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NEAR UNITY WITH PANEL DATA**

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## GMM ESTIMATION OF AUTOREGRESSIVE ROOTS NEAR UNITY WITH PANEL DATA

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This paper investigates a generalized method of moments (GMM) approach to the estimation of autoregressive roots near unity with panel data and incidental deterministic trends. Such models arise in empirical econometric studies of firm size and in dynamic panel data modeling with weak instruments. The two moment conditions in the GMM approach are obtained by constructing bias corrections to the score functions under OLS and GLS detrending, respectively. It is shown that the moment condition under GLS detrending corresponds to taking the projected score on the Bhattacharya basis, linking the approach to recent work on projected score methods for models with infinite numbers of nuisance parameters (Waterman and Lindsay (1998)). Assuming that the localizing parameter takes a nonpositive value, we establish consistency of the GMM estimator and find its limiting distribution. A notable new finding is that the GMM estimator has convergence rate  $n^{1/6}$ , slower than  $\sqrt{n}$ , when the true localizing parameter is zero (i.e., when there is a panel unit root) and the deterministic trends in the panel are linear. These results, which rely on boundary point asymptotics, point to the continued difficulty of distinguishing unit roots from local alternatives, even when there is an infinity of additional data.

KEYWORDS: Bias, boundary point asymptotics, GMM estimation, local to unity, moment conditions, nuisance parameters, panel data, pooled regression, projected score.

### 1. INTRODUCTION

RECENT YEARS HAVE SEEN the introduction of several important panel data sets where the cross sectional dimension ( $n$ ) and the time series dimension ( $T$ ) are comparable in magnitude. Some of these panel data sets, like the Penn World Tables, involve time series that are manifestly nonstationary and have persistent or slowly decaying serial correlations. These features distinguish the new data from the characteristics that are conventionally assumed in the analysis of panel data where  $T$  is very small and  $n$  is very large.

Since the early 1990's, there has been ongoing theoretical and applied research on panels whose time series components are nonstationary or persistent. For large  $n$  and fixed  $T$  panels, see Hahn, Hausman, and Kuersteiner (2001) and Kruiniger (2000). For large  $n$  and  $T$  panels allowing for nonstationarity in the data over time, the theoretical research includes the study of asymptotically unbiased estimation of the dynamic panel model (e.g., Hahn

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and Kuersteiner (2002)), panel unit root tests (e.g., Quah (1994), Levin, Lin, and Chu (2002), Im, Pesaran, and Shin (1996), Maddala and Wu (1999), and Choi (2001)), panel cointegration tests (e.g., Pedroni (1999), Binder, Hsiao, and Pesaran (1999)), and the development of linear regression theories for panel estimators under nonstationarity (e.g., Pesaran and Smith (1995) and Phillips and Moon (1999)). Applied research includes tests of growth convergence theories (Bernard and Jones (1996)), purchasing power parity relations (MacDonald (1996), Oh (1996), Pedroni (1996), Wu (1996), and Wu (1997)), and studies of the international links between savings and investment (Coakley, Kulaski, and Smith (1996) and Moon and Phillips (1998)).

Two recent papers by the authors (Moon and Phillips (1999, 2000)) study panel regression models that allow for both deterministic trends and stochastic trends with roots local to unity. As we discuss in Section 2 of the present paper, such models are important empirically in studying Gibrat's law and they have received attention recently in the weak instrument literature. When the deterministic trends in nonstationary panel data are heterogeneous across individuals, Moon and Phillips (1999) show that the maximum likelihood estimator (MLE) of the local to unity parameter in the stochastic trend is inconsistent. They call this phenomenon, which arises because of the presence of an infinite number of nuisance parameters, an incidental trend problem because it is analogous to the well-known incidental parameter problem in dynamic panels when  $T$  is fixed.<sup>2</sup> To solve the incidental trend problem, Moon and Phillips (2000) propose various methods, including an iterative ordinary least squares (OLS) procedure and a double bias corrected estimator, and establish limit theories for these consistent estimators that can be used for statistical inference about the localizing parameter.

As a continuation of the two studies just mentioned, the present paper investigates a generalized method of moments (GMM) estimator of autoregressive roots near unity with panel data. We establish two moment conditions that form the basis for inference. The first moment condition is obtained by adjusting for the bias of the score function after conventional OLS detrending. The second moment condition is constructed by adjusting for the bias of the score function following quasi-difference (QD) detrending. Interestingly, the second moment condition is shown to correspond to the Gaussian projected score, where the projection is taken on the so-called Bhattacharya basis that has been studied recently in the conventional incidental parameter problem by Waterman and Lindsay (1996, 1998) and Hahn and Kuersteiner (2002). Unlike the conventional moment conditions used in estimating dynamic panel data models, these moment conditions do not suffer the weak instrument problem that is discussed, for example, in Kruiniger (2000) and Hahn, Hausman, and Kuersteiner (2001).

<sup>2</sup>Lancaster (2000) provides a recent general survey of the incidental parameter problem in econometrics.

Consistency of the GMM estimator is proved under the assumption that the localizing parameter takes a nonpositive value. This condition is not too restrictive because most econometric models consider nonexplosive autoregressive regression models. Nevertheless, the restriction does matter in deriving the limiting distribution of the estimator because it is possible that the true parameter lies on the boundary of the parameter set. The most interesting case is, of course, the pure unit root case where the true localizing parameter is zero. In this case, in establishing the limiting distribution we cannot use the conventional approach that approximates the first-order condition because the true parameter could be on the boundary of the parameter set. To avoid this difficulty, we use the approach that takes a quadratic approximation of the nonlinear objective function and optimize it on the parameter set (cf. Andrews (1999) for some recent developments of estimation and inference in boundary problems).

One of the most interesting findings in the present paper is that the GMM estimator has slower convergence rate than  $\sqrt{n}$  when the time series components in the panel have unit roots (i.e., the true localizing parameter is zero), and the deterministic trends are linear. In this case the convergence rate is actually  $O(n^{1/6})$  rather than  $O(\sqrt{n})$ . This slow convergence rate arises because of lack of information in the moment conditions when there is a unit root, i.e., at the point  $c = 0$  in the space of the localizing parameter. It points to the continued difficulty of distinguishing unit roots from local alternatives in the presence of heterogeneous deterministic trends even when there is an infinity of additional data from a cross section.

The paper is organized as follows. Section 2 lays out the model and gives the basic assumptions that are maintained throughout the paper. We also discuss the empirical relevance of the model and the conventional moment conditions used in dynamic panel models of this type. In Section 3 we introduce two new moment conditions and prove that the second of these moment conditions corresponds to a Gaussian projected score on the Bhattacharya basis. In Section 4 we establish consistency of the GMM estimator and obtain the limiting distributions of the GMM estimator when the true parameter is less than zero and equal to zero. The Appendix contains technical derivations and proofs of the results in the main text.

## 2. PERSISTENT DYNAMIC PANELS

### 2.1. *The Model and Assumptions*

We study panel data that may show characteristics of time trends and persistent temporal shocks and whose dimension is large in both cross section ( $n$ ) and time series ( $T$ ) dimensions. To model such data, we extend the conventional dynamic panel model by taking the components formulation

$$(1) \quad z_{it} = \beta_i' g_{pt} + y_{it},$$

where  $g_{pt} = (t, t^2, \dots, t^p)'$ , the coefficients  $\beta_i$  are  $p$ -vectors that could be random, and the residuals  $y_{it}$  follow

$$y_{it} = \rho y_{it-1} + \varepsilon_{it}$$

with a common autoregressive coefficient  $\rho$  that is close to one defined in (4) below. Let  $y_{i0} = z_{i0}$  be the panel observations at the initial time period. In (1), the first term  $\beta_i' g_{pt}$  represents deterministic trends in the data, omitting an intercept because this is not consistently estimable from time series data when  $y_{it}$  is near integrated (e.g., Phillips and Lee (1996)) and can be incorporated in the initial condition  $y_{i0}$ . Assuming that the coefficient  $\beta_i$  varies across  $i$ , we may treat the trends  $\beta_i' g_{pt}$  as systematic individual effects or fixed effects in the panel. Since  $\rho$  is in the vicinity of unity, the components  $y_{it}$  have stochastic trends with persistent innovations  $\varepsilon_{it}$ .

The components model (1) can also be written in the more familiar format of an augmented regression form as

$$(2) \quad z_{it} = \rho z_{it-1} + \delta_i + \gamma_i' g_{pt} + \varepsilon_{it},$$

where

$$\begin{aligned} \delta_i &= \rho \beta_i' \iota_p, & \iota_p &= -(-1, (-1)^2, \dots, (-1)^p)', \\ \gamma_i' &= \beta_i' Y(\rho), & Y(\rho) & \text{is a } (p \times p) \text{ matrix depending on } \rho. \end{aligned}$$

For example, when  $p = 1$ , the deterministic panel trends are linear and the augmented model (2) is

$$(3) \quad z_{it} = \rho \beta_i + (1 - \rho) \beta_i t + \rho z_{it-1} + \varepsilon_{it}.$$

This linear trend model (3) is an extended version of the standard model for dynamic panels in which the individual effects are the incidental trends  $\rho \beta_i + (1 - \rho) \beta_i t$  and the autoregressive parameter is assumed to be close to one.<sup>3</sup>

The augmented format (2) has the drawback that linear regression leads to inefficient trend elimination, but the advantage that the detrended data is invariant to the trend parameters in (2). In the next section, we use the augmented formation (2) to define the first moment condition, and the component model (1) for the second moment condition.

To enable a rigorous development when  $\rho$  is close to one, we take the specific near unity formulation,

$$(4) \quad \rho = 1 + \frac{c}{T},$$

<sup>3</sup>For other examples of incidental trend models, see Section 11.2.1 of Wooldridge (2001) and the references therein.

or equivalently,

$$T(\rho - 1) = c,$$

in which the standardized deviation of the coefficient  $\rho$  from unity remains constant ( $c$ ). In this case, the stochastic trends  $y_{it}$  are near integrated and they are characterized by the parameter  $c$  instead of the autoregressive coefficient  $\rho$ . The time series properties of the near integrated process  $y_{it}$  are well known from the nonstationary time series literature (e.g., Phillips (1987) and Stock (1994)). Recently, Hahn, Hausman, and Kuersteiner (2001) and Kruiniger (2000) use the related specification  $\rho = 1 + (c/n)$  to model an autoregressive coefficient near unity in a panel with large  $n$  and fixed  $T$ .

In a conventional time series autoregression (AR), the probabilistic features (and the asymptotics) are discontinuous with respect to the AR coefficient as it passes through unity. When  $|\rho| < 1$ , the process is stationary, reverts to its mean, converges to a steady state, and has no stochastic trend, whereas when  $\rho = 1$ , the process is nonstationary, not mean-reverting, and contains stochastic trends. Models with near unit roots as in (4) have probabilistic features that are continuous with respect to parameter  $c$ , while still retaining some of the implications of the three different cases:  $\rho < 1$  ( $c < 0$ ),  $\rho = 1$  ( $c = 0$ ), and  $\rho > 1$  ( $c > 0$ ). More specifically, a direct calculation shows that  $\text{var}(y_{it})$  increases at the rate  $t$ , regardless of the sign of the parameter  $c$ . Thus,  $y_{it}$  is nonstationary and has a stochastic trend regardless of the sign of the parameter  $c$ . On the other hand, when  $t = [Tr]$  with  $0 < r \leq 1$ , it is well known that

$$(5) \quad \frac{y_{it}}{\sqrt{T}} \Rightarrow J_c(r) \quad \text{as } T \rightarrow \infty,$$

where  $J_c(r) = \int_0^r e^{c(r-s)} dW(s)$  is the Ornstein–Uhlenbeck process, and  $W(s)$  is a Brownian motion (e.g., Phillips (1987)). So, the marginal asymptotic distribution of the standardized process  $y_{it}/\sqrt{T}$  is continuous in  $c$ . Also, when  $c < 0$ , the limit process  $J_c(r)$  is stationary and mean reverting, while for  $c = 0$ , the limit process is Brownian motion. Thus, the standardized process  $y_{it}/\sqrt{T}$  preserves some of the probabilistic implications implied by the conventional AR(1) model for  $|\rho| < 1$  and  $\rho = 1$ . One benefit of the continuity property is that it is possible to produce confidence intervals for the AR coefficient  $\rho$  from estimates of  $c$  (Stock (1991)) even though consistent estimation of  $c$  from time series observations is not possible. In this paper we use notation  $c_0$  to denote the true coefficient for  $c$ .

In later sections of the paper, as part of the asymptotic development, we need to verify some properties of complicated nonlinear functions of  $c$  that depend on the trend  $g_{pt}$ . These functions are so complicated that it is very difficult to establish analytic results under the general polynomial trend set up with  $g_{pt} = (t, \dots, t^p)'$ . Instead, we rely on numerical methods for this part of

the analysis. To assist the analytic development, we restrict our attention to the following two cases: (i)  $g_{1t} = t$  and (ii)  $g_{2t} = (t, t^2)'$ . This restriction is hardly restrictive in practice because the linear and quadratic trends are the most widely used in empirical applications. The set up is formalized as follows:

ASSUMPTION 1 (Trend formulation): *The polynomial trend in model (1) is either (i)  $g_{1t} = t$  or (ii)  $g_{2t} = (t, t^2)'$ .*

ASSUMPTION 2 (Initial condition): *The initial observations  $z_{i0} = y_{i0}$  are independent across  $i$  with  $\sup_i E y_{i0}^4 < \infty$ .*

ASSUMPTION 3 (Error condition): *The error terms  $\varepsilon_{it} \sim \text{iid}(0, \sigma^2)$  across  $i$  and  $t$  with  $E \varepsilon_{it}^4 < \infty$  and  $\varepsilon_{it}$  are independent of  $y_{i0}$  for all  $i$ .*

ASSUMPTION 4 (Parameter set): (a) *The localizing parameter  $c$  takes a value in a compact subset  $\mathbb{C} = [\bar{c}, 0] \subset \mathbb{R}$ , where  $\bar{c} < 0$ .* (b) *The true localizing parameter  $c_0$  is in the set  $\mathbb{C}_0 = [\bar{c}, 0]$ .*

Assumption 4(a) restricts the parameter set  $\mathbb{C} = [\bar{c}, 0]$  to be nonpositive. This restriction is made because in most econometric applications, the cases  $|\rho| < 1$  and  $\rho = 0$  are of most interest. When the true parameter  $c_0 = 0$ , the model becomes nonstandard in the sense that the true parameter is on the boundary of the parameter set. Section 4.3 explores the implications of the boundary point aspect of this case.

The practical implication of the restriction of  $c_0$  to  $\mathbb{C} = [\bar{c}, 0]$  is of some interest. One of the advantages of pooling is that consistent estimation of  $c_0$  is achievable with panel data while it is not with a single time series. Take the case where the true value  $\rho_0 = 1 + (c_0/T)$  lies in the interior of the compact interval  $[1 + (\bar{c}/T), 1]$  for some  $\bar{c} < 0$ . Theorem 3 shows that the GMM estimator  $\hat{c}$  is consistent for  $c_0$  and has a limit normal distribution  $\sqrt{n}(\hat{c} - c_0) \rightarrow_d Z$ . The corresponding autoregressive coefficient estimate is  $\hat{\rho} = 1 + (\hat{c}/T)$  and  $\sqrt{n}T(\hat{\rho} - \rho_0) \rightarrow_d Z$ . So, panel pooling affects the “usual asymptotic distribution” of the autoregressive coefficient in near integrated models. The distribution is normal and lives in a shrunken neighborhood within the compact set  $[1 + (\bar{c}/T), 1]$ . In effect,  $\hat{\rho}$  is distributed in an  $O(1/\sqrt{n})$  neighborhood of the true value  $\rho_0 \in [1 + (\bar{c}/T), 1]$ . The main thrust of the present work is to put panel pooling to good effect, so that we get this increased precision about the value of  $\rho_0$  even when it is already in the locality of unity. To the extent that many economic time series are near integrated but also may have idiosyncratic deterministic trend components, this process increases precision in estimation, confidence interval construction, and local discriminatory power at unity for  $\rho_0$  (where it increases from  $O(T^{-1})$  to  $O(T^{-1}n^{-1/6})$ —see Section 4.3), but also near unity (where it increases from  $O(T^{-1})$  to  $O(T^{-1}n^{-1/2})$ —see Section 4.2).

ASSUMPTION 5:  $(n, T \rightarrow \infty)$  with  $(\log n / \log T) < 2 + O(1 / \log T)$ .

Assumption 5 restricts the relative size of the cross-section and temporal dimensions in the panel in the asymptotic theory. It is assumed that  $n$  increases at a lesser speed than  $T^2$  as  $T \rightarrow \infty$ . So if  $n = T^\alpha$ , then  $(\log n / \log T) = \alpha < 2$ , thereby allowing for panels in which  $n$  and  $T$  are of comparable size satisfying  $(n/T) \rightarrow \kappa$ ,  $0 < \kappa < \infty$ , as in Hahn and Kuersteiner (2002), and some panels where the size of the cross section dominates the time series count, i.e.,  $(n/T) \rightarrow \infty$ . This assumption is required for the derivation of the limit distribution of the estimator of  $c$  in Section 4. Consistency of the estimator does not require this restriction.

