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COWLES FOUNDATION PAPER NO. 1166

COWLES FOUNDATION FOR RESEARCH IN ECONOMICS
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2006

http://cowles.econ.yale.edu/

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First Version received July 2004

Abstract. First order autoregression is shown to satisfy a limit theory which is uniform over stationary values of the autoregressive coefficient $\rho = \rho_n \in [0, 1)$ provided $(1 - \rho_n)n \to \infty$. This extends existing Gaussian limit theory by allowing for values of stationary ρ that include neighbourhoods of unity provided they are wider than $O(n^{-1})$, even by a slowly varying factor. Rates of convergence depend on ρ and are at least $\sqrt{n}$ but less than n. Only second moments are assumed, as in the case of stationary autoregression with fixed ρ.

Keywords. Autoregression; Gaussian limit theory; local to unity; uniform limit.
AMS subject classifications. Primary 62M10; secondary 60F15.

1. INTRODUCTION

In pioneering work on limit theory for autoregressions, Mann and Wald (1943) showed consistency and asymptotic normality of least squares regression in stationary models. Anderson (1959) confirmed that these results hold in scalar models under weaker conditions requiring only second moments. Lai and Wei (1982) extended the results further to stochastic regression models with martingale difference errors having homoscedastic variance and moments of order greater than two. In contrast, it is well known that in unit root autoregressions (White, 1958) and in models whose roots are local to unity (Phillips, 1987; Chan and Wei, 1987) the limit distribution is non-Gaussian and involves functionals of stochastic processes.

The present note shows that in stationary regions that are further removed from unity than $O(n^{-1})$ for samples of size n the Gaussian limit theory still applies. In particular, in first order autoregression a Gaussian limit theory holds uniformly over stationary values of the autoregressive coefficient $\rho = \rho_n \in [0, 1)$, which includes local vicinities of unity that satisfy $(1 - \rho_n)\, n \to \infty$. Thus, even for $\rho_n = 1 - L_n/n$, where $L_n \to \infty$ is slowly varying at infinity the usual Gaussian limit theory applies. Rates of convergence depend on ρ_n and are at least $\sqrt{n}$ but less than n. Only second moments are assumed, as in the case of stationary autoregression with fixed ρ.

The results given here provide a supplement to those of Lai and Wei (1982). Theorem 3 of Lai and Wei gives the asymptotic normality of a suitably standardized and centred least squares estimator in regression models with stochastic regressors under known conditions that enable the use of a standard martingale central limit theorem (CLT). These conditions, which need to be checked in individual cases, involve a stability condition on the sample variance of the regressors, a uniform negligibility condition on the standardized regressors and uniform error moments of order greater than two. The present work provides a direct proof of asymptotic normality under primitive conditions on ρ_n and the errors in an autoregression, allowing for roots in the vicinity of unity of the form ρ_n. These conditions appear to be near minimal for Gaussianity in an autoregression. Our proof of asymptotic normality uses an asymptotic truncation argument and a martingale CLT that applies when only second moments are finite. Least squares regression in moderate neighbourhoods of unity for stationary and explosive cases under independent and identically distributed (i.i.d.) errors was first analysed in Phillips and Magdalinos (2004) using a strong approximation technique.

2. MAIN RESULTS

We consider the model

$$y_t = \rho_n y_{t-1} + u_t, \quad t = 1, \ldots, n \tag{1}$$

where u_t, $t \in \mathbb{Z}_+$ is a stationary and ergodic martingale difference sequence with respect to the natural filtration $F_{t-1} = \sigma(u_{t-1}, u_{t-1}, \ldots)$ with finite conditional variance $E(u_t^2|F_{t-1}) = \sigma^2$ a.s. and initialization y_0. Formally, $y_t = y_{tn}$, $t = 1, \ldots, n$ is a triangular array but it is unnecessary to add the additional index in what follows.

