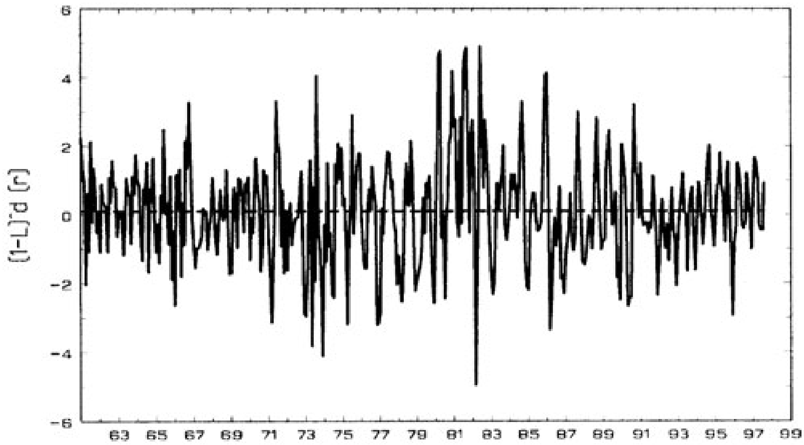
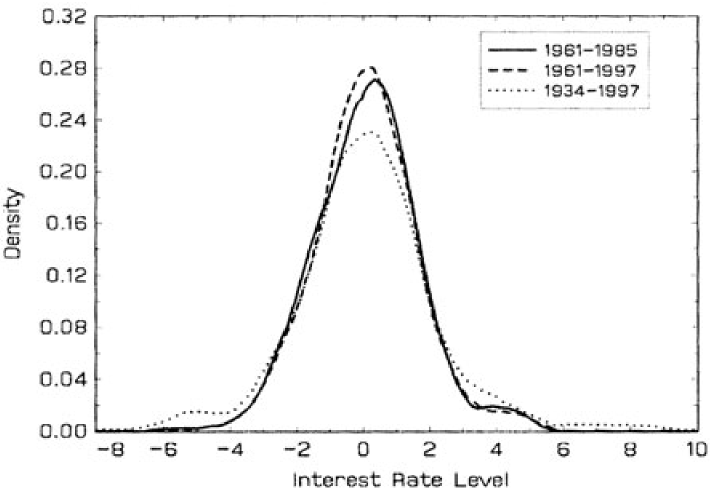


Figure 13

Densities of $(1 - L)^{\hat{d}} r_t$.



in Equation (14)—so that they are directly comparable and correspond to conventional kernel estimates for a stationary density. As is apparent from the figure, the fitted densities are symmetric and quite close, although there is apparently greater dispersion in the case of the longer data set 1934–1997.

In a similar way, we found semi-parametric estimates of d for the nominal rate and inflation over the three time periods 1961–1985, 1961–1997, and 1934–1997. The results are given in Tables 2 and 3. The results in Table 2 corroborate many earlier findings that the nominal rate of interest is nonstationary. Unit root nonstationarity is included in the confidence band for d over the longer periods 1961–1997 and 1934–1997. The point estimates of d increase from 0.82 to 0.96 as the time period lengthens.

The estimates reported in Table 3 show that inflation appears to be nonstationary over 1961–1985, but long memory stationary dependence in inflation is not rejected for the longer periods 1961–1997 and 1934–1997. For all time periods, unit root nonstationarity and short

Table 2

Empirical Estimates of d: Nominal Rate

Period	$\hat{d}$	$s_{\hat{d}}$	95% Confidence Interval
1961–1985	0.823	0.076	[0.673,0.971]
1961–1997	0.968	0.065	[0.840,1.096]
1934–1997	0.967	0.053	[0.863,1.071]

Table 3

Empirical Estimates of d: Inflation

Period	$\hat{d}$	$s_{\hat{d}}$	95% Confidence Interval
1961–1985	0.775	0.076	[0.626,0.924]
1961–1997	0.663	0.065	[0.535,0.792]
1934–1997	0.538	0.053	[0.434,0.643]

memory dependence are rejected. These estimates therefore do not support results like those in Miskin (1992), where inflation is treated as an $I(1)$ variable. The point estimates of d decrease as the time period lengthens, although in each case these exceed 0.5.