## 2.2. Discussion of the Model

### 2.2.1. Empirical Relevance in Modeling Firm Size

In the empirical industrial organization literature on firm size, dynamic panel models have been used widely to describe firm growth in terms of a simple formulation that follows the spirit of Gibrat's law of proportional effect (Gibrat (1931)). Gibrat's law states that the expected value of the increment in a firm's size each time period is proportional to the current size of the firm. Let  $Z_{it}$  denote the size of firm  $i$  at time  $t$  and  $e_{it}$  denote the (stochastic) proportionate rate of growth of firm  $i$  between time  $t$  and  $t - 1$ . Then Gibrat's law is formalized as

$$Z_{it} - Z_{it-1} = Z_{it-1}e_{it}$$

(e.g., Steindl (1965) and Sutton (1997)). Let  $z_{it} = \log Z_{it}$  and, using the approximation  $\log(1 + a) \approx a$  for small  $a$ , we may write the proportional law in autoregressive form as

$$(6) \quad z_{it} = z_{it-1} + e_{it},$$

which McCloughan (1995) calls Gibrat's process—a law in which the growth rate of a firm is independent of its initial size. When  $e_{it} \sim \text{iid}(\mu, \sigma_e^2)$ , set  $\varepsilon_{it} = e_{it} - \mu$ , and then we may rewrite Gibrat's process (6) in the component form

$$(7) \quad \begin{aligned} z_{it} &= \mu t + y_{it}, \\ y_{it} &= y_{it-1} + \varepsilon_{it}, \end{aligned}$$

which is a special case of model (1). The panel model (1) (or (2)) can therefore be interpreted as a generalized version of Gibrat's process when applied to firm growth data. To motivate (1), we now discuss more detailed features of the model in the context of this application and indicate how we can explain in terms of this model (1) some recent empirical findings relating to Gibrat's law.



First, according to (1) and (4), the stochastic shock in the logarithm of firm size is nearly integrated and involves the parameter  $c$ . As discussed earlier, for  $T$  large this formulation ensures that the distribution of firm size is asymptotically lognormal, as indicated by Gibrat's law.<sup>4,5</sup> Also, since the autoregressive coefficient  $\rho$  is close to unity for large  $T$ , the serial correlations of  $z_{it}$  will not decay. These properties coincide with what Hall and Mairesse (2000) observe in their firm panel data. Secondly, since the random growth rate process is  $\Delta z_{it} = \delta_i + \gamma_i' g_{pt} + (c/T)z_{it-1} + \varepsilon_{it}$ , we have  $\partial \Delta z_{it} / \partial z_{it-1} = (c/T) < 0$  if  $c < 0$ . Thus, firm growth decreases as firm size increases, giving the size effect. Also, if the deterministic trends are quadratic and the coefficient of the quadratic coefficient is negative, we have the age effect because  $\partial E(\Delta z_{it}) / \partial t = 2\beta_{i2} < 0$  when  $z_{i0} = y_{i0} = 0$  (see Evans (1987), Jovanovic (1982), and McCloughan (1995)). Thirdly, when  $y_{i0}$  is nonrandom, a simple calculation shows that  $\text{var}(z_{it}) = \sigma^2(1 + \rho^2 + \dots + \rho^{2(t-1)}) \sim \sigma^2 \sum_{s=0}^{t-1} \exp(2c(s/T))$  and so  $\Delta \text{var}(z_{it}) \simeq \sigma^2 \exp(2c(t-1)/T)$ . These indicate that inequalities in firm growth grow over time, as implied by Gibrat's law, but the rate of change in the inequality depends on the value of the growth parameter  $c$ . Fourthly, model (2) introduces individual specific effects through the trend coefficients  $\beta_i$  and the initial conditions  $z_{i0}$ . These effects may explain certain heterogeneous factors such as the  $x$ -inefficiency of some firms.

In view of these characteristics, the panel model (2) helps to bridge the gap between the pure form of Gibrat's law and the empirical evidence in the literature indicating certain systematic violations of the law. In addition, consistent estimation of the systematic growth parameter  $c$  makes it possible to measure the size effect and changes in growth inequality or firm concentration.

### 2.2.2. Conventional Moment Conditions and Weak Instruments

In recent years, one of the most widely used methods of estimating dynamic panel regression models with fixed effects, such as

$$z_{it} = (1 - \rho)\beta_i + \rho z_{it-1} + \varepsilon_{it},$$

is to utilize the moment conditions implied by the assumptions imposed on the model. (For details and references, see the recent survey by Arellano and

<sup>4</sup>The more recent literature argues that firm size distributions do not follow a "typical" distributional shape but nevertheless have skewness as their central characteristic (e.g., Sutton (1997) and Schmalensee (1989)).

<sup>5</sup>In empirical studies, a commonly used model for representing a generalized Gibrat process is the conventional AR(1) model  $z_{it} = \mu + \rho z_{it-1} + e_{it}$  (e.g., McCloughan (1995) and Hall and Mairesse (2000)). In this case, when  $-1 < \rho < 1$ , the size distribution depends on the distribution of the error  $e_{it}$  and skewness of the size distribution is not guaranteed. By contrast, skewness is guaranteed, at least asymptotically, in the near integrated case.

Honore (2001).) Among these moment conditions,<sup>6</sup> the moment restrictions known as the basic moment conditions, viz.,

$$(8) \quad E(z_{is}\Delta\varepsilon_{it}) = 0 \quad \text{for } s = 0, \dots, t - 2,$$

are the most widely applied. This section considers the properties of the basic form of moment conditions implied by the model (1).

For simplicity, we consider only the linear trend case, that is,

$$(9) \quad z_{it} = \rho\beta_i + (1 - \rho)\beta_it + \rho z_{it-1} + \varepsilon_{it}.$$

Due to the presence of the incidental trends, instead of (8) the basic moment conditions in the model (1) are

$$(10) \quad E(z_{is}\Delta^2\varepsilon_{it}) = 0 \quad \text{for } s = 0, \dots, t - 3,$$

which provide the orthogonality conditions in an instrumental variable regression of

$$(11) \quad \Delta^2 z_{it} = \rho\Delta^2 z_{it-1} + \Delta^2 \varepsilon_{it},$$

with instrumental variables  $z_{is}$ ,  $s = 0, \dots, t - 3$ . Notice from (11) that the basic moment conditions (10) are linear in the parameter  $\rho$ . To evaluate the orthogonality conditions in (10) in terms of their information content, we calculate the first derivative of the moments with respect to the parameter  $\rho$ . Then,

$$\frac{\partial E(z_{is}(\Delta^2 z_{it} - \rho\Delta^2 z_{it-1}))}{\partial \rho} = E(z_{is}\Delta^2 z_{it-1}), \quad s = 0, \dots, t - 3,$$

which is the covariance between the instrumental variable  $z_{is}$  and regressor in (11).

First, when  $\rho = 1$ , as is well known in the conventional dynamic panel model, the information content of the basic moment conditions is zero because  $E(z_{is}\Delta^2 z_{it-1}) = 0$  for  $s = 0, \dots, t - 3$ , and so the instruments are not correlated with the regressors of (11). In this case, the parameter  $\rho$  is not identifiable from the basic moment conditions.

When  $\rho = 1 + (c/T)$  with  $c < 0$ , from (9), we have

$$\Delta z_{it-1} = \left(1 + \frac{c}{T}\right)\beta_i - \frac{c}{T}\beta_i(t-1) + \frac{c}{T}z_{it-2} + \varepsilon_{it-1},$$

<sup>6</sup>Other types of moment conditions used in conventional dynamic panel models are derived under extra assumptions on the initial conditions, the fixed effect parameter, and the covariance stationarity restrictions.

and

$$\begin{aligned}\Delta^2 z_{it-1} &= -\frac{c}{T}\beta_i + \frac{c}{T}\Delta z_{it-2} + \Delta \varepsilon_{it-1} \\ &= -\frac{c}{T}\beta_i + \frac{c}{T}\left\{\left(1 + \frac{c}{T}\right)\beta_i - \frac{c}{T}\beta_i(t-2) + \frac{c}{T}z_{it-3} + \varepsilon_{it-2}\right\} \\ &\quad + \Delta \varepsilon_{it-1} \\ &= \left(\frac{c}{T}\right)^2 (y_{it-3} + \varepsilon_{it-2}) + \Delta \varepsilon_{it-1}.\end{aligned}$$

Then, it follows that

$$E(z_{it}\Delta^2 z_{it-1}) = \left(\frac{c}{T}\right)^2 E[(\beta_i s + y_{it})y_{it-3}] \sim O\left(\frac{1}{\sqrt{T}}\right).$$

This implies that if the coefficient is near unity as in (4) and the time dimension is large, the information in the basic moment conditions is small, and the instruments  $z_{it}$  become weak.<sup>7</sup> In consequence, GMM estimation involving only conventional moment conditions cannot estimate the model consistently. This has become an issue of some importance in empirical studies of dynamic panel regression.

We therefore proceed to consider a different approach that augments the information content of the basic moment conditions. This approach arises from a consideration of the incidental trends problem, which we now discuss.

### 2.2.3. Incidental Trends Problem

Two recent papers by the authors (Moon and Phillips (1999, 2000)) find that an incidental trend problem arises in estimating the local to unity parameter  $c$  in the panel regression model (1) or (2) where an infinite number of nuisance parameters are present. Moon and Phillips (1999) show that the score of the (pseudo) likelihood that concentrates out the incidental trend parameters  $\beta_i$  is biased (even in the limit). In consequence, the maximum likelihood estimator (MLE) of  $c$  in (1) is inconsistent. Also, according to Moon and Phillips (2000), when the incidental trend parameters  $\beta_i$  are eliminated by ordinary least squares (OLS) projection, the normal equation of the pooled OLS estimator is biased as well (even in the limit) due to the correlation between the OLS detrended error term and the OLS detrended regressor. It follows that the pooled OLS estimator of  $c$  obtained from OLS detrended data is also biased.

<sup>7</sup>When  $T$  is finite and  $n$  is large, Hahn, Hausman, and Kuersteiner (2001) and Kruiniger (2000) investigate similar weak instrument effects in the conventional dynamic panel regression model with near unit root specifications  $\rho = 1 + (c/n)$ .

3. MOMENT CONDITIONS

We now propose an estimation procedure that eliminates the effects of the incidental trend problem. The approach involves GMM and minimizes a distance criterion based on a vector of sample moment functions. The moment conditions are designed to have a limit that will identify the true localizing parameter  $c_0$  even in the presence of the incidental trend coefficients  $\beta_i$ .

The principle we use for constructing moment conditions with this property is to adjust for the bias that arises in the usual score functions, the latter being explained in Moon and Phillips (1999, 2000). The adjustments are based on formulae from the explicit computation of the bias functions. The next sections explain these constructions in detail.<sup>8</sup>

3.1. *The First Moment Condition*

The following notation is defined to assist with the analysis of the trend function asymptotics and it will be used throughout the rest of the paper. Let

$$\begin{aligned} \tilde{\gamma}_i &= (\delta_i, \gamma_i)', \\ \tilde{g}_{pt} &= (1, g'_{pt})', \\ g_p(r) &= (r, \dots, r^p)' \quad \text{with } r \in [0, 1], \\ \tilde{g}_p(r) &= (1, g_p(r))', \\ G_{p,T} &= (g'_{p1}, \dots, g'_{pT})', \\ G_{p,T,-1} &= (g'_{p0}, \dots, g'_{pT-1})', \\ \tilde{G}_{p,T} &= (\tilde{g}'_{p1}, \dots, \tilde{g}'_{pT})', \\ \tilde{M}_{p,T} &= I_T - \tilde{G}_{p,T}(\tilde{G}'_{p,T} \tilde{G}_{p,T})^{-1} \tilde{G}'_{p,T}, \\ D_{p,T} &= \text{diag}(T, \dots, T^p), \\ \tilde{D}_{p,T} &= \text{diag}(1, D_{p,T}), \\ h_{pT}(t, s) &= g'_{pt} D_{p,T}^{-1} \left( \frac{1}{T} \sum_{t=1}^T D_{p,T}^{-1} g_{pt} g'_{pt} D_{p,T}^{-1} \right)^{-1} D_{p,T}^{-1} g_{ps}, \\ \tilde{h}_{pT}(t, s) &= \tilde{g}'_{pt} \tilde{D}_{p,T}^{-1} \left( \frac{1}{T} \sum_{t=1}^T \tilde{D}_{p,T}^{-1} \tilde{g}_{pt} \tilde{g}'_{pt} \tilde{D}_{p,T}^{-1} \right)^{-1} \tilde{D}_{p,T}^{-1} \tilde{g}_{ps}, \end{aligned}$$

<sup>8</sup>The principle employed can be applied to any biased score function for transformed data that is invariant to the incidental trend coefficients. A referee suggested such a biased score function that leads to an alternative moment condition that turns out to be closely related to the first moment condition described below. In view of this similarity and space constraints, we do not investigate this alternative moment condition here.

$$h_p(r, s) = g'_p(r) \left( \int_0^1 g_p(r) g_p(r)' dr \right)^{-1} g_p(s),$$

$$\tilde{h}_p(r, s) = \tilde{g}'_p(r) \left( \int_0^1 \tilde{g}_p(r) \tilde{g}_p(r)' dr \right)^{-1} \tilde{g}_p(s).$$

Write  $z_i = (z_{i1}, \dots, z_{iT})'$ ,  $z_{i,-1} = (z_{i0}, \dots, z_{iT-1})'$ , and  $\varepsilon_i = (\varepsilon_{i1}, \dots, \varepsilon_{iT})'$ . Define

$$\underline{z}_i = \tilde{M}_{p,T} z_i, \quad \underline{\varepsilon}_i = \tilde{M}_{p,T} \varepsilon_i, \quad \underline{z}_{i,-1} = \tilde{M}_{p,T} z_{i,-1}.$$

Then, it is straightforward to show that  $\underline{z}_i = \underline{y}_i$  and  $\underline{z}_{i,-1} = \underline{y}_{i,-1}$ , where  $\underline{y}_i = \tilde{M}_{p,T} y_i$ ,  $\underline{y}_{i,-1} = \tilde{M}_{p,T} y_{i,-1}$ ,  $y_i = (y_1, \dots, y_T)'$ , and  $y_{i,-1} = (y_0, \dots, y_{T-1})'$ . Let  $\underline{z}_{i,-1,t} = z_{it-1} - \frac{1}{T} \sum_{s=1}^T \tilde{h}_{pT}(t, s) z_{is-1}$  be the  $t$ th element of  $\underline{z}_{i,-1}$ .

One straightforward procedure of estimating  $c_0$  (equivalently  $\rho_0 = 1 + (c_0/T)$ ) is to eliminate the unknown trends  $\delta_i + \gamma'_i g_{pt}$  by taking OLS regression residuals and then applying pooled least squares with an appropriate bias correction for the serial correlation of  $\varepsilon_{it}$ . We call this method iterative OLS. However, as noted in Moon and Phillips (2000), this iterative OLS procedure yields an inconsistent estimator of  $c_0$  due to a nondegenerating asymptotic bias between the detrended regressor and the detrended error term. The first moment condition is obtained simply by subtraction of this asymptotic bias term in iterative OLS from the normal equation for the OLS detrended data.

More specifically, write the model (2) in vector notation as

$$(12) \quad z_i = \rho_0 z_{i,-1} + \tilde{G}_{p,T} \tilde{\gamma}_i + \varepsilon_i, \quad \rho_0 = 1 + \frac{c_0}{T}.$$

Applying  $\tilde{M}_{pT}$  to both sides, we have

$$(13) \quad \underline{z}_i = \rho_0 \underline{z}_{i,-1} + \underline{\varepsilon}_i,$$

where  $\underline{z}_i$ ,  $\underline{z}_{i,-1}$ , and  $\underline{\varepsilon}_i$  are OLS detrended versions of  $z_i$ ,  $z_{i,-1}$ , and  $\varepsilon_i$ , respectively. In general, the detrended regressor vector  $\underline{z}_{i,-1}$  and the detrended error vector  $\underline{\varepsilon}_i$  are correlated and the first moment condition is found by correcting for the bias due to the correlation between  $\underline{z}_{i,-1}$  and  $\underline{\varepsilon}_i$ .

Before we define the first moment condition, we discuss the estimation of the error variance  $\sigma^2$ . Let  $\hat{\rho}$  be the pooled OLS estimator of (13),  $\hat{\rho} = (\sum_{i=1}^n \underline{z}'_{i,-1} \underline{z}_{i,-1})^{-1} (\sum_{i=1}^n \underline{z}'_{i,-1} \underline{z}_i)$ , and  $\hat{\varepsilon}_i$  be the OLS residual of (13),  $\hat{\varepsilon}_i = \underline{z}_i - \hat{\rho} \underline{z}_{i,-1}$ . The estimator of  $\sigma^2$  is

$$(14) \quad \hat{\sigma}^2 = \frac{1}{nT} \sum_{i=1}^n \hat{\varepsilon}'_i \hat{\varepsilon}_i.$$

The first moment condition formula is defined as

$$\begin{aligned}
 (15) \quad m_{1,iT}(c) &= \frac{1}{T} \left( z_i - \left( 1 + \frac{c}{T} \right) z_{i,-1} \right)' z_{i,-1} + \hat{\sigma}^2 \omega_{pT}(c) \\
 &= \frac{1}{T} \sum_{t=1}^T \varepsilon_{it} y_{i,t-1} - \frac{1}{T^2} \sum_{t=1}^T \sum_{s=1}^T \varepsilon_{it} y_{i,s-1} \tilde{h}_{pT}(t, s) \\
 &\quad - (c - c_0) \frac{1}{T^2} \sum_{t=1}^T y_{i,-1,t}^2 + \hat{\sigma}^2 \omega_{pT}(c),
 \end{aligned}$$

where

$$\omega_{pT}(c) = \frac{1}{T^2} \sum_{t=2}^T \sum_{s=1}^{t-1} \left[ \left( 1 + \frac{c}{T} \right)^T \right]^{\frac{t-s-1}{T}} \tilde{h}_{pT}(t, s),$$

and  $\varepsilon_{it}$  and  $y_{i,-1,t}$  are the  $t$ th elements of  $\varepsilon_i$  and  $y_{i,-1}$ , respectively. The term  $\hat{\sigma}^2 \omega_{pT}(c)$  corrects for the bias that arises from the correlation between  $\varepsilon_{it}$  and  $y_{i,-1,t}$ .

This first moment condition was utilized in the double bias corrected estimator of Moon and Phillips (2000). Suppose that a preliminary consistent estimate of  $c$  is available. By linearizing the bias correction term  $\hat{\sigma}^2 \omega_{pT}(c)$  in (15) around the consistent estimate of  $c$ , we may approximate the first moment condition as a linear function of  $(c - c_0)$ . The double bias corrected estimator solves this linear approximation equation. In this paper, to estimate the parameter  $c$ , we continue to use the nonlinear moment condition (15) rather than work with a linear approximation.