The coefficient ρ is fitted least squares, giving the estimator

$$\hat{\rho} = \frac{\sum_{j=1}^n y_t y_{t-1}}{\sum_{j=1}^n y_{t-1}^2},$$

and

$$\hat{\rho} - \rho_n = \frac{\sum_{j=1}^n u_t y_{t-1}}{\sum_{j=1}^n y_{t-1}^2}. \tag{2}$$

The following conditions are imposed on ρ_n and u_t throughout the remainder of this paper.

A.1. $\rho_n \in [0, 1)$ may depend on n and is such that $v_n = 1 - \rho_n$ has property

$$v_n n \to \infty. \tag{3}$$

A.2. y_0 is independent of $\{u_t : t = 1, 2, \ldots\}$ and

$$Ey_0^2 = o(n^{1/2}). \tag{4}$$

Note that distribution of u_t does not depend on n.

THEOREM 1. *Under* A.1 *and* A.2,

$$\frac{n^{1/2}}{(1-\rho_n^2)^{1/2}}(\hat{\rho}-\rho_n) \Rightarrow N(0,1) \tag{5}$$

and

$$\frac{n^{1/2}}{|1-\hat{\rho}^2|^{1/2}}(\hat{\rho}-\rho_n) \Rightarrow N(0,1). \tag{6}$$

In both cases the limit distribution is Gaussian uniformly in ρ_n satisfying A1, although the convergence rate depends directly on how close ρ_n is to unity. The asymptotic distribution of the sample mean of y_t is similarly Gaussian, again with convergence rate that depends on ρ_n.

In what follows, $\Rightarrow$ denotes convergence in distribution and $\rightarrow_p$ convergence in probability.

THEOREM 2. *Under* A.1–A.2,

$$\frac{(1-\rho_n)}{n^{1/2}}\sum_{t=1}^{n} y_t \Rightarrow N(0,\sigma^2) \tag{7}$$

and

$$\left(\frac{1+\rho_n}{1-\rho_n}\right)^{1/2}\frac{\sum_{t=1}^{n} y_t}{\left(\sum_{t=1}^{n} y_t^2\right)^{1/2}} \Rightarrow N(0,1). \tag{8}$$

The proofs use the following lemmas.

LEMMA 1.

$$\frac{(1-\rho_n^2)^{1/2}}{n^{1/2}}\sum_{t=1}^{n} u_t y_{t-1} \Rightarrow N(0,\sigma^4). \tag{9}$$

LEMMA 2.

$$\frac{1-\rho_n^2}{n}\sum_{t=1}^{n} y_{t-1}^2 \rightarrow_p \sigma^2. \tag{10}$$

Result (10) proves the 'stability' condition which Lai and Wei [1982, theorem 3, condition (4.2)] use for their limit theory corresponding to (5).

3. PROOFS

PROOF OF THEOREM 1. The convergence (5) follows from Lemmas 2 and 1. To prove (6), note from (5) that

$$\hat{\rho} - \rho_n = O_P(v_n^{1/2} n^{-1/2}) = o_P(v_n),$$

since $n^{-1} = o(v_n)$ by (3). Thus

$$|1 - \hat{\rho}|^{-1/2} = |1 - \rho_n + o_P(v_n)|^{-1/2} = |1 - \rho_n|^{-1/2} |1 + o_P(v_n)|^{-1/2}.$$

This and (5) imply (6). □

Proof of Lemma 1. Denote by $\zeta_t = v_n^{1/2} n^{-1/2} u_t y_{t-1}$ the sequence of martingale differences with respect to the sigma algebra F_{t-1} generated by $u_1, \ldots, u_{t-1}$. Recall that $v_n = 1 - \rho_n$.