Comparing the results in Table 1 with those in Tables 2–3, the point estimates of d appear to be more stable over the three subperiods for the real rate than for either the nominal rate or inflation. Also, there is some incompatibility between the three sets of estimates. In a model with three fractionally integrated variables $y_t \equiv I(d_y)$, $x_t \equiv I(d_z)$, $z_t \equiv I(d_z)$, satisfying the simple linear relation $y_t = x_t - z_t$, the behaviour of y_t is necessarily dominated by the dominant component of x_t and z_t. Thus, if $d_x > d_z$, then $d_y = d_x$. In the present case, this suggests that the fractional integration of the nominal rate should dominate the real rate. However, the separate empirical estimates indicate that the fractional integration of the real rate is significantly less than that of the nominal rate. There are several possible explanations for this type of empirical incompatibility, including model misspecification and scale differences between the variables. It seems likely in the present case that the greater volatility of inflation over the nominal rate of interest is a contributing factor in these differences. Similar phenomena have been found to arise in the context of exchange-rate data, where there are major differences in scale between spot returns and the forward premium (see Maynard and Phillips 1998). The data are graphed in Figures 14–15 and seem to support this explanation.

VII

Conclusion

THIS PAPER INTRODUCES SOME NEW statistical procedures for describing and analyzing nonstationary data and applies these techniques to the study of Fisher's equation for the real rate of interest. Under weak conditions that assure some form of functional limit theorem for standardized forms of the data, we construct asymptotically valid spatial density and hazard function measures for the series, both of which take the form of random processes rather than nonrandom functions as in the case of stationary time series. Consistent techniques for

Figure 14

Nominal 90-day TB rate.

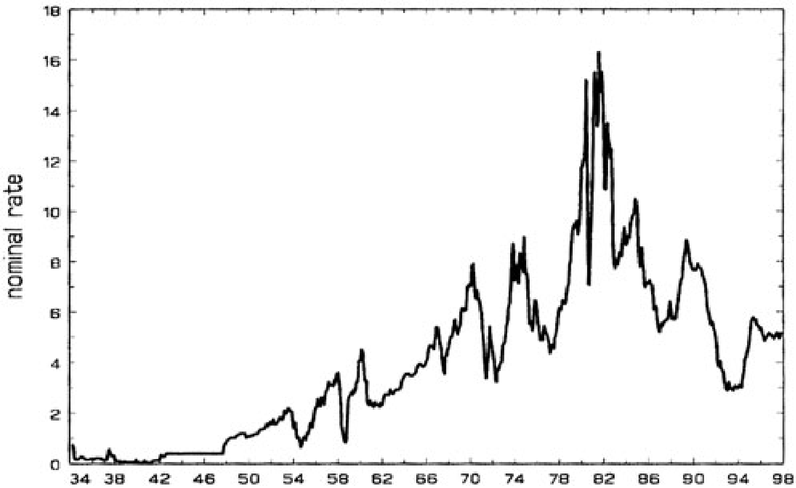
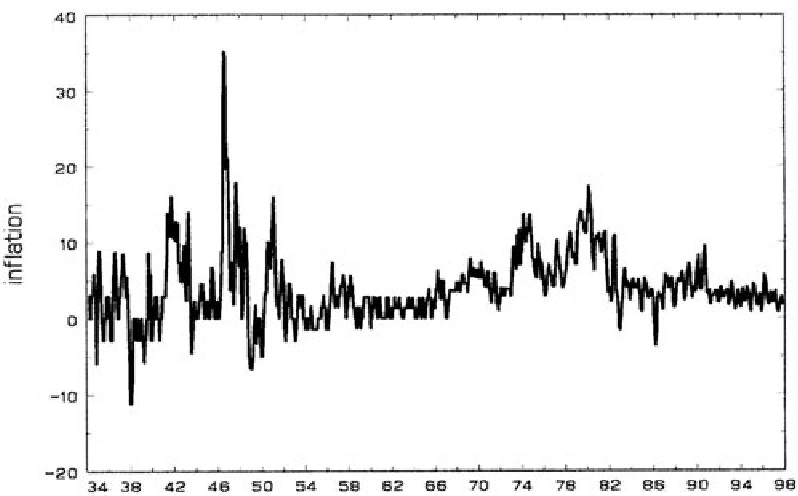


Figure 15

CPI three-month inflation rate.



estimating these quantities are given, together with a limit theory that enables the measures to be used in inference.