### 3.2. The Second Moment Condition

Before discussing the second moment condition, we introduce some additional notation. Let  $\Delta_c = (1 - (1 + (c/T))L)$ , where  $L$  is the lag operator. Define

$$\begin{aligned}
 F_{p,T} &= \text{diag}(1, T, \dots, T^{p-1}) = \frac{1}{T} D_{p,T}, & \widehat{\Delta_c g_{pt}} &= F_{p,T}^{-1} \Delta_c g_{pt}, \\
 \dot{g}_p(r) &= \frac{d}{dr} g_p(r) = (1, 2r, \dots, pr^{p-1})', & \dot{g}_{pc}(r) &= \dot{g}_p(r) - c g_p(r), \\
 A_{pT}(c) &= \frac{1}{T} \sum_{t=1}^T \widehat{\Delta_c g_{pt}} \widehat{\Delta_c g_{pt}}', & A_p(c) &= \int_0^1 \dot{g}_{pc}(r) \dot{g}_{pc}(r)' dr, \\
 B_{pT}(c) &= \frac{1}{T} \sum_{t=1}^T \widehat{\Delta_c g_{pt}} g'_{pt-1} D_{pT}^{-1}, & B_p(c) &= \int_0^1 \dot{g}_{pc}(r) g_p(r)' dr.
 \end{aligned}$$

The second moment condition is obtained from the efficiently detrended regression equation. According to Canjels and Watson (1997) and Phillips and Lee (1996), the trend coefficient in the model (1) can be efficiently estimated in the time domain by employing a procedure that amounts to quasi-differencing the data with the operator  $\Delta_c$ . That is, when the localizing parameter  $c$  is known, an estimator of  $\beta_i$  in (1) that is asymptotically more efficient than the OLS estimator of  $\beta_i$  is

$$\hat{\beta}_i(c) = \left( \sum_{t=1}^T \Delta_c g_{pt} \Delta_c g'_{pt} \right)^{-1} \left( \sum_{t=1}^T \Delta_c g_{pt} \Delta_c z_{it} \right).$$

Denoting  $y_{it}(\beta_i) = z_{it} - \beta_i' g_{pt}$ , we write

$$\hat{\beta}_i(c) = \beta_i + \left( \sum_{t=1}^T \Delta_c g_{pt} \Delta_c g'_{pt} \right)^{-1} \left( \sum_{t=1}^T \Delta_c g_{pt} \Delta_c y_{it}(\beta_i) \right).$$

Define  $\varepsilon_{it}(c, \beta_i) = \Delta_c z_{it} - \beta_i' \Delta_c g_{pt}$ .

The second moment function  $m_{2,iT}(c)$  is defined as

$$(16) \quad m_{2,iT}(c) = \frac{1}{T} \sum_{t=1}^T \varepsilon_{it}(c, \hat{\beta}_i(c)) y_{it-1}(\hat{\beta}_i(c)) + \hat{\sigma}^2 \lambda_{pT}(c), \quad \text{where}$$

$$\lambda_{pT}(c) = \frac{1}{T^2} \sum_{t=2}^T \sum_{s=1}^{t-1} \left[ \left( 1 + \frac{c}{T} \right)^T \right]^{\frac{t-s-1}{T}} \widehat{\Delta_c g_{ps}}' A_{pT}(c)^{-1} \widehat{\Delta_c g_{pt}}.$$

Notice that  $y_{it-1}(\hat{\beta}_i(c))$  is the induced residual of the regression equation  $z_{it} = \beta_i' g_{pt} + y_{it}$  and  $\varepsilon_{it}(c, \hat{\beta}_i(c))$  is the residual of the quasi-differenced equation  $\Delta_c z_{it} = \beta_i' \Delta_c g_{pt} + \Delta_c y_{it}$ . In the second moment function  $m_{2,iT}(c)$  we correct for the asymptotic bias of  $(1/T) \sum_{t=1}^T \varepsilon_{it}(c, \hat{\beta}_i(c)) y_{it-1}(\hat{\beta}_i(c))$  by subtracting off the estimate  $\hat{\sigma}^2 \lambda_{pT}(c)$ .

As mentioned above, Moon and Phillips (1999) showed that the Gaussian MLE of the panel regression model (3) with linear incidental trends is inconsistent. The main reason for inconsistency of the MLE is that the concentrated score of the (standardized) Gaussian likelihood function,  $(1/n) \sum_{i=1}^n (1/T) \times \sum_{t=1}^T \varepsilon_{it}(c, \hat{\beta}_i(c)) y_{it-1}(\hat{\beta}_i(c))$ , has nonzero mean in the limit. In the second moment formulation of  $m_{2,iT}(c)$ , by subtracting off the estimate  $\hat{\sigma}^2 \lambda_{pT}(c)$ , we eliminate the asymptotic bias of the concentrated Gaussian score function.

### 3.3. The Relationship Between the Second Moment Condition and the Projected Score

This section shows that the second moment function  $m_{2,iT}(c)$  is a projected score of the panel regression model (1) under Gaussian errors. Suppose that

the error process  $\varepsilon_{it}$  in the model (1) is an iid standard normal process across  $i$  and over  $t$ . For convenience we assume that  $z_{i0} = y_{i0} = 0$  for all  $i$ .

Under general regularity conditions, it is well known that the asymptotic properties of the MLE, and most notably its consistency, are closely related to the unbiasedness of the score function at the true parameter. However, it is also well known that in dynamic panel regression models with incidental parameters the MLE is not consistent (e.g., see Neyman and Scott (1948) and Nickell (1981)) as  $n \rightarrow \infty$  with  $T$  fixed. Recently, Moon and Phillips (1999) found that this incidental parameter problem also arises in nonstationary panel regression models with incidental trends and roots local to unity when both  $n \rightarrow \infty$  and  $T \rightarrow \infty$ , covering models such as (1).

The main reason for the inconsistency of the MLE is that the score function in an incidental trend model has a bias at the true parameter. Therefore, in order to obtain a consistent estimate, one needs to correct for the bias in the score function. One recently investigated method to correct for this bias is to use a projected score function, where the projection is taken onto the so-called Bhattacharyya basis. The resulting approach is called “a projected score method.”

To define a projected score in the present case, assuming that  $\varepsilon_{it}$  are iid  $N(0, 1)$ , we introduce the following notation. Denote the joint density of  $z_i$

$$(17) \quad f_i(z_i; c, \beta_i) = \left( \frac{1}{\sqrt{2\pi}} \right)^T \exp\left( -\frac{1}{2} \sum_{t=1}^T (\Delta_c z_{it} - \beta_i' \Delta_c g_{pt})^2 \right)$$

and set

$$U_{1i}(c, \beta_i) = \frac{\partial f_i / \partial c}{f_i}, \quad V_{1i}(c, \beta_i) = \frac{\partial f_i / \partial \beta_i}{f_i},$$

$$V_{2i}(c, \beta_i) = \frac{\partial^2 f_i / \partial \beta_i \partial \beta_i'}{f_i}, \quad V_i(c, \beta_i) = \begin{pmatrix} V_{1i}(c, \beta_i) \\ D_p^+ \text{vec } V_{2i}(c, \beta_i) \end{pmatrix},$$

where  $D_p^+ = (D_p' D_p)^{-1} D_p'$  and  $D_p$  is the duplication matrix.

For convenience, we use the notation  $U_{1i}, V_i$ , and  $V_{ki}$  for  $U_{1i}(c, \beta_i), V_i(c, \beta_i)$ , and  $V_{ki}(c, \beta_i), k = 1, 2$ , respectively. In the statistics literature,  $V_{1i}$  and  $V_{2i}$  are known as the Bhattacharyya basis of order 1 and 2, respectively (see Bhattacharyya (1946) and Waterman and Lindsay (1996)). The projected score  $U_{2i}$  is defined as the residual in the  $L_2$ -projection of  $U_{1i}$  on the closed linear space spanned by  $V_{1i}$  and  $V_{2i}$ , i.e.,

$$(18) \quad U_{2i} = U_{1i} - \xi_1' V_{1i} - \xi_2' D_p^+ (\text{vec } V_{2i}).$$

Recently, using the projected score method, Waterman and Lindsay (1998) and Hahn (1998) were able to solve similar nuisance parameter problems in the



classical Neyman and Scott panel regression model and in a simple dynamic panel regression model with fixed effects, respectively.

When the joint density of  $z_i$  is given in (17),  $U_{1i}$ ,  $V_{1i}$ , and  $V_{2i}$  are found to be

$$\begin{aligned}
 U_{1i}(c, \beta_i) &= \frac{1}{T} \sum_{t=1}^T \varepsilon_{it}(c, \beta_i) y_{it-1}(\beta_i), \\
 V_{1i}(c, \beta_i) &= \sum_{t=1}^T \varepsilon_{it}(c, \beta_i) \Delta_c g_{pt}, \\
 V_{2i}(c, \beta_i) &= - \sum_{t=1}^T \Delta_c g_{pt} \Delta_c g'_{pt} \\
 &\quad + \left( \sum_{t=1}^T \varepsilon_{it}(c, \beta_i) \Delta_c g_{pt} \right) \left( \sum_{t=1}^T \varepsilon_{it}(c, \beta_i) \Delta_c g_{pt} \right)'.
 \end{aligned}$$

Some algebra verifies that  $EV_{1i}U_{1i} = 0$  and  $EV_{1i}[D_p^+(\text{vec } V_{2i})]' = 0$ .<sup>9</sup> Hence, the two  $L_2$ -projection coefficients  $\xi_1$  and  $\xi_2$  in (18) are given by

$$\xi_1 = [EV_{1i}V'_{1i}]^{-1}EV_{1i}U_{1i} = 0,$$

and

$$\xi_2 = [D_p^+E(\text{vec } V_{2i})(\text{vec } V_{2i})'D_p^+]^{-1}D_p^+E(\text{vec } V_{2i})U_{1i}.$$

Next

$$\begin{aligned}
 E(\text{vec } V_{2i})(\text{vec } V_{2i})' &= \sum_{t=1}^T \sum_{s=1}^T (\Delta_c g_{pt} \Delta_c g'_{pt} \otimes \Delta_c g_{ps} \Delta_c g'_{ps}) \\
 &\quad + \sum_{t=1}^T \sum_{s=1}^T (\Delta_c g_{pt} \Delta_c g'_{ps} \otimes \Delta_c g_{ps} \Delta_c g'_{pt}),
 \end{aligned}$$

and

$$\begin{aligned}
 E(\text{vec } V_{2i})U_{1i} &= \frac{1}{T} \sum_{t=2}^T \sum_{s=1}^{t-1} [\Delta_c g_{pt} \otimes \Delta_c g_{ps} + \Delta_c g_{ps} \otimes \Delta_c g_{pt}] \\
 &\quad \times \left[ \left( 1 + \frac{c}{T} \right)^T \right]^{\frac{t-s-1}{T}}.
 \end{aligned}$$

<sup>9</sup>The expectation is evaluated at the parameters  $c$  and  $\beta_i$ .

Therefore, the projected score  $U_{2i}(c, \beta_i)$  is

$$U_{2i}(c, \beta_i) = \frac{1}{T} \sum_{t=1}^T \varepsilon_{it}(c, \beta_i) y_{it-1}(\beta_i) + \xi_2' D_p^+ \sum_{t=1}^T (\Delta_c g_{pt} \otimes \Delta_c g_{pt}) - \xi_2' D_p^+ \left( \sum_{t=1}^T \varepsilon_{i,t}(c, \beta_i) \Delta_c g_{pt} \right) \otimes \left( \sum_{s=1}^T \varepsilon_{is}(c, \beta_i) \Delta_c g_{ps} \right),$$

where

$$\begin{aligned} \xi_2 = & \left[ \sum_{t=1}^T \sum_{s=1}^T D_p^+ \{ (\Delta_c g_{pt} \Delta_c g'_{pt} \otimes \Delta_c g_{ps} \Delta_c g'_{ps}) \right. \\ & \left. + (\Delta_c g_{pt} \Delta_c g'_{ps} \otimes \Delta_c g_{ps} \Delta_c g'_{pt}) \} (D_p^+)' \right]^{-1} \\ & \times \frac{1}{T} \sum_{t=2}^T \sum_{s=1}^{t-1} D_p^+ [\Delta_c g_{pt} \otimes \Delta_c g_{ps} \\ & + \Delta_c g_{ps} \otimes \Delta_c g_{pt}] \left[ \left( 1 + \frac{c}{T} \right)^T \right]^{\frac{t-s-1}{T}}. \end{aligned}$$

Since  $\beta_i$  in  $U_{2i}$  is unknown, we replace it with the estimate

$$\hat{\beta}_i(c) = \left( \sum_{t=1}^T \Delta_c g_{pt} \Delta_c g'_{pt} \right)^{-1} \left( \sum_{t=1}^T \Delta_c g_{pt} \Delta_c z_{it} \right).$$

Then, we have the following concentrated projected score,

$$(19) \quad U_{2i}(c, \hat{\beta}_i(c)) = \frac{1}{T} \sum_{t=1}^T \varepsilon_{it}(\hat{\beta}_i(c), c) y_{it-1}(\hat{\beta}_i(c)) + \xi_2' D_p^+ \sum_{t=1}^T (\Delta_c g_{pt} \otimes \Delta_c g_{pt}),$$

because  $\sum_{t=1}^T \varepsilon_{it}(c, \hat{\beta}_i(c)) \Delta_c g_{pt} = 0$ .

When the error process  $\varepsilon_{it}$  is iid(0, 1) across  $i$  and over  $t$ , the second moment function  $m_{2,iT}(c)$  is

$$m_{2,iT}(c) = \frac{1}{T} \sum_{t=1}^T \varepsilon_{it}(c, \hat{\beta}_i(c)) y_{it-1}(\hat{\beta}_i(c)) + \lambda_{pT}(c).$$

The following lemma states that the bias correction term  $\lambda_{pT}(c)$  in  $m_{2,iT}(c)$  is equivalent to  $\xi_2' D_p^+ \sum_{t=1}^T (\Delta_c g_{pt} \otimes \Delta_c g_{pt})$ . So, the second moment function actually corresponds to the concentrated projected score function of the Gaussian model.

LEMMA 1 (Equivalence): *Suppose that the errors in model (1) are iid normal with mean zero and variance 1 across  $i$  and over  $t$  and  $y_{i0} = z_{i0} = 0$  for all  $i$ . Then, the second moment condition  $m_{2,iT}(c)$  is equivalent to the concentrated projected score function  $U_{2i}(c, \hat{\beta}_i(c))$ .*

We omit the proof of Lemma 1, which is given in Moon and Phillips (2002).

#### 4. GMM ESTIMATION AND ASYMPTOTICS

This section investigates the asymptotic properties of a GMM estimator of  $c$  that is based on the two moment conditions introduced in the previous section. Let

$$M_{nT}(c) = \frac{1}{n} \sum_{i=1}^n m_{iT}(c),$$

where  $m_{iT}(c)' = (m_{1,iT}(c), m_{2,iT}(c))$ , and  $m_{1,iT}(c)$  and  $m_{2,iT}(c)$  are defined in (15) and (16), respectively. Let  $\hat{W}$  be a  $(2 \times 2)$  random weight matrix and  $B_{nT}$  be a sequence of real numbers that converges to infinity as  $(n, T \rightarrow \infty)$ . The GMM estimator  $\hat{c}$  for the unknown parameter  $c_0$  in (12) is defined as the extremum estimator for which

$$(20) \quad Z_{nT}(\hat{c}) \leq \min_{c \in \mathbb{C}} Z_{nT}(c) + o_p(B_{nT}^{-2}),$$

where

$$Z_{nT}(c) = M_{nT}(c)' \hat{W} M_{nT}(c).$$

Since the objective function  $Z_{nT}(c)$  is continuous in  $c$  and the parameter set  $\mathbb{C} = [\bar{c}, 0]$  for some  $\bar{c} < 0$  that contains  $c_0$  is assumed to be compact, it is possible to find a global minimum of  $Z_{nT}(c)$  over  $\mathbb{C}$ . The main purpose in allowing for an  $o_p(B_{nT}^{-1})$  deviation bound from the global minimum  $\min_{c \in \mathbb{C}} Z_{nT}(c)$  is to reduce the computational burden and allow for potential numerical computational errors within a range of  $o_p(B_{nT}^{-1})$ . Later in the paper, depending on the convergence rate of  $\hat{c}$  to  $c_0$ , we will determine the sequence  $B_{nT}$ .