(a) We first prove convergence (9) in the case where fourth moments are finite: $Eu_t^4 < \infty$ and $Ey_0^4 < \infty$. It suffices to show that, as $n \to \infty$,

$$(1 + \rho_n) \sum_{t=1}^{n} E[\zeta_t^2 | F_{t-1}] \to_p \sigma^4 \tag{11}$$

and

$$\sum_{t=1}^{n} E[|\zeta_t|^2 1_{\{|\zeta_t| \geq \delta\}} | F_{t-1}] \to_p 0, \tag{12}$$

for all $\delta > 0$, which in turn implies the convergence

$$(1 + \rho_n)^{1/2} \sum_{t=1}^{n} \zeta_t \Rightarrow N(0, \sigma^4)$$

and yields (9). We note that (12) corresponds to the uniform negligibility condition used by Lai and Wei [1982, condition (4.3)]. To check (11), we show that

$$(1 + \rho_n) \sum_{t=1}^{n} E\zeta_t^2 \to \sigma^4 \tag{13}$$

and

$$E\left(\sum_{t=1}^{n} E[(\zeta_t^2 - E\zeta_t^2) | F_{t-1}] \right)^2 \to 0. \tag{14}$$

Since

$$E[\zeta_t^2 | F_{t-1}] = (v_n/n) E[u_t^2 y_{t-1}^2 | F_{t-1}] = (v_n/n) \sigma^2 y_{t-1}^2,$$

where $v_n = 1 - \rho_n$, we have

$$(1 + \rho_n) E \sum_{t=1}^{n} E[\zeta_t^2 | F_{t-1}] = (1 - \rho_n^2) n^{-1} \sum_{t=1}^{n} \sigma^2 E[y_{t-1}^2] \to \sigma^4$$

by Lemma 3 below.

To show (14) note that

$$E[(\zeta_t^2 - E\zeta_t^2)|F_{t-1}] = \sigma^2\left(\frac{v_n}{n}\right)(y_{t-1}^2 - Ey_{t-1}^2), \quad t = 1, 2, \ldots.$$

Then, using C to denote a generic constant in what follows, we have by Lemma 4 below

$$Ey_{t-1}^4 \leq Cv_n^{-2}, \tag{15}$$

and therefore

$$\begin{aligned} E\left(\sum_{t=1}^n E[(\zeta_t^2 - E\zeta_t^2)|F_{t-1}]\right)^2 &= v_n^2 n^{-2}\sum_{t,s=1}^n E[(u_t^2-\sigma^2)(u_s^2-\sigma^2)]E[y_{t-1}^2 y_{s-1}^2] \\ &= v_n^2 n^{-2}\sum_{t=1}^n E[(u_t^2-\sigma^2)^2]E[y_{t-1}^4] \\ &\leq Cv_n^2 n^{-2}\sum_{t=1}^n (1-\rho_n)^{-2} \leq Cn^{-1} \to 0, \end{aligned}$$

proving (14).

Finally, to show (12), note that by (15)

$$q_n := E\sum_{t=1}^n E[|\zeta_t|^2 1_{\{|\zeta_t|\geq\delta\}}|F_{t-1}] \leq \delta^{-2}\sum_{t=1}^n E|\zeta_t|^4 \leq C\delta^{-2}v_n^2 n^{-2}\sum_{t=1}^n E|y_{t-1}|^4 \leq Cn^{-1} \to 0. \tag{16}$$

(b) Suppose now that either or both of $Eu_1^4 = \infty$ and $Ey_0^4 = \infty$ apply. Let $K > 0$ be a fixed constant. Set

$$u_t^{(1)} = u_t 1(|u_t| \leq K) - E[u_t 1(|u_t| \leq K)|F_{t-1}], \;\; u_t^{(2)} = u_t 1(|u_t| > K) - E[u_t 1(|u_t| > K)|F_{t-1}],$$

$$y_0^{(1)} = y_0 1(|y_0| \leq K) - E[y_0 1(|y_0| \leq K)|F_{t-1}], \;\; y_0^{(2)} = y_0 1(|y_0| > K) - E[y_0 1(|y_0| > K)|F_{t-1}],$$

and then by recursion define

$$y_t^{(1)} = \rho_n y_{t-1}^{(1)} + u_t^{(1)}, \;\; y_t^{(2)} = \rho_n y_{t-1}^{(2)} + u_t^{(2)}, \quad t = 1, 2, \ldots$$