Using these spatial density techniques, we analyze the *ex post* real rate of interest in the United States over the period 1934–1997. Results over the subperiod 1961–1985 provide some corroborating evidence to support the conclusion of a recent study by Garcia and Perron (1994) that the real rate fluctuates about constant levels that change over regimes. However, over the longer period 1934–1997, the regime change approach requires many change points, seems artificial, and lacks parsimony.

An alternative semiparametric model is considered that allows for fractional integration in the real rate, including unit root nonstationary and stationary long-range dependence as special cases, to account for various types of long-run behavior and that retains generality with respect to the modeling of the short-run component of the series. Through the fractional integration parameter in this model the extent of nonstationarity in the real rate can be directly measured. A new modified log periodogram estimation of the fractional integration parameter is performed using recently obtained asymptotic results by Kim and Phillips (1999a) that apply in the nonstationary case. Empirical estimates of the fractional differencing parameter are computed for the *ex post* rate over 1934–1997 and two shorter subperiods. The results confirm evidence of nonstationarity in the real rate for all time periods, and therefore generally support the conclusion reached by Rose (1988). Our estimates also confirm Mishkin's (1992) rejection of unit root nonstationarity in the real rate of interest. However, although unit root nonstationarity and short memory can be rejected and point estimates indicate nonstationarity, confidence intervals for the fractional differencing parameter are in the region [0.4, 0.6] and therefore do not completely rule out the possibility of a stationary real rate of interest with very long-range dependence.

The fractional integration model for the real rate of interest seems to be successful in transforming the data to stationarity over all three periods 1934–1997, 1961–1985, and 1961–1997, all with a very similar fractional integration parameter. In this respect, the model is at once more parsimonious and more generally applicable than a model with many regime shifts.

VIII

Technical Appendix and Proofs

OUR APPROACH TO AN ASYMPTOTIC THEORY for spatial density and hazard rate estimation relies on recent work in Phillips and Park (1998), hereafter P^2, for kernel density estimation and regression for nonstationary time series. We only sketch the derivations we need here.

Suppose X_t is a unit root time series with differences $\Delta X_t = u_t$ and initialization X_0 that satisfy Conditions VIII.A and VIII.B below. Distant initial conditions are permitted in VIII.B and play a role in the spatial density asymptotics. Conditions VIII.C and VIII.D relate to the kernel function and restrictions on the bandwidth h, and they are used in P^2. Note the important difference between VIII.D and conventional assumptions about bandwidth in kernel density estimation; here, the bandwidth can be of the form

$$h_n = cn^k,\ k \in \left[-\frac{1}{4}+\delta, \frac{1}{12}-\delta\right],$$

so that bandwidths that increase with n are permissible. The bandwidth cannot decrease too fast or increase too fast, as $n \to \infty$. Condition VIII.A allows for differences u_t that follow a linear process and are standard (Phillips and Solo 1992). The higher moment condition in VIII.A is useful in assuring the validity of a strong approximation to partial sums of u_t.

A. Assumption

(a) *u_t is a linear process $u_t = C(L)\varepsilon_t = \Sigma_{j=0}^{\infty} c_j \varepsilon_{t-j}$ with $C(1) \neq 0$ and*

$$\sum\nolimits_{j=0}^{\infty} j^{\frac{1}{2}} |c_j| < \infty.$$

(b) *ε_t is iid $(0, \sigma^2)$ with $E(|\varepsilon_j|^q) < \infty$ for some $q > 2p > 4$.*

B. Assumption

The initial conditions of X_t are set at $t = 0$, and X_0 has the following general form allowing for effects in the distant past

$$X_0 = u + \sum_{j=0}^{[n\kappa]} u_{-j}, \text{ for some } \kappa \geq 0,$$

where u is an $O_{a.s.}(1)$ *random variable with* $E(|u|^p)$ *for some* $p > 2$.

C. Assumption

The kernel $K(\cdot)$ *is a symmetric and nonnegative density with integrable characteristic function* φ_K *and satisfies the following conditions for some* $r > 2$:

$$\int_{-\infty}^{\infty} K(s) = 1, \int_{-\infty}^{\infty} s^{2r} K(s) ds < \infty, \quad \sup_s K(s) < \infty.$$

D. Assumption

$n^{1-\delta} h_n^4 \to \infty$, *and* $h_n / n^{(1-\delta)/12} \to 0$ *for some* $\delta > 0$.