##### 4.1. Consistency of the GMM Estimator

Define

$$M(c) = (m_1(c), m_2(c))',$$

where

$$\begin{aligned}
 m_1(c) &= \omega_p(c) - \omega_p(c_0) - (c - c_0)\psi_p(c_0), \\
 \omega_p(c) &= \int_0^1 \int_0^r e^{c(r-s)} \tilde{h}_p(r, s) ds dr, \\
 \psi_p(c_0) &= -\frac{1}{2c_0} \left( 1 + \frac{1}{2c_0} (1 - e^{2c_0}) \right) \\
 &\quad - \int_0^1 \int_0^1 e^{c_0(r+s)} \frac{1}{2c_0} (1 - e^{-2c_0(r \wedge s)}) \tilde{h}_p(r, s) ds dr,
 \end{aligned}$$

and

$$\begin{aligned}
 m_2(c) &= -(c - c_0) \left( \int_0^1 \int_0^r e^{2c_0(r-s)} ds dr \right) \\
 &\quad + (c - c_0) \int_0^1 \int_0^1 \int_0^{r \wedge s} e^{c_0(r+s-2v)} \dot{g}_{pc}(s)' \\
 &\quad \quad \quad \times A_p(c)^{-1} \dot{g}_{pc}(r) dv ds dr \\
 &\quad + (c - c_0) \int_0^1 \int_0^r e^{c_0(r-s)} \dot{g}_{pc}(r)' A_p(c)^{-1} g_p(s) ds dr \\
 &\quad + (c - c_0) \int_0^1 \int_0^r e^{c_0(r-s)} \dot{g}_{pc}(s)' A_p(c)^{-1} g_p(r) ds dr \\
 &\quad - (c - c_0)^2 \int_0^1 \int_0^1 \int_0^{r \wedge s} e^{c_0(r+s-2v)} \dot{g}_{pc}(s)' \\
 &\quad \quad \quad \times A_p(c)^{-1} g_p(r) dv ds dr \\
 &\quad - (c - c_0) \int_0^1 \int_0^r e^{c_0(r-s)} \dot{g}_{pc}(r)' A_p(c)^{-1} B_p(c) \\
 &\quad \quad \quad \times A_p(c)^{-1} \dot{g}_{pc}(s) ds dr \\
 &\quad - (c - c_0) \int_0^1 \int_0^r e^{c_0(r-s)} \dot{g}_{pc}(s)' A_p(c)^{-1} B_p(c)' \\
 &\quad \quad \quad \times A_p(c)^{-1} \dot{g}_{pc}(r) ds dr
 \end{aligned}$$

$$\begin{aligned}
 &+ (c - c_0)^2 \int_0^1 \int_0^1 \int_0^{r \wedge s} e^{c_0(r+s-2v)} \dot{g}_{pc}(s)' A_p(c)^{-1} B_p(c) \\
 &\quad \times A_p(c)^{-1} \dot{g}_{pc}(r) dv ds dr \\
 &- \int_0^1 \int_0^r e^{c_0(r-s)} \dot{g}_{pc}(s)' A_p(c)^{-1} \dot{g}_{pc}(r) ds dr + \lambda_p(c),
 \end{aligned}$$

where

$$\lambda_p(c) = \int_0^1 \int_0^r e^{c(r-s)} \dot{g}_{pc}(s)' A_p(c)^{-1} \dot{g}_{pc}(r) ds dr.$$

The following lemma derives the convergence rate of  $\hat{\sigma}^2$  defined in (14).<sup>10</sup>

LEMMA 2: *Suppose that Assumptions 1–3 hold. Define*

$$\tilde{\sigma}^2 = \frac{1}{nT} \sum_{i=1}^n \sum_{t=1}^T \varepsilon_{it}^2.$$

*Then, as  $(n, T \rightarrow \infty)$ ,  $\hat{\sigma}^2 = \tilde{\sigma}^2 + O_p(1/T)$  and  $\tilde{\sigma}^2 = \sigma^2 + O_p(1/\sqrt{nT})$ .*

The next lemma shows that the sample moment condition  $M_{nT}(c)$  has a uniform limit in  $c$ .

LEMMA 3 (Uniform convergence): *Under Assumptions 1–3,*

$$M_{nT}(c) \rightarrow_p \sigma^2 M(c) \quad \text{uniformly in } c, \text{ as } (n, T \rightarrow \infty).$$

ASSUMPTION 6: *As  $(n, T \rightarrow \infty)$ ,  $\hat{W} \rightarrow_p W$ , where  $W$  is positive definite.*

Observe that the limit function  $M(c)$  is continuous on the compact parameter set  $\mathbb{C}$ . Also, note that  $M(c) = 0$  at the true parameter  $c = c_0$ . In Appendix D, we confirm numerically that  $M(c) = 0$  only when  $c = c_0$ . Then, by a standard result (e.g., Theorem 2.1 of Newey and McFadden (1994)), the GMM estimator  $\hat{c}$  is consistent for the true parameter  $c_0$ . Summarizing, we have the following theorem.

<sup>10</sup>In this paper, we assume that  $\varepsilon_{it}$  are iid (Assumption 3). If the error process  $\varepsilon_{it}$  has both serial correlation and heteroskedasticity, we may need to estimate the one-side long-run variance ( $\Lambda_i$ ) and the long-run variance ( $\Omega_i$ ) of  $\varepsilon_{it}$ . Letting  $\hat{\Lambda}_i$  and  $\hat{\Omega}_i$  denote the estimates of  $\Lambda_i$  and  $\Omega_i$ , respectively, we need  $(1/\sqrt{n}) \sum_{i=1}^n (\hat{\Lambda}_i - \Lambda_i) = o_p(1)$  and  $(1/\sqrt{n}) \sum_{i=1}^n (\hat{\Omega}_i - \Omega_i) = o_p(1)$ . Typically, we use nonparametric kernel estimation for  $\hat{\Lambda}_i$  and  $\hat{\Omega}_i$ . In this case, in order to have the desired property, we may need to choose a proper bandwidth parameter and a kernel as well as strengthen the restriction on the relative convergence rates between  $n$  and  $T$  to be  $(n/T) \rightarrow 0$ . For details, the reader is referred to Moon and Perron (2003).

THEOREM 1 (Consistency): *Suppose Assumptions 1–3 and 6 hold. Then, as  $(n, T \rightarrow \infty)$ ,  $\hat{c} \rightarrow_p c_0$ .*

#### 4.2. Limiting Distribution of the GMM Estimator when $c_0 < 0$

By inspection the objective function  $Z_{nT}(c)$  is differentiable in  $c$  on the region  $c \in (\bar{c}, 0)$ , and it has right and left derivatives at  $c = \bar{c}$  and 0, respectively. To derive the limit distribution of the GMM estimator, we employ an approach that approximates the objective function  $Z_{nT}(c)$  uniformly in terms of a quadratic function in a shrinking neighborhood of the true parameter.

For this purpose, we define

$$dM_{nT}(c) = \frac{1}{n} \sum_{i=1}^n dm_{iT}(c),$$

where  $dm_{iT}(c)$  denotes the derivative of  $m_{iT}(c)$  with respect to  $c$  when  $c \in (\bar{c}, 0)$  and the right and left derivatives when  $c = \bar{c}$  and 0, respectively. By the mean value theorem, for  $c \neq c_0$ ,

$$M_{nT}(c) = M_{nT}(c_0) + dM_{nT}(c_0)(c - c_0) + r_{nT}(c, c_0)(c - c_0),$$

where

$$r_{nT}(c, c_0) = (r_{1nT}(c, c_0), r_{2nT}(c, c_0))',$$

$$r_{knT}(c, c_0) = \frac{1}{n} \sum_{i=1}^n (dm_{kiT}(c_k^+) - dm_{kiT}(c_0)),$$

and  $c_k^+$  lies between  $c$  and  $c_0$  for  $k = 1, 2$ .

Define

$$\mathcal{S}_{nT} = dM_{nT}(c_0)' \hat{W} M_{nT}(c_0) \quad \text{and} \quad \mathcal{H}_{nT} = dM_{nT}(c_0)' \hat{W} dM_{nT}(c_0).$$

Then, we can write

$$\begin{aligned} Z_{nT}(c) &= M_{nT}(c_0)' \hat{W} M_{nT}(c_0) + 2(c - c_0) \mathcal{S}_{nT} + (c - c_0)^2 \mathcal{H}_{nT} \\ &\quad + (c - c_0) \mathcal{R}_{1nT}(c, c_0) + (c - c_0)^2 \mathcal{R}_{2nT}(c, c_0), \end{aligned}$$

where

$$\mathcal{R}_{1nT}(c, c_0) = 2M_{nT}(c_0)' \hat{W} r_{nT}(c, c_0),$$

and

$$\mathcal{R}_{2nT}(c, c_0) = 2dM_{nT}(c_0)' \hat{W} r_{nT}(c, c_0) + r_{nT}(c, c_0)' \hat{W} r_{nT}(c, c_0).$$

Next, we give some asymptotic results that are useful in establishing the limit distribution of  $\hat{c}$ .

LEMMA 4: *Suppose that Assumptions 1–3 hold. When the true parameter is  $c_0$ ,*

$$\begin{aligned} dM_{nT}(c) &\rightarrow_p \sigma^2 dM(c) \\ &= \sigma^2(dM_1(c), dM_2(c))' \quad \text{uniformly in } c \text{ as } (n, T \rightarrow \infty) \end{aligned}$$

for some continuous function  $dM(c)$  with

$$dM_1(c_0) = \int_0^1 \int_0^r e^{c_0(r-s)}(r-s)\tilde{h}_p(r,s) ds dr - \psi_p(c_0),$$

and

$$\begin{aligned} dM_2(c_0) &= - \int_0^1 \int_0^r e^{2c_0(r-s)} ds dr \\ &\quad + \int_0^1 \int_0^1 \int_0^{r \wedge s} e^{c_0(r+s-2v)} \dot{g}_{pc_0}(s)' A_p(c_0)^{-1} \dot{g}_{pc_0}(r) dv ds dr \\ &\quad + \int_0^1 \int_0^r e^{c_0(r-s)} \dot{g}_{pc_0}(r)' A_p(c_0)^{-1} g_p(s) ds dr \\ &\quad + \int_0^1 \int_0^r e^{c_0(r-s)} \dot{g}_{pc_0}(s)' A_p(c_0)^{-1} g_p(r) ds dr \\ &\quad - \int_0^1 \int_0^r e^{c_0(r-s)} \dot{g}_{pc_0}(r)' A_p(c_0)^{-1} B_p(c_0) \\ &\quad \quad \quad \times A_p(c_0)^{-1} \dot{g}_{pc_0}(s) ds dr \\ &\quad - \int_0^1 \int_0^r e^{c_0(r-s)} \dot{g}_{pc_0}(s)' A_p(c_0)^{-1} B_p(c_0)' \\ &\quad \quad \quad \times A_p(c_0)^{-1} \dot{g}_{pc_0}(r) ds dr \\ &\quad + \int_0^1 \int_0^r (r-s)e^{c_0(r-s)} \dot{g}_{pc_0}(r)' A_p(c_0)^{-1} \dot{g}_{pc_0}(s) ds dr. \end{aligned}$$

LEMMA 5: *Suppose that Assumptions 1–3 hold and set  $B_{nT} = \sqrt{n}$ . Then, as  $(n, T \rightarrow \infty)$  following Assumption 5,*

$$B_{nT}M_{nT}(c_0) = \frac{1}{\sqrt{n}} \sum_{i=1}^n m_{iT}(c_0) \Rightarrow N(0, \sigma^4 J' \Phi(c_0) J),$$

where

$$J = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & -1 & 1 \end{pmatrix}'$$

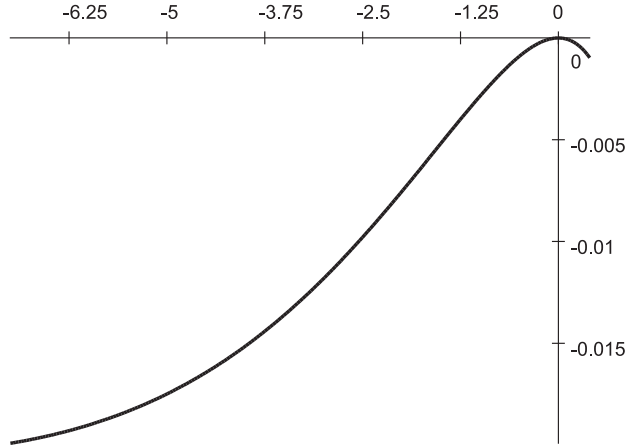


FIGURE 1.—When  $\tilde{g}_{1t} = (1, t)'$ .

and  $\Phi(c_0)$  is defined in (37).

REMARKS: (a) The proof of Lemma 5 is similar to that of Lemma 3 and is omitted.

(b) Figures 1 and 2 plot the graphs of  $dM_1(c_0)$  in the cases  $\tilde{g}_{1t} = (1, t)'$  and  $\tilde{g}_{2t} = (1, t, t^2)'$ , respectively. The graphs reveal that  $dM_1(c_0) < 0$  for  $c_0 < 0$ , and, therefore,  $\mathcal{H}_{nT} > 0$  for  $c_0 < 0$ .

(c) According to Moon and Phillips (2000), when  $c_0 = 0$ ,  $dM_1(c_0) = 0$  holds for all polynomial trends  $\tilde{g}_{pt} = (1, \dots, t^p)'$ . Also, for  $c_0 = 0$ , direct calculations show that  $dM_2(c_0) = 0$  for  $g_{1t} = t$  and  $dM_2(c_0) = 0$  for  $g_{2t} = (t, t^2)'$ . Therefore,  $\mathcal{H}_{nT} \rightarrow_p 0$  when  $c_0 = 0$ ,  $g_{1t} = t$ , and  $g_{2t} = (t, t^2)'$ .

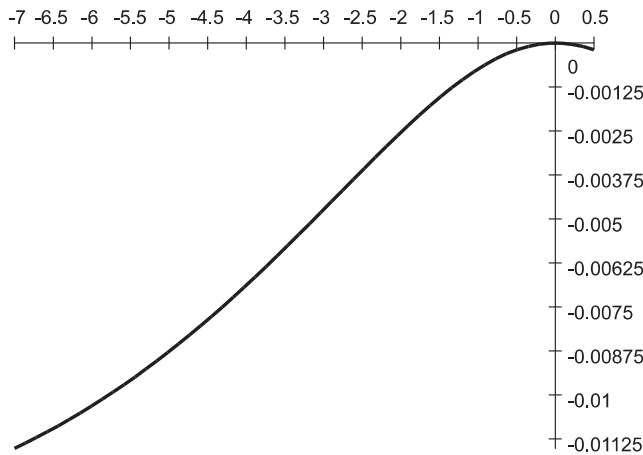


FIGURE 2.—When  $\tilde{g}_{2t} = (1, t, t^2)'$ .



From Lemma 4 and the following remarks and by Assumption 6, it follows that  $\mathcal{H}_{nT}$  has a positive limit as  $(n, T \rightarrow \infty)$  when  $c_0 < 0$ . Thus,  $\mathcal{H}_{nT}^{-1} = O_p(1)$ . Then, we can write

$$\begin{aligned}
 (21) \quad B_{nT}^2 Z_{nT}(c) &= M_{nT}(c_0)' \hat{W} M_{nT}(c_0) - \frac{(B_{nT} \mathcal{S}_{nT})^2}{\mathcal{H}_{nT}} \\
 &\quad + \mathcal{H}_{nT} \left( B_{nT}(c - c_0) - \frac{B_{nT} \mathcal{S}_{nT}}{\mathcal{H}_{nT}} \right)^2 \\
 &\quad + B_{nT}(c - c_0) B_{nT} \mathcal{R}_{1nT}(c, c_0) \\
 &\quad + (B_{nT}(c - c_0))^2 \mathcal{R}_{2nT}(c, c_0).
 \end{aligned}$$

LEMMA 6: *Under Assumptions 1–3 and 6, for every sequence  $\gamma_{nT} \rightarrow 0$ , we have as  $(n, T \rightarrow \infty)$  following Assumption 5:*

- (a)  $\sup_{c \in \mathbb{C}: |c - c_0| \leq \gamma_{nT}} |B_{nT} \mathcal{R}_{1nT}(c, c_0)| = o_p(1)$ , and
- (b)  $\sup_{c \in \mathbb{C}: |c - c_0| \leq \gamma_{nT}} |\mathcal{R}_{2nT}(c, c_0)| = o_p(1)$ .

THEOREM 2: *Suppose that Assumptions 1–3 and 6 hold. Then, as  $(n, T \rightarrow \infty)$  following Assumption 5,  $B_{nT}(\hat{c} - c_0) = O_p(1)$ .*

Lemma 6 establishes that the two remainder terms  $B_{nT} \mathcal{R}_{1nT}(c, c_0)$  and  $\mathcal{R}_{2nT}(c, c_0)$  converge in probability to zero uniformly in the shrinking neighborhood of the true parameter. Also, Theorem 2 shows that the GMM estimator is  $B_{nT} (= \sqrt{n})$ -consistent. This implies that in the shrinking neighborhood of the true parameter, the scaled objective function  $B_{nT}^2 Z_{nT}(c)$  is uniformly approximated by the following quadratic function:

$$\begin{aligned}
 B_{nT}^2 Z_{q,nT}(c) &= M_{nT}(c_0)' \hat{W} M_{nT}(c_0) - \frac{(B_{nT} \mathcal{S}_{nT})^2}{\mathcal{H}_{nT}} \\
 &\quad + \mathcal{H}_{nT} \left( B_{nT}(c - c_0) - \frac{B_{nT} \mathcal{S}_{nT}}{\mathcal{H}_{nT}} \right)^2.
 \end{aligned}$$

The heuristic ideas of the limit theory are as follows. Let  $B_{nT}(\hat{c}_q - c_0) = \arg \min_{c \in \mathbb{C}} B_{nT}^2 Z_{q,nT}(c)$ . Then, we may expect that a minimizer of  $B_{nT}^2 Z_{nT}(c)$  will be close to the minimizer of  $B_{nT}^2 Z_{q,nT}(c)$ , suggesting that the GMM estimator  $B_{nT}(\hat{c} - c_0)$  will be close to

$$\begin{aligned}
 B_{nT}(\hat{c}_q - c_0) &= \frac{B_{nT} \mathcal{S}_{nT}}{\mathcal{H}_{nT}} && \text{if } \left\{ B_{nT}(\bar{c} - c_0) \leq \frac{B_{nT} \mathcal{S}_{nT}}{\mathcal{H}_{nT}} \leq -B_{nT} c_0 \right\}, \\
 &= B_{nT}(\bar{c} - c_0) && \text{if } \left\{ B_{nT}(\bar{c} - c_0) > \frac{B_{nT} \mathcal{S}_{nT}}{\mathcal{H}_{nT}} \right\}, \\
 &= -B_{nT} c_0 && \text{if } \left\{ \frac{B_{nT} \mathcal{S}_{nT}}{\mathcal{H}_{nT}} > -B_{nT} c_0 \right\}.
 \end{aligned}$$

Notice that  $B_{nT}S_{nT}/\mathcal{H}_{nT} = O_p(1)$  and recall that it is assumed that the true parameter satisfies  $\bar{c} < c_0 < 0$ . In this case, the probabilities of the events  $\{B_{nT}(\bar{c} - c_0) > (B_{nT}S_{nT}/\mathcal{H}_{nT})\}$  and  $\{(B_{nT}S_{nT}/\mathcal{H}_{nT}) > -B_{nT}c_0\}$  will be very small and the scaled and centered estimator  $B_{nT}(\hat{c}_q - c_0)$  will therefore be close with high probability to the random variable

$$\hat{\phi}_{nT} = \mathcal{H}_{nT}^{-1}(B_{nT}S_{nT}).$$

In view of Lemmas 4 and 5 and Assumption 6,

$$B_{nT}S_{nT} \Rightarrow \mathcal{S} \stackrel{d}{=} N(0, \sigma^8[dM(c_0)'WJ'\Phi(c_0)JWdM(c_0)])$$

and

$$\mathcal{H}_{nT} \rightarrow_p \mathcal{H} = \sigma^4 dM(c_0)'WdM(c_0) > 0$$

as  $(n, T \rightarrow \infty)$  with  $n/T \rightarrow 0$ . Thus, when  $c_0 \in \mathbb{C}_0 \setminus \{0\}$ ,

$$\hat{\phi}_{nT} \Rightarrow \phi \stackrel{d}{=} \mathcal{H}^{-1}\mathcal{S} \stackrel{\text{let}}{=} \mathcal{Z}.$$

The proof of the following theorem confirms the heuristic argument above.