Note that

$$E[u_t 1(|u_t| \leq K)|F_{t-1}] + E[u_t 1(|u_t| > K)|F_{t-1}] = E[u_t|F_{t-1}] = 0$$

and therefore $y_t^{(1)} + y_t^{(2)} = y_t, t = 0, 1, 2, \ldots$. Moreover, the random variables $u_t^{(j)}$ and $y_0^{(j)}$ satisfy A.2 when $j = 1, 2$. Then we can write

$$\sum_{t=1}^{n} u_t y_{t-1} = \sum_{t=1}^{n} u_t^{(1)} y_{t-1}^{(1)} + \sum_{t=1}^{n} u_t^{(2)} y_{t-1}^{(1)} + \sum_{t=1}^{n} u_t y_{t-1}^{(2)}$$
$$=: S_{n,1} + S_{n,2} + S_{n,3}.$$

Since $y_t^{(1)}$ is a sequence of martingale differences, by part (a) we have,

$$(1-\rho_n^2)^{1/2} n^{-1/2} S_{n,1} \Rightarrow N(0, \sigma_K^4), \tag{17}$$

where

$$\sigma_K^2 = E(u_t^{(1)})^2 = Eu_t^2 1(|u_t| \le K) \to \sigma^2 = Eu_t^2,$$

as $K \to \infty$. We show that for $l = 2, 3$, uniformly in $n \ge 1$,

$$(1-\rho_n^2) n^{-1} ES_{n,l}^2 \le \delta_K, \tag{18}$$

where $\delta_K \to 0$ as $K \to \infty$ which together with (17) proves convergence (9) in case (b).

Set $\delta_K = E(u_t^{(2)})^2$. Note that

$$\delta_K \to 0, \quad \text{as } K \to \infty. \tag{19}$$

By Lemma 3,

$$\sum_{t=1}^{n} E(y_{t-1}^{(1)})^2 \le Cnv_n^{-1}, \quad \sum_{t=1}^{n} E(y_{t-1}^{(2)})^2 \le C\delta_K n v_n^{-1}$$

where C does not depend on n, K. Thus, since $u_t^{(2)} y_{t-1}^{(1)}$ and $u_t y_{t-1}^{(2)}$ are uncorrelated sequences,

$$ES_{n,2}^2 + ES_{n,3}^2 \le \sum_{t=1}^{n} \left[E(u_t^{(2)})^2 E(y_{t-1}^{(1)})^2 + Eu_t^2 E(y_{t-1}^{(2)})^2 \right]$$
$$\le C\delta_K 2nv_n^{-1},$$

which, together with (19), implies (18). □

Proof of Lemma 2. Since $y_t = \rho_n y_{t-1} + u_t$ we can write

$$\sum_{t=1}^{n} y_t^2 = \rho_n^2 \sum_{t=1}^{n} y_{t-1}^2 + 2\rho_n \sum_{t=1}^{n} y_{t-1} u_t + \sum_{t=1}^{n} u_t^2.$$

Thus

$$\sum_{t=1}^{n} y_{t-1}^2 = (1-\rho_n^2)^{-1} \left(-y_n^2 + y_0^2 + 2\rho_n \sum_{t=1}^{n} y_{t-1} u_t + \sum_{t=1}^{n} u_t^2 \right). \tag{20}$$

By Lemma 1,

$$Z_n := (1-\rho_n^2)^{1/2} n^{-1/2} \sum_{t=1}^{n} y_{t-1} u_t \Rightarrow N(0, \sigma^4).$$

Moreover, because u_t is an ergodic sequence with a finite second moment σ^2, we have

$$\sum_{t=1}^{n} u_t^2 = n\sigma^2 + \sum_{t=1}^{n} (u_t^2 - \sigma^2) = n\sigma^2 + o_P(n).$$