E. Proof of Theorem 3.1

The proof follows the same lines as that of Theorem 3.1 of P^2. As in that theorem, we need to augment the probability space so that a strong approximation to the limit Brownian motion $B(r)$ of

$$n^{-\frac{1}{2}} X_{[nr]}$$

can be used. The only changes in the proof are:

(i) Since u_t is a linear process, we need to rely on extended versions of the preliminary Lemmas 5.5 and 5.7 in P^2 that apply for linear processes instead of iid random variables. These extensions follow in precisely the same way but make use of Lemma D of Phillips (1999b), which provides a strong approximation result for linear processes.

(ii) We use the consistent scale estimate $\hat{\omega}^2$ of the long-run variance $\omega^2 = 2\pi f_{\Delta x}(0)$ in the definition of the spatial density estimate. This is not needed in Theorem 3.1 of P^2 because chronological local time $\bar{L}_B(r, a) = \omega^{-2} L_B(r, a)$ is used in P^2 in place of local time.

(iii) Since $\hat{\omega}^2 \to_p \omega^2$, the resulting limit holds in probability rather than almost surely.

F. Proof of Theorem 3.2

From Lemma 2.9(d) of P^2 we have

$$2^{-1}\lambda^{1/2}\left\{L_B\left(t, r+\frac{s}{\lambda}\right)-L_B(t, r)\right\}\xrightarrow{d} Q\left(L_B(t, r), s\right),$$

where $Q(a, b)$ is a standard Brownian sheet. Then, using Theorem III.A of P^2, we have

$$\begin{aligned}
&\sqrt{c_n}\left[\hat{L}_B\left(r, n^{-\frac{1}{2}}s\right)-L_B(r, a-B_0(\kappa))\right]\\
=\ &\sqrt{c_n}\left[\frac{\hat{\omega}^2}{\sqrt{n}h_n}\sum_{t=1}^{[nr]}K\left(\frac{s-X_t}{h_n}\right)-L_B(r, a-B_0(\kappa))\right]\\
=\ &\sqrt{c_n}\left[c_n\omega^2\int_0^r K\left(c_n\left\{\frac{s-X_0}{\sqrt{n}}-B(g)\right\}\right)dg-L_B(r, a-B_0(\kappa))\right]+o_p(1)\\
=\ &\sqrt{c_n}\left[c_n\int_{-\infty}^{\infty}K\left(c_n\left\{\frac{s-X_0}{\sqrt{n}}-p\right\}\right)L_B(r, p)dp-L_B(r, a-B_0(\kappa))\right]+o_p(1)\\
=\ &\sqrt{c_n}\int_{-\infty}^{\infty}K(q)\left[L_B\left(r, \frac{s-X_0}{\sqrt{n}}-\frac{q}{c_n}\right)-L_B(r, a-B_0(\kappa))\right]dq+o_p(1)\\
=\ &\int_{-\infty}^{\infty}K(q)\sqrt{c_n}\left[L_B\left(r, \frac{s-X_0}{\sqrt{n}}-\frac{q}{c_n}\right)-L_B(r, a-B_0(\kappa))\right]dq+o_p(1)\\
\to_d\ &2\int_{-\infty}^{\infty}K(q)Q(L_B(r, a-B_0(\kappa)), q)dq\\
=_d\ &2L_B(r, a-B_0(\kappa))^{\frac{1}{2}}\int_{-\infty}^{\infty}K(q)Q(1, q)dq\\
=_d\ &2L_B(r, a-B_0(\kappa))^{\frac{1}{2}}N(0, V),
\end{aligned}$$

where

$$V=\int_0^{\infty}\int_0^{\infty}K(q)(q\wedge p)K(p)dqdp+\int_0^{\infty}\int_0^{\infty}K(-q)(q\wedge p)K(-p)dqdp,$$

which gives the stated mixed normal distribution

$$MN\left(0, 8L_B(r, a-B_0(\kappa))\int_0^{\infty}\int_0^{\infty}K(q)(q\wedge p)K(p)dqdp\right).$$