**THEOREM 3:** *Suppose that Assumptions 1–3 and 6 hold. Suppose that  $c_0 \in \mathbb{C}_0 \setminus \{0\}$  and  $\hat{c}$  is the GMM estimator defined in (20). Then, as  $(n, T \rightarrow \infty)$  following Assumption 5,  $\sqrt{n}(\hat{c} - c_0) \Rightarrow \mathcal{Z}$ , where*

$$\mathcal{Z} \stackrel{d}{=} N\left(0, \frac{dM(c_0)'WJ'\Phi(c_0)JWdM(c_0)}{[dM(c_0)'WdM(c_0)]^2}\right).$$

**REMARKS:** (a) When  $c_0 \in \mathbb{C}_0 \setminus \{0\}$  and  $J'\Phi(c_0)J$  is invertible, the optimal weight matrix is found as  $\hat{W}_{opt} = (J'\Phi(\hat{c})J)^{-1}$ . The limiting distribution of  $\sqrt{n}(\hat{c} - c_0)$  is then

$$(22) \quad \sqrt{n}(\hat{c} - c_0) \Rightarrow \mathcal{Z}_{opt} \stackrel{d}{=} N\left(0, \frac{\sigma^4}{[dM(c_0)'WdM(c_0)]^2}\right).$$

(b) Figures 3 and 4 plot the graphs of the minimum eigenvalue of  $J'\Phi(c_0)J$  as a function of  $c_0$ . As we see through the graph,  $J'\Phi(c_0)J$  is positive definite except for the case of  $c_0 = 0$ .

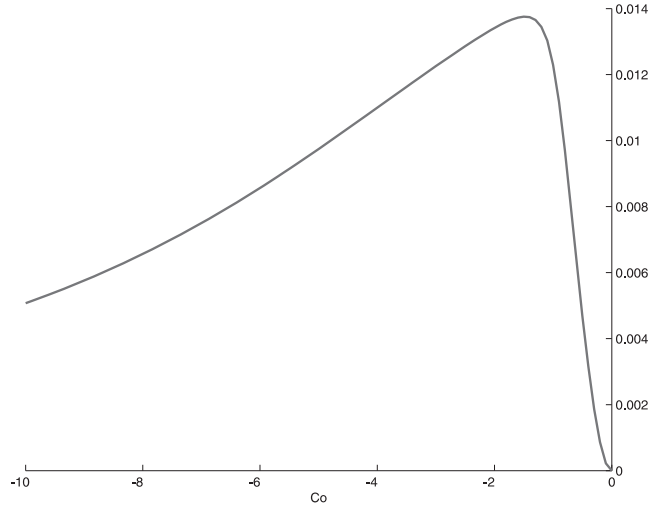


FIGURE 3.—Graph of the minimum eigenvalue of  $J'\Phi(c_0)J$  when  $g_{1t} = t$ .

4.3. *Limiting Distribution of the GMM Estimator when  $c_0 = 0$*

An important special case of model (1) occurs when  $c_0 = 0$ . The time series components of  $y_{it}$  in (1) then have a unit root, and the deterministic trend is linear, so

$$(23) \quad \begin{aligned} z_{it} &= \beta_{i0}t + y_{it}, \\ y_{it} &= \rho_0 y_{it-1} + \varepsilon_{it}, \end{aligned}$$

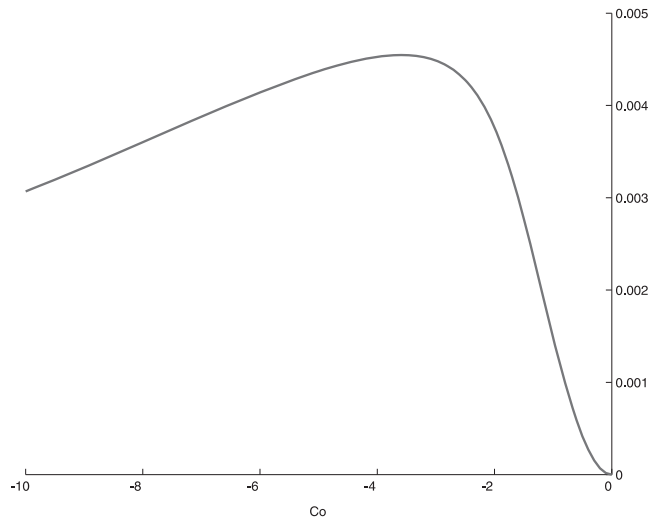


FIGURE 4.—Graph of the minimum eigenvalue of  $J'\Phi(c_0)J$  when  $g_{2t} = (t, t^2)'$ .

where  $\rho_0 = 1$ , i.e.,  $c_0 = 0$ . According to Remark (c) below Lemma 5, in this case, the information from the moment conditions is zero because  $\mathcal{H}_{nT} \rightarrow_p 0$ . We cannot then use a conventional quadratic approximation approach, as in the previous section, and need instead to employ a higher-order approximation to extract the limit theory (cf., Sargan (1983)).

This section develops asymptotics for the GMM estimator when the true localizing parameter is zero, so throughout this section we set  $c_0 = 0$ . The following lemmas find the limits of the first and the second moment conditions and their higher-order derivatives at  $c = 0$ .

LEMMA 7: *Suppose that the panel data is generated by (23). Under Assumptions 2 and 3, the following hold as  $(n, T \rightarrow \infty)$  following Assumption 5:*

- (a)  $\sqrt{n}M_{1nT}(0) \Rightarrow N(0, \sigma^4/60) \equiv \sqrt{\sigma^4/60}\mathcal{Z}$ , where  $\mathcal{Z} \equiv N(0, 1)$ ,
- (b)  $\sqrt{nd}M_{1nT}(0) \Rightarrow N(0, \sigma^4(11/6300))$ ,
- (c)  $\sqrt{nd^2}M_{1nT}(0) = o_p(1)$ ,
- (d)  $d^3M_{1nT}(c) \rightarrow_p \sigma^2 d^3M_1(c)$  uniformly in  $c$  with  $d^3M_1(0) = -1/70$ , where  $d^k M_{1nT}(c)$  is the  $k$ th left derivative of  $M_{1nT}(c)$ , and  $d^3M_1(c)$  is the third left derivative of  $M_1(c)$ , the probability limit of  $M_{1nT}(c)$ .

LEMMA 8: *Suppose that the assumptions in Lemma 7 hold. Then, when  $(n, T \rightarrow \infty)$  following Assumption 5:*

- (a)  $\sqrt{n}M_{2nT}(0) = o_p(1)$ ,
- (b)  $\sqrt{nd}M_{2nT}(0) \Rightarrow N(0, \sigma^4/45)$ ,
- (c)  $\sqrt{nd^2}M_{2nT}(0) = o_p(1)$ ,
- (d)  $d^3M_{2nT}(c) \rightarrow_p \sigma^2 d^3M_2(c)$  uniformly in  $c$  with  $d^3M_2(0) = -1/15$ , where  $d^k M_{2nT}(0)$  is the  $k$ th left derivative of  $M_{2nT}(c)$  at  $c = 0$ , and  $d^3M_2(0)$  is the third left derivative of  $d^3M_2(c)$  at  $c = 0$ .

REMARK: Since the higher-order derivatives of  $M_{2nT}(0)$  are complicated and involve extremely lengthy expressions, we omit the details of their derivation and instead give a sketch of the proof in the Appendix.<sup>11</sup>

Using the left derivatives of the moment condition  $M_{nT}(c)$  at  $c = 0$ , we approximate  $M_{nT}(c)$  around the true parameter  $c_0 = 0$  with a third order polynomial as follows:

$$M_{nT}(c) = M_{nT}(0) + c(dM_{nT}(0)) + \frac{1}{2}c^2(d^2M_{nT}(0)) \\ + \frac{1}{6}c^3(d^3M_{nT}(0)) + c^3\tilde{r}_{nT}(c, 0),$$

<sup>11</sup>Detailed derivations can be obtained from the authors upon request.

where

$$\begin{aligned}\tilde{r}_{nT}(c, 0) &= (\tilde{r}_{1nT}(c, 0), \tilde{r}_{2nT}(c, 0))', \\ \tilde{r}_{knT}(c, 0) &= d^3 M_{knT}(c_k^+) - d^3 M_{knT}(0), \quad k = 1 \text{ and } 2.\end{aligned}$$

Then,

$$Z_{nT}(c) = M_{nT}(c)' \hat{W} M_{nT}(c) = \sum_{k=0}^6 c^k \mathcal{A}_{k,nT} + \mathcal{N}_{nT}(c, 0),$$

where

$$\begin{aligned}\mathcal{A}_{0,nT} &= M_{nT}(0)' \hat{W} M_{nT}(0), \\ \mathcal{A}_{1,nT} &= 2M_{nT}(0)' \hat{W} dM_{nT}(0), \\ \mathcal{A}_{2,nT} &= M_{nT}(0)' \hat{W} d^2 M_{nT}(0) + dM_{nT}(0)' \hat{W} dM_{nT}(0), \\ \mathcal{A}_{3,nT} &= \frac{1}{3} M_{nT}(0)' \hat{W} d^3 M_{nT}(0) + dM_{nT}(0)' \hat{W} d^2 M_{nT}(0), \\ \mathcal{A}_{4,nT} &= \frac{1}{3} dM_{nT}(0)' \hat{W} d^3 M_{nT}(0) + \frac{1}{4} d^2 M_{nT}(0)' \hat{W} d^2 M_{nT}(0), \\ \mathcal{A}_{5,nT} &= \frac{1}{6} d^2 M_{nT}(0)' \hat{W} d^3 M_{nT}(0), \\ \mathcal{A}_{6,nT} &= \frac{1}{36} d^3 M_{nT}(0)' \hat{W} d^3 M_{nT}(0),\end{aligned}$$

and

$$\begin{aligned}\mathcal{N}_{nT}(c, 0) &= \sum_{k=3}^6 c^k \mathcal{N}_{k,nT}(c, 0), \\ \mathcal{N}_{k,nT}(c, 0) &= \alpha_k d^{(k-3)} M_{nT}(0)' \hat{W} \tilde{r}_{nT}(c, 0) \quad \text{for } k = 3, 4, 5, \\ \mathcal{N}_{6,nT}(c, 0) &= \alpha_6 d^3 M_{nT}(0)' \hat{W} \tilde{r}_{nT}(c, 0) + \tilde{r}_{nT}(c, 0)' \hat{W} \tilde{r}_{nT}(c, 0), \\ \alpha_3, \alpha_4 &= 2, \quad \alpha_5 = 1, \quad \alpha_6 = \frac{1}{3},\end{aligned}$$

where  $d^0 M_{nT}(0)$  denotes  $M_{nT}(0)$ .

In view of Lemmas 7 and 8, it is easy to find that as  $(n, T \rightarrow \infty)$  following Assumption 5,

$$(24) \quad n^{5/6} \mathcal{A}_{1,nT} = o_p(1),$$

$$(25) \quad n^{2/3} \mathcal{A}_{2,nT} = o_p(1),$$

$$(26) \quad n^{1/3} \mathcal{A}_{4,nT} = o_p(1),$$

$$(27) \quad n^{1/6} \mathcal{A}_{5,nT} = o_p(1),$$

and

$$(28) \quad \mathcal{A}_{6,nT} \rightarrow_p \frac{\sigma^4}{36} \left( \frac{W_{11}}{4900} + \frac{2W_{12}}{1050} + \frac{W_{22}}{225} \right) > 0,$$

$$(29) \quad n^{1/2} \mathcal{A}_{3,nT} \Rightarrow \mathcal{A}_3 \mathcal{Z},$$

$$(30) \quad n \mathcal{A}_{0,nT} \Rightarrow \mathcal{A}_0 \mathcal{Z}^2,$$

where  $\mathcal{Z} \equiv N(0, 1)$  and

$$\mathcal{A}_3 = -\frac{\sigma^2}{3} \left( \frac{W_{11}}{70} + \frac{W_{12}}{15} \right) \sqrt{\frac{\sigma^4}{60}} \quad \text{and} \quad \mathcal{A}_0 = W_{11} \frac{\sigma^4}{60}.$$

Also, using Lemmas 7 and 8 and following similar lines of proof to Lemma 6, we can show that

$$(31) \quad \sup_{c \in \mathbb{C}: |c| \leq \gamma_{nT}} |n^{(6-k)/6} \mathcal{N}_{k,nT}(c, 0)| = o_p(1) \quad \text{for} \quad k = 3, \dots, 6,$$

for any sequence  $\gamma_{nT}$  tending to zero as  $(n, T \rightarrow \infty)$ . Then, we have the following limit theory for  $\hat{c}$  at the origin.

**THEOREM 4:** *Under the assumptions in Lemmas 7 or 8, and as  $(n, T \rightarrow \infty)$  following Assumption 5,  $n^{1/6}(\hat{c} - c_0) = O_p(1)$ , where  $c_0 = 0$ .*

So, when the true localizing parameter is  $c_0 = 0$ , the GMM estimator  $\hat{c}$  is  $n^{1/6}$ -consistent, which is slower than the regular case of  $\sqrt{n}$  that applies for  $c_0 < 0$  as shown in Section 4. To find the limiting distribution of  $\hat{c}$ , we use an argument similar to that of the previous section. Consequently, we sketch the derivation and give the final result in Theorem 5.

In view of (24)–(30) and (31), the standardized objective function  $nZ_{nT}(c)$  is approximated by

$$Z_{q,nT}(c) = n \mathcal{A}_{0,nT} + (n^{1/6}c)^3 \sqrt{n} \mathcal{A}_{3,nT} + (n^{1/6}c)^6 \mathcal{A}_{6,nT}.$$

Notice that the probability limit of  $\mathcal{A}_{6,nT}$  is positive, as shown in (28). Then, it is easy to see that the approximate objective function  $Z_{q,nT}(c)$  is minimized at

$$\begin{aligned} n^{1/6} \hat{c}_q &= - \left( \frac{\sqrt{n} \mathcal{A}_{3,nT}}{2 \mathcal{A}_{6,nT}} \right)^{1/3} & \text{if} & \quad \left\{ n^{1/6} \bar{c} \leq - \frac{\sqrt{n} \mathcal{A}_{3,nT}}{2 \mathcal{A}_{6,nT}} \leq 0 \right\}, \\ &= 0 & \text{if} & \quad \left\{ - \frac{\sqrt{n} \mathcal{A}_{3,nT}}{2 \mathcal{A}_{6,nT}} > 0 \right\}, \\ &= -(n^{1/6}(-\bar{c}))^{1/3} & \text{if} & \quad \left\{ n^{1/6} \bar{c} > - \frac{\sqrt{n} \mathcal{A}_{3,nT}}{2 \mathcal{A}_{6,nT}} \right\}. \end{aligned}$$

Using arguments similar to those in the proof of Theorem 3, we can prove that the standardized GMM estimator  $n^{1/6}\hat{c}$  is approximated by  $n^{1/6}\hat{c}_q$ , the minimizer of  $Z_{q,nT}(c)$ , that is,

$$n^{1/6}\hat{c} = n^{1/6}\hat{c}_q + o_p(1),$$

and the estimator  $n^{1/6}\hat{c}_q$  is approximated by

$$\hat{\phi}_{nT} = -\left(\frac{\sqrt{n}\mathcal{A}_{3,nT}}{2\mathcal{A}_{6,nT}}\right)^{1/3} \mathbf{1}\left\{-\frac{\sqrt{n}\mathcal{A}_{3,nT}}{2\mathcal{A}_{6,nT}} \leq 0\right\},$$

where  $\mathbf{1}\{A\}$  is the indicator of  $A$ . In view of (28) and (29), as  $(n, T \rightarrow \infty)$  following Assumption 5, it follows by the continuous mapping theorem that

$$\hat{\phi}_{nT} \Rightarrow \mathcal{Z}_0^{1/3} \mathbf{1}\{\mathcal{Z}_0 \leq 0\},$$

where

$$(32) \quad \mathcal{Z}_0 = V_0 \mathcal{Z},$$

$$(33) \quad V_0 = \left| \frac{\sqrt{\frac{1}{15}\left(\frac{W_{11}}{70} + \frac{W_{12}}{15}\right)}}{\frac{1}{3}\left(\frac{W_{11}}{4900} + \frac{2W_{12}}{1050} + \frac{W_{22}}{225}\right)} \right|,$$

and  $W_{ij}$  are the  $(i, j)$ th element of the weight matrix  $W$ . Thus, we have the following theorem.

**THEOREM 5:** *Under the assumptions in Lemmas 7 and 8, as  $(n, T \rightarrow \infty)$  following Assumption 5,  $n^{1/6}\hat{c} \Rightarrow \mathcal{Z}_0^{1/3} \mathbf{1}\{\mathcal{Z}_0 \leq 0\}$ , where  $\mathcal{Z}_0$  is defined in (32).*

**REMARKS 1:** (a) Theorem 4 shows that when the true parameter  $c_0 = 0$ , i.e., in the case of a panel unit root, the GMM estimator is  $n^{1/6}$ -consistent and that its limit distribution is nonstandard, involving the cube root of a truncated normal. The truncation in the limiting distribution arises because the true parameter is on the boundary of the parameter set.