Therefore

$$\begin{aligned}\sum_{t=1}^{n} y_{t-1}^2 &= (1-\rho_n^2)^{-1}\Big(-y_n^2 + y_0^2 + 2\rho_n(1-\rho_n^2)^{-1/2} n^{1/2} Z_n + n\sigma^2 + o_P(n)\Big)\\ &= (1-\rho_n^2)^{-1} n\Big(-n^{-1}(y_n^2 - y_0^2) + 2\rho_n((1-\rho_n^2)n)^{-1/2} Z_n + \sigma^2 + o_P(1)\Big)\\ &= (1-\rho_n^2)^{-1} n(\sigma^2 + o_P(1)),\end{aligned}$$

because $(1-\rho_n^2)n \to \infty$ by assumption (3), and therefore $((1-\rho_n^2)n)^{-1/2} Z_n \to_p 0$, whereas

$$n^{-1} y_n^2 = O_P((nv_n)^{-1}) \to_p 0,$$

by (22), and $n^{-1} y_0^2 \to_p 0$ by Assumption A.2, proving (10). □

Proof of Theorem 2. We can write

$$\sum_{t=1}^{n} y_t = \rho_n \sum_{t=1}^{n} y_{t-1} + \sum_{t=1}^{n} u_t.$$

Thus

$$\begin{aligned}\sum_{t=1}^{n} y_{t-1} &= (1-\rho_n)^{-1}\left(-y_n + y_0 + \sum_{t=1}^{n} u_t\right)\\ &= (1-\rho_n)^{-1} n^{1/2}\left((y_0 - y_n)n^{-1/2} + n^{-1/2}\sum_{t=1}^{n} u_t\right). \qquad (21)\end{aligned}$$

Then

$$(1-\rho_n)n^{-1/2}\sum_{t=1}^{n} y_t = (y_0 - y_n)n^{-1/2} + n^{-1/2}\sum_{t=1}^{n} u_t.$$

Since u_t is a martingale difference sequence with $Eu_t^2 < \infty$,

$$n^{-1/2}\sum_{t=1}^{n} u_t \Rightarrow N(0,1).$$

By A.2, $n^{-1/2}E|y_0| = o(1)$. Writing

$$y_t = \rho_n y_{t-1} + u_t = \rho_n^2 y_{t-2} + \rho_n u_{t-1} + u_t = \rho_n^t y_0 + \sum_{j=0}^{t-1} \rho_n^j u_{t-j},$$

we have

$$\begin{aligned} n^{-1/2}E|y_t| &= n^{-1/2}E\left|\rho_n^t y_0 + \sum_{j=0}^{t-1}\rho_n^j u_{t-j}\right| \\ &\le n^{-1/2}E|y_0| + n^{-1/2}\left(E\left(\sum_{j=0}^{t-1}\rho_n^j u_{t-j}\right)^2\right)^{1/2} \\ &\le o(1) + n^{-1/2}\left(\sum_{j=0}^{t-1}\rho_n^{2j}Eu_{t-j}^2\right)^{1/2} \\ &\le o(1) + Cn^{-1/2}(1-\rho_n^2)^{-1/2} \le o(1) + C(nv_n)^{-1/2} = o(1), \end{aligned}$$

by (3). Thus $n^{-1/2}(y_0 - y_n) \rightarrow_p 0$ and

$$(1-\rho_n)n^{-1/2}\sum_{t=1}^{n} y_t = o_P(1) + n^{-1/2}\sum_{t=1}^{n} u_t \Rightarrow N(0,\sigma^2),$$

proving (7). Finally, (8) follows from (7) and Lemma 2. □

LEMMA 3. *Suppose* A.1–A.2 *hold. Then*

$$Ey_n^2 \le C\sigma^2(1-\rho_n)^{-1}, \tag{22}$$

where C does not depend on ρ_n and σ^2. Further,

$$(1-\rho_n^2)n^{-1}\sum_{j=0}^{n-1} Ey_j^2 = \sigma^2 + o(1). \tag{23}$$

PROOF OF LEMMA 3. Since $Ey_s u_t = 0$ for $t > s$, by (1), for $t \ge 1$

$$\begin{aligned} Ey_t^2 &= \rho_n^2 Ey_{t-1}^2 + Eu_t^2 = \rho_n^2 Ey_{t-1}^2 + \sigma^2 = \rho_n^4 Ey_{t-2}^2 + (\rho_n^2 + \rho_n^0)\sigma^2 \\ &= \rho_n^{2t}Ey_0^2 + (\rho_n^{2(t-1)} + \cdots + \rho_n^0)\sigma^2 = \rho_n^{2t}Ey_0^2 + \frac{1-\rho_n^{2t}}{1-\rho_n^2}\sigma^2, \end{aligned} \tag{24}$$