G. Proof of Theorem 4.1

By virtue of Theorem III.A and since $s=s_0+\sqrt{n}\,a$, we have

$$\hat{H}_B\left(t, \frac{s}{\sqrt{n}}\right) = \frac{\hat{L}_B\left(t, \frac{s}{\sqrt{n}}\right)}{\int_{\frac{s}{\sqrt{n}}}^{\infty} \hat{L}_B\left(t, \frac{p}{\sqrt{n}}\right)\frac{dp}{\sqrt{n}}}$$

$$\rightarrow_p \frac{L_B(t, a - B_0(\kappa))}{\int_a^{\infty} L_B(t, b - B_0(\kappa))db}$$

$$= \frac{L_B(t, a - B_0(\kappa))}{\int_{a-B_0(\kappa)}^{\infty} L_B(t, c)dc} = H_B(t, a - B_0(\kappa))$$

giving the stated result.

H. Proof of Theorem 4.2

We have

$$\sqrt{c_n}\left[\hat{H}_B\left(r, n^{-\frac{1}{2}}s\right) - H_B(r, a - B_0(\kappa))\right]$$

$$= \sqrt{c_n}\left[\frac{\hat{L}_B\left(t, \frac{s}{\sqrt{n}}\right)}{\int_{\frac{s}{\sqrt{n}}}^{\infty} \hat{L}_B\left(t, \frac{p}{\sqrt{n}}\right)\frac{dp}{\sqrt{n}}} - \frac{L_B(t, a - B_0(\kappa))}{\int_{a-B_0(\kappa)}^{\infty} L_B(t, c)dc}\right]. \tag{24}$$

Observe that

$$\int_{\frac{s}{\sqrt{n}}}^{\infty} \hat{L}_B\left(t, \frac{p}{\sqrt{n}}\right)\frac{dp}{\sqrt{n}} = \frac{\hat{\omega}^2}{\sqrt{n}h_n}\sum_{t=1}^{[nr]}\int_{\frac{s}{\sqrt{n}}}^{\infty} K\left(\frac{p - X_t}{h_n}\right)\frac{dp}{\sqrt{n}}$$

$$= \frac{\hat{\omega}^2}{\sqrt{n}h_n}\sum_{t=1}^{[nr]}\int_{\frac{s}{\sqrt{n}}}^{\infty} K\left(c_n\left[\frac{p - X_t}{\sqrt{n}}\right]\right)\frac{dp}{\sqrt{n}}$$

$$= \frac{\hat{\omega}^2}{\sqrt{n}h_n}\sum_{t=1}^{[nr]}\int_a^{\infty} K\left(c_n\left\{b - \frac{X_t}{\sqrt{n}}\right\}\right)db$$

$$= \frac{\hat{\omega}^2}{\sqrt{n}h_n}\sum_{t=1}^{[nr]}\frac{1}{c_n}\overline{\mathbb{K}}\left(c_n\left\{a - \frac{X_t}{\sqrt{n}}\right\}\right)$$

$$= \frac{\hat{\omega}^2}{\sqrt{n}h_n}\sum_{t=1}^{[nr]}\overline{\mathbb{K}}\left(c_n\left\{a - \frac{X_t}{\sqrt{n}}\right\}\right). \tag{25}$$

Now, under the assumption that $\sqrt{c_n}(\hat{\omega}^2 - \omega^2) = o_p(1)$, using the strong approximation

$$\frac{X_{[nr]}}{\sqrt{n}} = B_0(\kappa) + B(r) + o_{a.s.}\left(n^{-\frac{1}{2}+\frac{1}{P}}\right)$$

as in P[2] and proceeding as in the proof of Theorem 3.1 of P[2], we get

$$\begin{aligned}\frac{\hat{\omega}^2}{n}\sum_{t=1}^{[nr]}\overline{\mathbb{K}}\left(c_n\left\{a - \frac{X_t}{\sqrt{n}}\right\}\right) &= \omega^2\int_0^r \overline{\mathbb{K}}(c_n\{a - B_0(\kappa) - B(s)\})ds + o_p\left(\frac{1}{\sqrt{c_n}}\right)\\ &= \omega^2\int_{-\infty}^{\infty} \overline{\mathbb{K}}(c_n\{a - B_0(\kappa) - c\})L_B(r, c)dc +\\ &\quad o_p\left(\frac{1}{\sqrt{c_n}}\right)\\ &= \omega^2\int_{a-B_0(\kappa)}^{\infty} L_B(r, c)dc + o_p\left(\frac{1}{\sqrt{c_n}}\right) \qquad (26)\end{aligned}$$