(b) The reason for the slower convergence rate in the panel unit root case is that first order information in the moment condition (from the first derivative of the moment condition) is asymptotically zero at the true parameter. In order to obtain nonnegligible information from the moment condition, we need to pass to third order derivatives of the moment condition. Taking the higher-order approximation slows down the convergence rate because the rate at which information in the moment condition is passed to the estimator is slowed down at the origin because of the zero lower derivatives.

(c) In view of Lemmas 7(a) and 8(a), we find that  $\sqrt{n}M_{2nT}(0) = o_p(1)$ , while  $\sqrt{n}M_{1nT}(0)$  converges in distribution to a normal random variable with positive variance. Because of the convergence rate difference between  $\sqrt{n}M_{2nT}(0)$

and  $\sqrt{n}M_{1nT}(0)$ , we have only  $W_{11}$  and  $W_{12}$  but not  $W_{22}$  in the limiting scale  $V_0$  of (33). In this case, setting  $W_{11} = W_{12} = 0$ , i.e., not considering the first moment condition, causes the variance of the limit variate  $Z_0$  in (32) to vanish, from which one might expect that the GMM estimator from the second moment condition alone would have a faster convergence rate than  $n^{1/6}$ . The reason for using the first moment condition is to identify the true parameter  $c_0$  when  $c_0 < 0$ . As we discuss in Appendix D, the second moment condition cannot identify the true parameter  $c_0$  unless  $c_0 = 0$ .

#### 4.4. On Testing for a Unit Root

This section briefly considers how the asymptotic results for the localizing coefficient given in the previous section may be used to test for a unit root in the panel. Suppose the null hypothesis is  $\mathbb{H}_0 : c_0 = 0$  and the alternative hypothesis is  $\mathbb{H}_1 : c_0 < 0$ . We discuss two types of panel unit root tests, one involving a  $t$ -test and the other an LM test.

First, Theorem 5 shows that to test  $\mathbb{H}_0$  we can use a suitably constructed  $t$ -statistic. Specifically, let  $\hat{V}_0$  be a consistent estimator of  $V_0$  and define

$$t_{gmm} = \frac{\sqrt{n}\hat{c}^3}{\hat{V}_0}.$$

Then, since  $\hat{V}_0 \rightarrow_p V_0$  and from Theorem 5 with  $(n, T \rightarrow \infty)$  as in Assumption 5, we get

$$t_{gmm} \Rightarrow Z\mathbf{1}\{Z \leq 0\},$$

where  $Z \equiv N(0, 1)$ . Under the alternative hypothesis  $c = c_A < 0$ , we have

$$t_{gmm} = \frac{\sqrt{n}(\hat{c}^3 - c_A^3)}{\hat{V}_0} + \frac{\sqrt{n}c_A^3}{\hat{V}_0} = O_p(1) + \frac{\sqrt{n}c_A^3}{\hat{V}_0}$$

by Theorem 3 and the delta method. So, under the alternative hypothesis,  $t_{gmm} \rightarrow -\infty$  and the test is consistent.

Another type of test is to use the asymptotic properties of the moment conditions in Lemmas 7 and 8 in conjunction with the restricted parameter estimator, which is zero in this case. For example, in Lemma 7 we observe that  $\sqrt{n}dM_{1nT}(0) = N(0, \sigma^4(11/6300))$ . Thus, a simple test can be based on

$$LM_{gmm} = \left( \sqrt{\frac{6300}{11}} \frac{\sqrt{n}dM_{1nT}(0)}{\hat{\sigma}^2} \right)^2.$$

Then, as  $(n, T \rightarrow \infty)$  as in Assumption 5, we have

$$LM_{gmm} \Rightarrow \chi^2(1).$$



Under the alternative  $c = c_A < 0$ , it is easy to show that  $(\sqrt{nd}M_{2nT}(0))^2 = nO(1)^2 \rightarrow \infty$ , while  $d^3M_{2nT}(0) \rightarrow_p -\sigma^2(1/15) < 0$ . Thus, under the alternative hypothesis,  $LM_{gmm} \rightarrow \infty$ . The same principle can be applied to the second moment condition  $M_{2nT}(0)$ .

## 5. CONCLUSION

Part of the richness of panel data is that it can provide information about features of a model on which time series and cross section data are uninformative when they are used on their own. In the context of nonstationary panels with near unit roots, an interesting new example of this “added information” feature of panel data is that consistent estimation of the common local to unity coefficient becomes possible. This means that panel data help to sharpen our capacity to learn from data about the precise form of nonstationarity where time series data alone are insufficient to do so. However, as the authors have shown in earlier work, the presence of individual deterministic trends in a panel model introduces a serious complication in this nice result on the consistent estimation of a root local to unity. The complication is that individual trends produce an incidental parameter problem as  $n \rightarrow \infty$  that does not disappear as  $T \rightarrow \infty$ . The outcome is that common procedures like pooled least squares and maximum likelihood are inconsistent. Thus, the presence of deterministic trends continues to confabulate inference about stochastic trends even in the panel data case.

One option is to adjust procedures like maximum likelihood to deal with the bias. The present paper shows how to make these adjustments. The theory is cast in the context of moment formulae that lead naturally to GMM based estimation. The paper has two important findings.

The first is that bias correction in the moment formulae arising from GLS estimation of the trend coefficients corresponds to taking the projected score (under Gaussian assumptions) on the Bhattacharyya basis. This correspondence relates the approach we take here to recent work on projected score methods by Waterman and Lindsay (1998) that deals with models that have infinite numbers of nuisance parameters like the original incidental parameters problem.

The second is that our limit theory validates GMM-based inference about the localizing coefficient in near unit root panels. A notable new result is that the GMM estimator has a convergence rate slower than  $\sqrt{n}$  when the true localizing parameter is zero (i.e., when there is a panel unit root) and the deterministic trends in the panel are linear. The asymptotic theory in this case provides a new example of limit theory on the boundary of a parameter space. The results point to the continued difficulty of distinguishing unit roots from local alternatives when there are deterministic trends in the data even when time series data is coupled with an infinity of additional data from a cross section.

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#### APPENDIX A: USEFUL RESULTS FOR JOINT ASYMPTOTICS

This section consists of two subsections. The first subsection introduces some useful results for joint asymptotic theories. Many of these are modified versions of results developed in Phillips and Moon (1999) so we report them only briefly here. The second subsection introduces some useful results that will be used repeatedly in the following sections of the proofs for the results in the main text.

##### Appendix A1

The following two theorems provide convenient conditions to find the joint probability limit of double indexed processes.

**THEOREM 6 (Joint probability limits):** *Suppose the  $(m \times 1)$  random vectors  $Y_{iT}$  are independent across  $i = 1, \dots, n$  for all  $T$  and integrable. Assume that  $Y_{iT} \Rightarrow Y_i$  as  $T \rightarrow \infty$  for all  $i$ . Let  $X_{nT} = 1/n \sum_{i=1}^n Y_{iT}$  and  $X_n = 1/n \sum_{i=1}^n Y_i$ .*

(a) *Let the following hold:*

- (i)  $\limsup_{n,T} 1/n \sum_{i=1}^n E \|Y_{iT}\| < \infty$ ,
- (ii)  $\limsup_{n,T} 1/n \sum_{i=1}^n \|E Y_{iT} - E Y_i\| = 0$ ,
- (iii)  $\limsup_{n,T} 1/n \sum_{i=1}^n E \|Y_{iT}\| \mathbf{1}\{\|Y_{iT}\| > n\varepsilon\} = 0 \forall \varepsilon > 0$ , and
- (iv)  $\limsup_n 1/n \sum_{i=1}^n E \|Y_i\| \mathbf{1}\{\|Y_i\| > n\varepsilon\} = 0 \forall \varepsilon > 0$ .

(b) *If  $\lim_{n \rightarrow \infty} 1/n \sum_{i=1}^n E Y_i (=:\tilde{\mu}_X)$  exists and  $X_n \rightarrow_p \tilde{\mu}_X$  as  $n \rightarrow \infty$ , then  $X_{nT} \rightarrow_p \tilde{\mu}_X$  as  $(n, T \rightarrow \infty)$ .*

**THEOREM 7:** *Suppose that  $Y_{iT} = C_i Q_{iT}$ , where the  $(m \times 1)$  random vectors  $Q_{iT}$  are iid across  $i = 1, \dots, n$  for all  $T$ , and the  $C_i$  are  $(m \times m)$  nonrandom matrices for all  $i$ . Assume that:*

- (i)  $Q_{iT} \Rightarrow Q_i$  as  $T \rightarrow \infty$  for all  $i$ ,
- (ii)  $\|Q_{iT}\|$  is uniformly integrable in  $T$  for all  $i$ .
- (iii)  $\sup_i \|C_i\| < \infty$ ,  $\inf_i \|C_i\| > 0$ , and  $C = \lim_n 1/n \sum_{i=1}^n C_i$ .

*Then  $1/n \sum_{i=1}^n Y_{iT} \rightarrow_p C E(Q_i)$  as  $(n, T \rightarrow \infty)$ .*

**THEOREM 8 (Joint limit CLT for scaled variates):** *Suppose that  $Y_{iT} = C_i Q_{iT}$ , where the  $(m \times 1)$  random vectors  $Q_{iT}$  are iid  $(0, \Sigma_T)$  across  $i = 1, \dots, n$  for all  $T$  and the  $C_i$  are  $(m \times m)$  nonzero and nonrandom matrices. Assume the following conditions hold:*

- (i) let  $\Sigma_T^2 = \lambda_{\min}(\Sigma_T)$  and  $\liminf_T \sigma_T^2 > 0$ ,
- (ii)  $\max_{i \leq n} \|C_i\|^2 / \lambda_{\min}(\sum_{i=1}^n C_i C_i') = O(\frac{1}{n})$  as  $n \rightarrow \infty$ ,
- (iii)  $\|Q_{iT}\|^2$  are uniformly integrable in  $T$ ,
- (iv)  $\lim_{n,T} \frac{1}{n} \sum_{i=1}^n C_i \sum_T C_i' = \Omega > 0$ .

*Then,*

$$X_{nT} = \frac{1}{\sqrt{n}} \sum_{i=1}^n Y_{iT} \Rightarrow N(0, \Omega) \quad \text{as } n, T \rightarrow \infty.$$

Appendix A2

Recall that

$$\tilde{h}_{pT}(t, s) = \tilde{g}'_{pt} \tilde{D}_{p,T} \left( \frac{1}{T} \sum_{i=1}^T \tilde{D}_{p,T} \tilde{g}_{pi} \tilde{g}'_{pi} \tilde{D}_{p,T} \right)^{-1} \tilde{D}_{p,T} \tilde{g}_{ps}.$$

It is easy to see that when  $t = [Tr]$  and  $s = [Tv]$ , as  $T \rightarrow \infty$

$$\tilde{h}_T(t, s) \rightarrow \tilde{g}'_p(r) \left( \int \tilde{g}_p \tilde{g}'_p \right)^{-1} \tilde{g}_p(v) = \tilde{h}_p(r, v)$$

uniformly in  $(r, v) \in [0, 1] \times [0, 1]$ . The following limit also holds:

$$\sup_{1 \leq t, s \leq T} \tilde{h}_{pT}(t, s) \rightarrow \sup_{0 \leq r, v \leq 1} \tilde{h}_p(r, v).$$

Next, define

$$(34) \quad x_{it} = \sum_{s=1}^t \left( 1 + \frac{c_0}{T} \right)^{(t-s)} \varepsilon_{is}$$

and  $x_{i0} = 0$ . Then, we can write  $y_{it}$  as

$$(35) \quad y_{it} = x_{it} + \left( 1 + \frac{c_0}{T} \right)^t y_{i0}.$$

The following Lemmas 10 and 11 contain useful results for the uniform convergence of the GMM objective function  $M_{nT}(c)$ . This section gives the results without proofs, which can be found in Moon and Phillips (2002).

LEMMA 9: *Let*

$$f_T(c) = \frac{1}{T^2} \sum_{i=1}^T \sum_{s=1}^T \left[ \left( 1 + \frac{c}{T} \right)^T \right]^{h_T\left(\frac{i}{T}, \frac{s}{T}\right)} g_T\left(\frac{i}{T}, \frac{s}{T}, c\right),$$

where  $c \in \mathbb{C}$ , a compact subset in  $\mathbb{R}$ ,  $g_T(t/T, s/T, c)$  is continuously differentiable in  $c$ , and  $\sup_{(r,p,c) \in [0,1]^2 \times \mathbb{C}} |g_T(r, p, c)|$ ,  $\sup_{(r,p,c) \in [0,1]^2 \times \mathbb{C}} |\partial g_T(r, p, c) / \partial c|$ ,  $\sup_{(r,p) \in [0,1]^2} |h_T(r, p)| < \bar{K}$ . Suppose that  $g_T([Tr]/T, [Tp]/T, c) \rightarrow g(r, p, c)$  and  $h_T([Tr]/T, [Tp]/T) \rightarrow h(r, p)$  uniformly in  $(r, p) \in [0, 1]^2$ , where  $g(r, p, c)$  and  $h(r, p)$  are continuous functions on  $[0, 1]^2 \times \mathbb{C}$  and  $[0, 1]^2$ , respectively, satisfying

$$\int_0^1 \int_0^1 e^{\bar{c}|h(r,p)|} dp dr < \infty,$$

where  $\bar{c} = \max_{c \in \mathbb{C}} |c|$ . Then

$$f_T(c) \rightarrow f(c) = \int_0^1 \int_0^1 e^{ch(r,p)} g(r, p, c) dr dp \quad \text{uniformly in } c$$

as  $T \rightarrow \infty$ .

LEMMA 10:

- (a)  $A_{pT}(c) \rightarrow A_p(c)$  uniformly in  $c \in \mathbb{C}$ ,
- (b)  $B_{pT}(c) \rightarrow B_p(c)$  uniformly in  $c \in \mathbb{C}$ ,
- (c)  $A_{pT}(c)^{-1} \rightarrow A_p^{-1}(c)$  uniformly in  $c \in \mathbb{C}$ ,
- (d)  $\omega_{pT}(c) \rightarrow \omega_p(c)$  uniformly in  $c \in \mathbb{C}$ ,
- (e)  $\lambda_{pT}(c) \rightarrow \lambda_p(c)$  uniformly in  $c \in \mathbb{C}$ , where  $p = 1, 2$ .

LEMMA 11: For  $j = 1, \dots, J$ , assume that  $h_j(c, \tilde{c})$  are real-valued continuous functions on the product of the compact parameter set  $\mathbb{C} \times \mathbb{C}$  with  $h_j(c, c) = 0$ . Also, for  $j = 1, \dots, J$ , assume that  $l_{jT}(x, y)$  are real-valued continuous functions on  $[0, 1] \times [0, 1]$ . Let  $f_T(x, c)$  and  $g_T(x, c)$  be continuously differentiable functions from  $[0, 1] \times \mathbb{C}$  to  $\mathbb{R}$  such that  $f_T(x, c)g_T(y, c) - f_T(x, \tilde{c})g_T(y, \tilde{c}) = \sum_{j=1}^J h_j(c, \tilde{c})l_{jT}(x, y)$ . Assume that  $f_T(x, c) \rightarrow f(x, c)$ ,  $g_T(y, c) \rightarrow g(y, c)$ , and  $\sup_T \sup_{(r,c) \in [0,1] \times \mathbb{C}} |g_T(r, c)| < \infty$ . Suppose that  $y_{it} = (1 + (c_0/T))y_{it-1} + \varepsilon_{it}$ , where  $\varepsilon_{it}$  follows Assumption 3. Let Assumption 2 hold for the initial condition  $y_{i0}$ . Then, as  $(n, T \rightarrow \infty)$ , the following hold:

- (a) 
$$\frac{1}{n} \sum_{i=1}^n \frac{1}{T^2} \sum_{t=1}^T y_{it-1}^2 \rightarrow_p \sigma^2 \int_0^1 \int_0^r e^{2c_0(r-s)} ds dr.$$
- (b) 
$$\begin{aligned} \frac{1}{n} \sum_{i=1}^n \left( \frac{1}{\sqrt{T}} \sum_{t=1}^T \varepsilon_{it} f_T \left( \frac{t}{T}, c \right) \right) \left( \frac{1}{T\sqrt{T}} \sum_{t=1}^T y_{it-1} g_T \left( \frac{t}{T}, c \right) \right) \\ \rightarrow_p \sigma^2 \int_0^1 \int_0^r e^{c_0(r-s)} g(r, c) f(s, c) ds dr \quad \text{uniformly in } c. \end{aligned}$$
- (c) 
$$\begin{aligned} \frac{1}{n} \sum_{i=1}^n \left( \frac{1}{T\sqrt{T}} \sum_{t=1}^T y_{it-1} f_T \left( \frac{t}{T}, c \right) \right) \left( \frac{1}{T\sqrt{T}} \sum_{t=1}^T y_{it-1} g_T \left( \frac{t}{T}, c \right) \right) \\ \rightarrow_p \sigma^2 \int_0^1 \int_0^1 f(r, c) g(s, c) \int_0^{r \wedge s} e^{c_0(r+s-2v)} dv ds dr \quad \text{uniformly in } c. \end{aligned}$$
- (d) 
$$\begin{aligned} \frac{1}{n} \sum_{i=1}^n \left( \frac{1}{\sqrt{T}} \sum_{t=1}^T \varepsilon_{it} f_T \left( \frac{t}{T}, c \right) \right) \left( \frac{1}{\sqrt{T}} \sum_{t=1}^T \varepsilon_{it} g_T \left( \frac{t}{T}, c \right) \right) \\ \rightarrow_p \sigma^2 \int_0^1 f(r, c) g(r, c) dr \quad \text{uniformly in } c. \end{aligned}$$