by virtue of the sum

$$\rho_n^{2(t-1)} + \cdots + \rho_n^0 = \frac{1-\rho_n^{2t}}{1-\rho_n^2}.$$

Note that inequality $\log(1-v) \le -v$ $(0 \le v < 1)$ implies that

$$\rho_n^n = \exp(n\log\rho_n) = \exp(n\log(1-v_n)) \le \exp(-nv_n) \le (v_n n)^{-1} \to 0, \qquad (25)$$

since $nv_n \to \infty$ by (10). Since $0 \le \rho_n < 1$ and, under A.1 and A.2, $\rho_n^{2n} E y_0^2 \le C(v_n n)^{-1} E y_0^2 = o(v_n^{-1})$, it follows that (24) implies (22).

To show (23), note that

$$(1-\rho_n^2)n^{-1}\sum_{j=0}^{n-1} E y_j^2 = (1-\rho_n^2)n^{-1}\left(\frac{1-\rho_n^{2n}}{1-\rho_n^2}E y_0^2 + \sigma^2\frac{1}{1-\rho_n^2}\left(n - \frac{1-\rho_n^{2n}}{1-\rho_n^2}\right)\right)$$
$$= o(1) + \sigma^2,$$

by virtue of the assumption $E y_0^2 = o(n^{1/2})$, to prove (23). □

LEMMA 4. *Suppose* $E u_t^4 < \infty$, $E y_0^4 < \infty$. *Then*

$$E y_t^2 \le C(1-\rho_n)^{-1}, \qquad (26)$$

$$E y_t^4 \le C(1-\rho_n)^{-2}, \quad t = 1, 2, \ldots \qquad (27)$$

uniformly in $0 \le \rho_n < 1$ *and* t.

PROOF OF LEMMA 4. Since $E y_0^2 < \infty$, (26) follows from (24).

By (1),

$$\begin{aligned} E y_t^4 &= E(\rho_n y_{t-1} + u_t)^4 = E(\rho_n^2 y_{t-1}^2 + 2\rho_n y_{t-1}u_t + u_t^2)^2 \\ &= E(\rho_n^2 y_{t-1}^2 + \sigma^2 + [2\rho_n y_{t-1}u_t + u_t^2 - \sigma^2])^2 \\ &= E(\rho_n^2 y_{t-1}^2 + \sigma^2)^2 + E(2\rho_n y_{t-1}u_t + u_t^2 - \sigma^2)^2 \\ &= \rho_n^4 E y_{t-1}^4 + 2\rho_n^2 E y_{t-1}^2\sigma^2 + \sigma^4 + 4\rho_n^2 E y_{t-1}^2 E u_t^2 \\ &\quad + 4\rho_n E y_{t-1} E u_t^3 + E(u_t^2 - \sigma^2)^2 \\ &\le \rho_n^4 E y_{t-1}^4 + C(E y_{t-1}^2 + |E y_{t-1}| + 1), \end{aligned}$$

for some $C > 0$ since $\rho_n < 1$. Since by (26), $E y_t^2 \le C/(1-\rho_n)$, then

$$E y_{t-1}^2 + |E y_{t-1}| + 1 \le C/(1-\rho_n)$$

uniformly in ρ_n, and

$$Ey_t^4 \leq C(\rho_n^{4t} + \sum_{j=1}^{t} \rho_n^{4(j-1)}(1-\rho_n^2)^{-1}) \leq C(\rho_n^{4t} + (1-\rho_n^4)^{-1}(1-\rho_n^2)^{-1})$$
$$\leq C(1 + (1-\rho_n)^{-2}) \leq C(1-\rho_n)^{-2}$$

proving (27). □

ACKNOWLEDGEMENTS

Giraitis thanks the ESRC for research support under Grant R000239538. Phillips thanks the NSF for research support under Grant SES 04-142254.

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