since

$$\begin{aligned}\overline{\mathbb{K}}(c_n\{a - B_0(\kappa) - c\}) &= \int_{c_n\{a-B_0(\kappa)-c\}}^{\infty} K(s)ds\\ &= \begin{cases} O\left(\dfrac{1}{c_n^{2r-1}}\right) & \text{for} \quad c < a - B_0(\kappa)\\ 1 + O\left(\dfrac{1}{c_n^{2r-1}}\right) & \text{for} \quad c > a - B_0(\kappa)\end{cases}\end{aligned}$$

from the given tail behavior of the kernel $K(s)$ and where $r > 2$ (see Assumption VIII.C).

It follows from Equation (24)–(25) and Theorem 3.2 that

$$
\begin{aligned}
&\sqrt{c_n}\left[\hat{H}_B\left(r, n^{-\frac{1}{2}}s\right) - H_B(r, a - B_0(\kappa))\right] \\
&= \sqrt{c_n}\left[\frac{\hat{L}_B\left(t, \frac{p}{\sqrt{n}}\right)}{\int_{a-B_0(\kappa)}^{\infty} L_B(t,c)dc + o_p\left(\frac{1}{\sqrt{c_n}}\right)} - \frac{L_B(t, a - B_0(\kappa))}{\int_{a-B_0(\kappa)}^{\infty} L_B(t,c)dc}\right] \\
&= \frac{1}{\int_{a-B_0(\kappa)}^{\infty} L_B(t,c)dc}\sqrt{c_n}\left[\hat{L}_B\left(t, \frac{s}{\sqrt{n}}\right) - L_B(t, a - B_0(\kappa))\right] \\
&\Rightarrow \frac{2}{\int_{a-B_0(\kappa)}^{\infty} L_B(t,c)dc}\int_{-\infty}^{\infty} K(q)Q(L_B(r, a - B_0(\kappa)), q)dq \\
&\equiv MN\left(0, 8K_2\frac{L_B(r, a - B_0(\kappa))}{\left(\int_{a-B_0(\kappa)}^{\infty} L_B(r,s)ds\right)^2}\right),
\end{aligned}
$$

giving the required result.

I. Notation

$\rightarrow_{a.s.}$	almost sure convergence	$[\cdot]$	integer part of
$=_d$	distributional equivalence	$r \wedge s$	$\min(r, s)$
$:=$	definitional equality	$\equiv$	equivalence in distribution
$o_{a.s.}(1)$	tends to zero almost surely	$o_p(1)$	tends to zero in probability
$\rightarrow_p$	convergence in probability		
$(a)_k$	$(a)(a+1)\ldots(a+k-1)$		
$\Rightarrow, \rightarrow_d$	weak convergence		

J. Data Sources

(a) **Consumer Price Index**

Not seasonally adjusted
Area: U.S. city average (i.e., urban)
Items: All items
Base: 1982–1984 = 100
Source: Bureau of Labor Statistics, *Monthly Labor Review*
Code: CUUR0000SA0

(b) **Three-Month Treasury Bill Rate**
Secondary market
Average of daily closing bid
Annualized using a 360-day year for bank interest
Quoted on a discount basis
Source: Board of Governors of the Federal Reserve System, *Federal Reserve Bulletin*
Code: TB3MS

Notes

1. Fisher (1896) credited Marshall (1895) for making the distinction between real and nominal interest. It appears the idea that expected inflation affects interest rates can be traced to earlier political speeches and political economy pamphlets. Howitt (1992) and Laidler (1991) provide some further information about the history of the concept and the distinction between real and nominal rates. Fisher seems to have been the first to conduct a sustained study and to explore the matter in serious empirical research.

2. The reader is referred to Revuz and Yor (1994) for much of the underlying stochastic process theory used here. Good introductions are Chung and Williams (1990) and Karatzas and Shreve (1991).

3. In these and in our other calculations we used a bandwidth of

$$h_n = n^{-\frac{1}{5}}.$$

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