The following lemma is important in establishing asymptotic normality of the GMM estimator  $\hat{c}$ . To simplify notation, let

$$\begin{aligned} l_{1pT}(t, s, c) &= \widehat{\Delta_c g_{pt}}' A_{pT}(c)^{-1} \widehat{\Delta_c g_{ps}}, \\ l_{2pT}(t, s, c) &= \widehat{\Delta_c g_{pt}}' A_{pT}(c)^{-1} g_{ps-1} D_{p,T}^{-1}, \\ l_{3pT}(t, s, c) &= \widehat{\Delta_c g_{pt}}' A_{pT}(c)^{-1} B_{pT}(c) A_{pT}(c)^{-1} \widehat{\Delta_c g_{ps}}, \end{aligned}$$

and

$$\begin{aligned} l_{1p}(r, s, c) &= \dot{g}_{pc}(r)' A_p(c)^{-1} \dot{g}_{pc}(s), \\ l_{2p}(r, s, c) &= \dot{g}_{pc}(r)' A_p(c)^{-1} g_p(s), \\ l_{3p}(r, s, c) &= \dot{g}_{pc}(r)' A_p(c)^{-1} B_p(c) A_p(c)^{-1} \dot{g}_{pc}(s), \\ l_{4p} &= \int_0^1 g_p(r) g_p(r)' dr. \end{aligned}$$

LEMMA 12: Suppose that  $x_{it} = \exp(c_0/T)x_{it-1} + \varepsilon_{it}$ , where  $\varepsilon_{it}$  are iid(0,  $\sigma^2$ ) with finite fourth moments and  $x_{i0} = 0$  for all  $i$ . Let

$$\begin{aligned}
 (36) \quad Q_{1iT} &= \frac{1}{T} \sum_{t=1}^T x_{it-1} \varepsilon_{it}, \\
 Q_{2iT} &= \frac{1}{\sqrt{T}} \sum_{t=1}^T \frac{1}{T\sqrt{T}} \sum_{s=1}^T \varepsilon_{it} x_{is-1} \tilde{h}_{pT}(t, s) - \sigma^2 \omega_{pT}(c_0), \\
 Q_{3iT} &= \frac{1}{\sqrt{T}} \sum_{t=1}^T \frac{1}{T\sqrt{T}} \sum_{s=1}^T \varepsilon_{it} x_{is-1} l_{1pT}(t, s, c_0) - \sigma^2 \lambda_{pT}(c_0), \\
 Q_{4iT} &= \frac{1}{\sqrt{T}} \sum_{t=1}^T \frac{1}{\sqrt{T}} \sum_{s=1}^T \varepsilon_{it} \varepsilon_{is} l_{2pT}(t, s, c_0) - \sigma^2 \text{tr}(A_{pT}(c_0)^{-1} B_{pT}(c_0)), \\
 Q_{5iT} &= \frac{1}{\sqrt{T}} \sum_{t=1}^T \frac{1}{\sqrt{T}} \sum_{s=1}^T \varepsilon_{it} \varepsilon_{is} l_{3pT}(t, s, c_0) - \sigma^2 \text{tr}(A_{pT}(c_0)^{-1} B_{pT}(c_0)), \quad \text{and} \\
 Q_{iT} &= (Q_{1iT}, Q_{2iT}, Q_{3iT}, Q_{4iT}, Q_{5iT})'.
 \end{aligned}$$

Then, as  $(n, T \rightarrow \infty)$ ,

$$\frac{1}{\sqrt{n}} \sum_{i=1}^n Q_{iT} \Rightarrow N(0, \sigma^4 \Phi(c_0)),$$

where

$$(37) \quad \Phi(c_0) = \begin{bmatrix} \Phi_{11}(c_0) & \Phi_{12}(c_0) & \Phi_{13}(c_0) & \Phi_{14}(c_0) & \Phi_{15}(c_0) \\ \Phi_{12}(c_0) & \Phi_{22}(c_0) & \Phi_{23}(c_0) & \Phi_{24}(c_0) & \Phi_{25}(c_0) \\ \Phi_{13}(c_0) & \Phi_{23}(c_0) & \Phi_{33}(c_0) & \Phi_{34}(c_0) & \Phi_{35}(c_0) \\ \Phi_{14}(c_0) & \Phi_{24}(c_0) & \Phi_{34}(c_0) & \Phi_{44}(c_0) & \Phi_{45}(c_0) \\ \Phi_{15}(c_0) & \Phi_{25}(c_0) & \Phi_{35}(c_0) & \Phi_{45}(c_0) & \Phi_{55}(c_0) \end{bmatrix}$$

and

$$\begin{aligned}
 \Phi_{11}(c_0) &= \int_0^1 \int_0^r e^{2c_0(r-s)} ds dr, \\
 \Phi_{12}(c_0) &= \int_0^1 \int_0^1 \int_0^{r \wedge s} e^{c_0(r+s-2v)} \tilde{h}_p(r, s) dv ds dr \\
 &\quad + \int_0^1 \int_0^r \int_0^s e^{c_0(r-v)} \tilde{h}_p(v, r) dv ds dr, \\
 \Phi_{13}(c_0) &= \int_0^1 \int_0^r \int_0^s e^{c_0(r-v)} l_{1p}(r, v, c_0) dv ds dr \\
 &\quad + \int_0^1 \int_0^1 \int_0^{r \wedge s} e^{c_0(r+s-2v)} l_{1p}(r, s, c_0) dv ds dr, \\
 \Phi_{14}(c_0) &= \int_0^1 \int_0^r e^{c_0(r-s)} l_{2p}(r, s, c_0) ds dr + \int_0^1 \int_0^r e^{c_0(r-s)} l_{2p}(s, r, c_0) ds dr, \\
 \Phi_{15}(c_0) &= \int_0^1 \int_0^r e^{c_0(r-s)} l_{3p}(r, s, c_0) ds dr + \int_0^1 \int_0^r e^{c_0(r-s)} l_{3p}(s, r, c_0) ds dr,
 \end{aligned}$$

$$\begin{aligned}
 \Phi_{22}(c_0) &= \int_0^1 \int_0^1 \int_0^{r \wedge s} e^{c_0(r+s-2v)} \tilde{h}_p(r, s) dv ds dr \\
 &\quad + \int_0^1 \int_0^1 \int_0^r \int_0^s e^{c_0(r-v)} e^{c_0(s-q)} \tilde{h}_p(r, q) \tilde{h}_p(s, v) dq dv ds dr, \\
 \Phi_{23}(c_0) &= \int_0^1 \int_0^1 \int_0^1 \int_0^{s \wedge v} e^{c_0(s+v-2q)} \tilde{h}_p(r, s) l_{1p}(r, v, c_0) dq dv ds dr \\
 &\quad + \int_0^1 \int_0^1 \int_0^r \int_0^s e^{c_0(r-v)} e^{c_0(s-q)} \tilde{h}_p(r, q) l_{1p}(s, v, c_0) dq dv ds dr, \\
 \Phi_{24}(c_0) &= \int_0^1 \int_0^r \int_0^1 e^{c_0(r-s)} \tilde{h}_p(r, v) l_{2p}(v, s, c_0) dv ds dr \\
 &\quad + \int_0^1 \int_0^r \int_0^1 e^{c_0(r-s)} \tilde{h}_p(r, v) l_{2p}(s, v, c_0) dv ds dr, \\
 \Phi_{25}(c_0) &= \int_0^1 \int_0^r \int_0^1 e^{c_0(r-s)} \tilde{h}_p(r, v) l_{3p}(v, s, c_0) dv ds dr \\
 &\quad + \int_0^1 \int_0^r \int_0^1 e^{c_0(r-s)} \tilde{h}_p(r, v) l_{3p}(s, v, c_0) dv ds dr, \\
 \Phi_{33}(c_0) &= \int_0^1 \int_0^1 \int_0^{r \wedge s} e^{c_0(r+s-2v)} l_{1p}(r, s, c_0) dv ds dr \\
 &\quad + \int_0^1 \int_0^1 \int_0^r \int_0^s e^{c_0(r-v)} e^{c_0(s-q)} l_{1p}(r, q, c_0) l_{1p}(s, v, c_0) dq dv ds dr, \\
 \Phi_{34}(c_0) &= \int_0^1 \int_0^r e^{c_0(r-s)} l_{2p}(r, s, c_0) ds dr + \int_0^1 \int_0^r e^{c_0(r-s)} l_{3p}(r, s, c_0) ds dr, \\
 \Phi_{35}(c_0) &= \int_0^1 \int_0^r e^{c_0(r-s)} l_{3p}(r, s, c_0) ds dr + \int_0^1 \int_0^r e^{c_0(r-s)} l_{3p}(s, r, c_0) ds dr, \\
 \Phi_{44}(c_0) &= (\text{vec } A_p(c_0)^{-1})' \text{vec } l_{4p}(c_0) + \text{tr}(A_p(c_0)^{-1} B_p(c_0)' A_p(c_0)^{-1} B_p(c_0)'), \\
 \Phi_{45}(c_0) &= \text{tr}(A_p(c_0)^{-1} B_p(c_0) A_p(c_0)^{-1} B_p(c_0)') + \text{tr}(A_p(c_0)^{-1} B_p(c_0) A_p(c_0)^{-1} B_p(c_0)), \\
 \Phi_{55}(c_0) &= \text{tr}(A_p(c_0)^{-1} B_p(c_0) A_p(c_0)^{-1} B_p(c_0)') + \text{tr}(A_p(c_0)^{-1} B_p(c_0) A_p(c_0)^{-1} B_p(c_0)).
 \end{aligned}$$

PROOF: The proof uses Theorem 8. First, a direct calculation shows that  $EQ_{iT} = 0$ . Let  $\Phi_{nT}(c_0) = EQ_{iT} Q'_{iT}$ . Notice that  $Q_{iT}$  are iid(0,  $\Phi_{nT}(c_0)$ ) across  $i$ . As  $T \rightarrow \infty$ ,  $Q_{iT} \Rightarrow \sigma^2 Q_i$ , where

$$\begin{aligned}
 Q_i &= (Q_{1i}, Q_{2i}, Q_{3i}, Q_{4i}, Q_{5i})', \\
 Q_{1i} &= \int_0^1 J_{c_0,i}(r) dW_i(r), \\
 Q_{2i} &= \int_0^1 \int_0^1 J_{c_0,i}(r) \tilde{h}_p(r, s) dW_i(s) dr - \omega_p(c_0), \\
 Q_{3i} &= \int_0^1 \int_0^1 l_{1p}(r, s, c_0) J_{c_0,i}(s) dW_i(r) ds - \lambda_p(c_0),
 \end{aligned}$$

$$Q_{4i} = \int_0^1 \int_0^1 l_{2p}(r, s, c_0) dW_i(r) dW_i(s) - \text{tr}(A_p(c_0)^{-1}B_p(c_0)),$$

$$Q_{5i} = \int_0^1 \int_0^1 l_{3p}(r, s, c_0) dW_i(r) dW_i(s) - \text{tr}(A_p(c_0)^{-1}B_p(c_0)).$$

Also, a direct calculation shows that as  $T \rightarrow \infty$ ,  $\Phi_{nT}(c_0) = EQ_{iT}Q'_{iT} \rightarrow \sigma^4EQ_iQ'_i = \sigma^4\Phi(c_0)$ . Let  $l$  be any  $(5 \times 1)$  vector with  $\|l\| = 1$ . We consider two cases.

*Case 1: If  $l'\Phi(c_0)l > 0$ .* To establish the desired result with a joint limit, we apply Theorem 7. Condition (i) holds because it is assumed that  $l'\Phi(c_0)l > 0$ . Conditions (ii) is trivial. Finally condition (iii), viz.,  $(l'Q_{iT})^2$  are uniformly integrable in  $T$ , holds because  $(l'Q_{iT})^2 \Rightarrow (l'Q_i)^2$  as  $T \rightarrow \infty$  by the continuous mapping theorem with  $E(l'Q_{iT})^2 = l'\Phi_{nT}(c_0)l \rightarrow \sigma^4l'\Phi(c_0)l = \sigma^4E(l'Q_i)^2$ , and by applying Theorem 5.4 of Billingsley (1968).

*Case 2: If  $l'\Phi(c_0)l = 0$ .* Since  $l'\Phi_{nT}(c_0)l \rightarrow l'\Phi(c_0)l = 0$ , we have  $E((1/\sqrt{n}) \sum_{i=1}^n l'Q_{iT})^2 = l'\Phi_{nT}(c_0)l \rightarrow 0$ , which leads to  $(1/\sqrt{n}) \sum_{i=1}^n l'Q_{iT} \rightarrow_p 0$ . By the Cramér–Wold device, it follows that  $(1/\sqrt{n}) \sum_{i=1}^n Q_{iT} \Rightarrow N(0, \sigma^4\Phi(c_0))$ . *Q.E.D.*

APPENDIX B: PROOFS OF SECTION 4

PROOF OF LEMMA 2: By definition,

$$\hat{\sigma}^2 - \tilde{\sigma}^2 = (\hat{\rho} - \rho_0)^2 \frac{1}{nT} \sum_{i=1}^n y'_{i-1} y_{i-1} - 2(\hat{\rho} - \rho_0) \frac{1}{nT} \sum_{i=1}^n \varepsilon'_i y_{i-1}.$$

Since  $\hat{\rho} - \rho_0 = O_p(1)$ ,  $(1/nT^2) \sum_{i=1}^n y'_{i-1} y_{i-1} = O_p(1)$ , and  $(1/nT) \sum_{i=1}^n \varepsilon'_i y_{i-1} = O_p(1)$  by Lemma 11 and Theorem 1 of Moon and Phillips (2000),  $\hat{\sigma}^2 - \tilde{\sigma}^2 = O_p(1/T)$ . Next, since  $E(\tilde{\sigma}^2 - \sigma^2)^2 = O(1/\sqrt{nT})$ , it follows that  $\tilde{\sigma}^2 - \sigma^2 = O_p(1/\sqrt{nT})$ . *Q.E.D.*

PROOF OF LEMMA 3: We show separately the following:

(38)  $\frac{1}{n} \sum_{i=1}^n (m_{1iT}(c) - m_1(c)) \rightarrow_p 0,$

and

(39)  $\frac{1}{n} \sum_{i=1}^n (m_{2iT}(c) - m_2(c)) \rightarrow_p 0,$

uniformly in  $c$ .

First, by definition and the triangle inequality, we have

$$\begin{aligned} m_{1,iT}(c) &= \frac{1}{T} \left( z_i - \left( 1 + \frac{c}{T} \right) z_{i-1} \right)' z_{i-1} + \hat{\sigma}^2 \omega_{pT}(c) \\ &= \frac{1}{T} \varepsilon'_i y_{i-1} - (c - c_0) \frac{1}{T^2} y'_{i-1} y_{i-1} + \hat{\sigma}^2 \omega_{pT}(c) \\ &= \frac{1}{T} \sum_{t=1}^T \varepsilon_{it} y_{i,t-1} - \frac{1}{T^2} \sum_{t=1}^T \sum_{s=1}^T \varepsilon_{it} y_{i,s-1} \tilde{h}_{pT}(t, s) \\ &\quad - (c - c_0) \frac{1}{T^2} \sum_{t=1}^T y_{i,t-1}^2 + \hat{\sigma}^2 \omega_{pT}(c). \end{aligned}$$

So,

$$\begin{aligned}
 & \left| \frac{1}{n} \sum_{i=1}^n (m_{1iT}(c) - m_1(c)) \right| \\
 & \leq \left| \frac{1}{n} \sum_{i=1}^n \left\{ \frac{1}{T} \sum_{t=1}^T \varepsilon_{it} y_{it-1} - \left( \frac{1}{T^2} \sum_{t=1}^T \sum_{s=1}^T \varepsilon_{it} y_{is-1} \tilde{h}_{pT}(t, s) - \sigma^2 \omega_p(c_0) \right) \right. \right. \\
 & \quad \left. \left. + \sigma^2 (\omega_{pT}(c) - \omega_p(c)) \right. \right. \\
 & \quad \left. \left. - (c - c_0) \left( \frac{1}{T^2} \sum_{t=1}^T y_{i,-1,t}^2 - \sigma^2 \psi_p(c_0) \right) \right\} \right| \\
 & \quad + |\hat{\sigma}^2 - \sigma^2| |\omega_{pT}(c)| \\
 & \leq \left| \frac{1}{n} \sum_{i=1}^n \frac{1}{T} \sum_{t=1}^T \varepsilon_{it} y_{it-1} \right| \\
 & \quad + \left| \frac{1}{n} \sum_{i=1}^n \left( \frac{1}{T^2} \sum_{t=1}^T \sum_{s=1}^T \varepsilon_{it} y_{is-1} \tilde{h}_{pT}(t, s) - \sigma^2 \omega_p(c_0) \right) \right| \\
 & \quad + |c - c_0| \left| \frac{1}{n} \sum_{i=1}^n \left( \frac{1}{T^2} \sum_{t=1}^T y_{i,-1,t}^2 - \sigma^2 \psi_p(c_0) \right) \right| \\
 & \quad + \sigma^2 |\omega_{pT}(c) - \omega_p(c)| + |\hat{\sigma}^2 - \sigma^2| |\omega_{pT}(c)| \\
 & = I + II + III + IV + V, \quad \text{say.}
 \end{aligned}$$

Notice that the two terms  $I$  and  $II$  are independent of  $c$ , and by Lemma 9 of Moon and Phillips (2000),  $I, II \rightarrow_p 0$  as  $(n, T \rightarrow \infty)$ . Next,  $III \rightarrow_p 0$  uniformly in  $c$  because  $|(1/n) \sum_{i=1}^n (1/T^2 \times \sum_{t=1}^T y_{i,-1,t}^2 - \sigma^2 \psi_p(c_0))|$  that is independent of  $c$  converges in probability to zero as  $(n, T \rightarrow \infty)$  by Lemma 9 of Moon and Phillips (2000). Next,  $IV \rightarrow 0$  uniformly in  $c$  by Lemma 10(d). Finally, since  $\hat{\sigma}^2 - \sigma^2 = o_p(1)$  by Lemma 2, and  $\sup_{c \in \mathbb{C}} \omega_{pT}(c) < \bar{K}$  for some finite  $\bar{K}$ ,  $V$  converges in probability to zero uniformly in  $c$ . Therefore,  $(1/n) \sum_{i=1}^n (m_{1iT}(c) - m_1(c)) \rightarrow_p 0$  uniformly in  $c$  as  $(n, T \rightarrow \infty)$ .

Next, to prove (39), noting by definition that  $\Delta_c z_{it} = \beta'_{i0} \Delta_c g_{pt} - (c - c_0)(y_{it-1}/T) + \varepsilon_{it}$ , we write

$$\begin{aligned}
 \frac{1}{n} \sum_{i=1}^n m_{2iT}(c) &= \frac{1}{n} \sum_{i=1}^n \left( \frac{1}{T} \sum_{t=1}^T \varepsilon_{it} y_{it-1} \right) - (c - c_0) \frac{1}{n} \sum_{i=1}^n \frac{1}{T^2} \sum_{t=1}^T y_{it-1}^2 \\
 & \quad - \frac{1}{n} \sum_{i=1}^n \left[ \left( \frac{1}{\sqrt{T}} \sum_{t=1}^T \widehat{\Delta_c g_{pt}} \varepsilon_{it} \right)' A_{pT}(c)^{-1} \left( \frac{1}{T\sqrt{T}} \sum_{t=1}^T \widehat{\Delta_c g_{pt}} y_{it-1} \right) \right] \\
 & \quad + \sigma^2 \lambda_{pT}(c) \\
 & \quad + (c - c_0) \frac{1}{n} \sum_{i=1}^n \left[ \left( \frac{1}{T\sqrt{T}} \sum_{t=1}^T \widehat{\Delta_c g_{pt}} y_{it-1} \right)' A_{pT}(c)^{-1} \right. \\
 & \quad \quad \left. \times \left( \frac{1}{T\sqrt{T}} \sum_{t=1}^T \widehat{\Delta_c g_{pt}} y_{it-1} \right) \right] \\
 & \quad - \frac{1}{n} \sum_{i=1}^n \left[ \left( \frac{1}{\sqrt{T}} \sum_{t=1}^T \widehat{\Delta_c g_{pt}} \varepsilon_{it} \right)' A_{pT}(c)^{-1} \left( \frac{1}{\sqrt{T}} \sum_{t=1}^T g_{pt-1} D_{p,T}^{-1} \varepsilon_{it} \right) \right]
 \end{aligned}$$



$$\begin{aligned}
 &+ (c - c_0) \frac{1}{n} \sum_{i=1}^n \left[ \left( \frac{1}{T\sqrt{T}} \sum_{t=1}^T \widehat{\Delta_c g_{pt}} y_{it-1} \right)' A_{pT}(c)^{-1} \right. \\
 &\quad \left. \times \left( \frac{1}{\sqrt{T}} \sum_{t=1}^T g_{pt-1} D_{p,T}^{-1} \varepsilon_{it} \right) \right] \\
 &+ (c - c_0) \frac{1}{n} \sum_{i=1}^n \left[ \left( \frac{1}{\sqrt{T}} \sum_{t=1}^T \widehat{\Delta_c g_{pt}} \varepsilon_{it} \right)' A_{pT}(c)^{-1} \right. \\
 &\quad \left. \times \left( \frac{1}{T\sqrt{T}} \sum_{t=1}^T g_{pt-1} D_{p,T}^{-1} y_{it-1} \right) \right] \\
 &- (c - c_0)^2 \frac{1}{n} \sum_{i=1}^n \left[ \left( \frac{1}{T\sqrt{T}} \sum_{t=1}^T \widehat{\Delta_c g_{pt}} y_{it-1} \right)' A_{pT}(c)^{-1} \right. \\
 &\quad \left. \times \left( \frac{1}{T\sqrt{T}} \sum_{t=1}^T g_{pt-1} D_{p,T}^{-1} y_{it-1} \right) \right] \\
 &+ \frac{1}{n} \sum_{i=1}^n \left[ \left( \frac{1}{\sqrt{T}} \sum_{t=1}^T \widehat{\Delta_c g_{pt}} \varepsilon_{it} \right)' A_{pT}(c)^{-1} B_{pT}(c) A_{pT}(c)^{-1} \right. \\
 &\quad \left. \times \left( \frac{1}{\sqrt{T}} \sum_{t=1}^T \widehat{\Delta_c g_{pt}} \varepsilon_{it} \right) \right] \\
 &- (c - c_0) \frac{1}{n} \sum_{i=1}^n \left[ \left( \frac{1}{T\sqrt{T}} \sum_{t=1}^T \widehat{\Delta_c g_{pt}} y_{it-1} \right)' A_{pT}(c)^{-1} B_{pT}(c) A_{pT}(c)^{-1} \right. \\
 &\quad \left. \times \left( \frac{1}{\sqrt{T}} \sum_{t=1}^T \widehat{\Delta_c g_{pt}} \varepsilon_{it} \right) \right] \\
 &- (c - c_0) \frac{1}{n} \sum_{i=1}^n \left[ \left( \frac{1}{\sqrt{T}} \sum_{t=1}^T \widehat{\Delta_c g_{pt}} \varepsilon_{it} \right)' A_{pT}(c)^{-1} B_{pT}(c) A_{pT}(c)^{-1} \right. \\
 &\quad \left. \times \left( \frac{1}{T\sqrt{T}} \sum_{t=1}^T \widehat{\Delta_c g_{pt}} y_{it-1} \right) \right] \\
 &+ (c - c_0)^2 \frac{1}{n} \sum_{i=1}^n \left[ \left( \frac{1}{T\sqrt{T}} \sum_{t=1}^T \widehat{\Delta_c g_{pt}} y_{it-1} \right)' A_{pT}(c)^{-1} B_{pT}(c) A_{pT}(c)^{-1} \right. \\
 &\quad \left. \times \left( \frac{1}{T\sqrt{T}} \sum_{t=1}^T \widehat{\Delta_c g_{pt}} y_{it-1} \right) \right] \\
 &+ (\hat{\sigma}^2 - \sigma^2) \lambda_{pT}(c).
 \end{aligned}$$

Notice that  $\widehat{\Delta_c g_{pt}}$  and  $D_{p,T}^{-1} g_{pt-1}$  satisfy the conditions for  $f_T(x, c)$  and  $g_T(x, c)$  in Lemma 11. The desired result then follows by Lemma 10 and by applying Lemma 11 together with  $\hat{\sigma}^2 - \sigma^2 = o_p(1)$  (see Lemma 2) and the boundedness of  $\lambda_{pT}(c)$  on the compact parameter set  $\mathbb{C}$ . *Q.E.D.*

PROOF OF LEMMA 4: The proof is similar to that of Lemma 3, and is omitted.

PROOF OF LEMMA 5: First, using (35) and by Lemma 2, we may write

$$(40) \quad \frac{1}{\sqrt{n}} \sum_{i=1}^n m_{1iT}(c_0) = \frac{1}{\sqrt{n}} \sum_{i=1}^n (Q_{1iT} - Q_{2iT}) + \frac{1}{\sqrt{n}} \sum_{i=1}^n R_{1iT} \\ + O_p\left(\frac{\sqrt{n}}{T}\right) + O_p\left(\frac{1}{\sqrt{T}}\right)$$

and

$$(41) \quad \frac{1}{\sqrt{n}} \sum_{i=1}^n m_{2iT}(c_0) = \frac{1}{\sqrt{n}} \sum_{i=1}^n (Q_{1iT} - Q_{3iT} - Q_{4iT} + Q_{5iT}) + \frac{1}{\sqrt{n}} \sum_{i=1}^n R_{2iT} \\ + O_p\left(\frac{\sqrt{n}}{T}\right) + O_p\left(\frac{1}{\sqrt{T}}\right),$$

where

$$R_{1iT} = y_{i0} \left( \frac{1}{T} \sum_{t=1}^T \left[ \left( 1 + \frac{c_0}{T} \right)^T \right]^{\frac{t-1}{T}} \varepsilon_{it} \right) \\ - y_{i0} \left( \frac{1}{\sqrt{T}} \sum_{t=1}^T \frac{1}{T\sqrt{T}} \sum_{s=1}^T \varepsilon_{it} \left[ \left( 1 + \frac{c_0}{T} \right)^T \right]^{\frac{s-1}{T}} \tilde{h}_{pT}(t, s) \right) \\ = R_{11iT} - R_{12iT},$$

and

$$R_{2iT} = y_{i0} \left( \frac{1}{T} \sum_{t=1}^T \left[ \left( 1 + \frac{c_0}{T} \right)^T \right]^{\frac{t-1}{T}} \varepsilon_{it} \right) \\ - y_{i0} \left( \frac{1}{\sqrt{T}} \sum_{t=1}^T \frac{1}{T\sqrt{T}} \sum_{s=1}^T \varepsilon_{it} \left[ \left( 1 + \frac{c_0}{T} \right)^T \right]^{\frac{s-1}{T}} l_{1pT}(t, s, c_0) \right) \\ = R_{21iT} - R_{22iT}.$$

Notice that

$$E \left( \frac{1}{\sqrt{n}} \sum_{i=1}^n R_{11iT} \right)^2 = \frac{1}{n} \sum_{i=1}^n ER_{11iT}^2 \\ \leq \frac{\sigma^2}{T} \left( \sup_i E y_{i0}^2 \right) \left( \frac{1}{T} \sum_{t=1}^T \left[ \left( 1 + \frac{c_0}{T} \right)^T \right]^{\frac{2(t-1)}{T}} \right) \\ = O\left(\frac{1}{T}\right),$$

where the first equality holds because  $ER_{11iT} = 0$  and  $R_{11iT}$  is independent across  $i$  (Assump-

tion 3) and the second equality holds by Assumptions 2. Similarly, it follows that

$$\begin{aligned} E\left(\frac{1}{\sqrt{n}} \sum_{i=1}^n R_{12iT}\right)^2 &= \frac{1}{n} \sum_{i=1}^n ER_{12iT}^2 \\ &\leq \frac{\sigma^2}{T} \left(\sup_i Ey_{i0}^2\right) \left(\frac{1}{T} \sum_{t=1}^T \left(\frac{1}{T} \sum_{s=1}^T \left[\left(1 + \frac{c_0}{T}\right)^T\right]^{\frac{s-1}{T}} \tilde{h}_{pT}(t, s)\right)^2\right) \\ &= O\left(\frac{1}{T}\right). \end{aligned}$$

Therefore,

$$(42) \quad \frac{1}{\sqrt{n}} \sum_{i=1}^n R_{1iT} = O_p\left(\frac{1}{\sqrt{T}}\right).$$

By using similar arguments, it is possible to show that

$$(43) \quad \frac{1}{\sqrt{n}} \sum_{i=1}^n R_{2iT} = O_p\left(\frac{1}{\sqrt{T}}\right).$$

In view of (40)–(43), as  $(n, T \rightarrow \infty)$  following Assumption 5,

$$\frac{1}{\sqrt{n}} \sum_{i=1}^n \begin{pmatrix} m_{1iT}(c_0) \\ m_{2iT}(c_0) \end{pmatrix} = J' \left(\frac{1}{\sqrt{n}} \sum_{i=1}^n Q_{iT}\right) J + o_p(1).$$

The required result follows by Lemma 5.

*Q.E.D.*

PROOF OF LEMMA 6: Part (a). By definition and by the Cauchy–Schwarz inequality,

$$\sup_{c \in \mathbb{C}: |c-c_0| \leq \gamma_{nT}} |B_{nT} \mathcal{R}_{1nT}(c, c_0)| \leq 2 \|B_{nT} M_{nT}(c_0)\| \|\hat{W}\| \sup_{c \in \mathbb{C}: |c-c_0| \leq \gamma_{nT}} \|r_{nT}(c, c_0)\|.$$

By Lemma 5 and Assumption 6, we have  $\|B_{nT} M_{nT}(c_0)\| \|\hat{W}\| = O_p(1)$ . Thus, to complete the proof, it is enough to show that  $\sup_{c \in \mathbb{C}: |c-c_0| \leq \gamma_{nT}} \|r_{nT}(c, c_0)\| = o_p(1)$ . Notice by definition and the triangle inequality that

$$\begin{aligned} \sup_{c \in \mathbb{C}: |c-c_0| \leq \gamma_{nT}} \|r_{nT}(c, c_0)\| &\leq \sup_{c \in \mathbb{C}: |c-c_0| \leq \gamma_{nT}} |r_{1nT}(c, c_0)| + \sup_{c \in \mathbb{C}: |c-c_0| \leq \gamma_{nT}} |r_{2nT}(c, c_0)| \\ &\leq \sup_{c \in \mathbb{C}: |c-c_0| \leq \gamma_{nT}} \left| \frac{1}{n} \sum_{i=1}^n (dm_{1iT}(c) - dm_{1iT}(c_0)) \right| \\ &\quad + \sup_{c \in \mathbb{C}: |c-c_0| \leq \gamma_{nT}} \left| \frac{1}{n} \sum_{i=1}^n (dm_{2iT}(c) - dm_{2iT}(c_0)) \right|, \end{aligned}$$

where the last inequality holds because  $c_k^+$  locates between  $c$  and  $c_0$  for  $k = 1, 2$ . Notice that

$$\begin{aligned} &\sup_{c \in \mathbb{C}: |c-c_0| \leq \gamma_{nT}} \left| \frac{1}{n} \sum_{i=1}^n (dm_{1iT}(c) - dm_{1iT}(c_0)) \right| \\ &\leq \sup_{c \in \mathbb{C}} \left| \frac{1}{n} \sum_{i=1}^n (dm_{1iT}(c) - \hat{\sigma}^2 dm_1(c)) \right| + \left| \frac{1}{n} \sum_{i=1}^n (dm_{1iT}(c_0) - \hat{\sigma}^2 dm_1(c_0)) \right| \\ &\quad + \hat{\sigma}^2 \sup_{c \in \mathbb{C}: |c-c_0| \leq \gamma_{nT}} |dm_1(c) - dm_1(c_0)|. \end{aligned}$$

Then, the first term and the second term in the last line are  $o_p(1)$  by Lemma 4 and the last term is also  $o_p(1)$  because  $dm(c)$  is continuous in  $c$  and  $\hat{\sigma}^2$  has a finite limit. Therefore  $\sup_{c \in \mathbb{C}: |c-c_0| \leq \gamma_{nT}} |r_{1nT}(c, c_0)| = o_p(1)$ . Similarly, it follows that  $\sup_{c \in \mathbb{C}: |c-c_0| \leq \gamma_{nT}} |r_{2nT}(c, c_0)| = o_p(1)$ , and we complete the proof. *Q.E.D.*

Part (b). The proof of Part (b) is similar to that of Part (a) and is omitted.

PROOF OF THEOREM 2: We employ similar arguments to the proof of Theorem 1 of Andrews (1999). Define  $\hat{\theta}_{nT} = B_{nT}(\hat{c} - c_0)$ . Then,

$$\begin{aligned} o_p(1) &\leq B_{nT}^2(Z_{nT}(c_0) - Z_{nT}(\hat{c})) \\ &= -\mathcal{H}_{nT} \hat{\theta}_{nT}^2 + 2\mathcal{H}_{nT}(B_{nT}\mathcal{S}_{nT})\hat{\theta}_{nT} \\ &\quad - \hat{\theta}_{nT}B_{nT}\mathcal{R}_{1nT}(\hat{c}, c_0) - \hat{\theta}_{nT}^2\mathcal{R}_{2nT}(\hat{c}, c_0). \end{aligned}$$

From Lemmas 4 and 5 and Assumption 6, we have  $\mathcal{H}_{nT}, \mathcal{H}_{nT}^{-1} = O_p(1)$  and positive with probability 1 and  $B_{nT}\mathcal{S}_{nT} = O_p(1)$ . Also, by Lemma 6,  $B_{nT}\mathcal{R}_{1nT}(\hat{c}, c_0) = o_p(1)$  and  $\mathcal{R}_{2nT}(\hat{c}, c_0) = o_p(1)$ . Then,

$$o_p(1) \leq -|\hat{\theta}_{nT}|^2 + 2O_p(1)|\hat{\theta}_{nT}| + |\hat{\theta}_{nT}|o_p(1) + |\hat{\theta}_{nT}|^2o_p(1),$$

which is rearranged as

$$|\hat{\theta}_{nT}|^2 \leq 2O_p(1)|\hat{\theta}_{nT}| + o_p(1).$$

Then, the required result,

$$\hat{\theta}_{nT} = O_p(1),$$

follows by relation (7.4) on page 1377 of Andrews (1999). *Q.E.D.*

PROOF OF THEOREM 3: To complete the proof, it is enough to show (a)  $B_{nT}(\hat{c} - c_0) = B_{nT}(\hat{c}_q - c_0) + o_p(1)$  and (b)  $B_{nT}(\hat{c}_q - c_0) = \hat{\phi}_{nT} + o_p(1)$ .

Part (a). Recall that  $B_{nT}\mathcal{S}_{nT}/\mathcal{H}_{nT} = O_p(1)$  by Lemmas 4 and 5 and Assumption 6. Then, it follows by the definition of  $B_{nT}(\hat{c}_q - c_0)$  that

$$\left( B_{nT}(\hat{c}_q - c_0) - \frac{B_{nT}\mathcal{S}_{nT}}{\mathcal{H}_{nT}} \right)^2 \leq \left( \frac{B_{nT}\mathcal{S}_{nT}}{\mathcal{H}_{nT}} \right)^2 = O_p(1),$$

which leads to

$$B_{nT}(\hat{c}_q - c_0) = \frac{B_{nT}\mathcal{S}_{nT}}{\mathcal{H}_{nT}} + O_p(1) = O_p(1).$$

From this we find that  $\hat{c}_q$  is also  $B_{nT}(= \sqrt{n})$ -consistent.

Notice that we have

$$\begin{aligned} o_p(1) &\leq B_{nT}^2 Z_{nT}(\hat{c}_q) - B_{nT}^2 Z_{nT}(\hat{c}) \\ &= \left( B_{nT}(\hat{c}_q - c_0) - \frac{B_{nT}\mathcal{S}_{nT}}{\mathcal{H}_{nT}} \right)^2 - \left( B_{nT}(\hat{c} - c_0) - \frac{B_{nT}\mathcal{S}_{nT}}{\mathcal{H}_{nT}} \right)^2 + o_p(1), \\ &\leq o_p(1), \end{aligned}$$

where the first line holds by the definition of  $\hat{c}$ , the second line holds since  $B_{nT}(\hat{c}_q - c_0), B_{nT}(\hat{c} - c_0) = O_p(1)$  and by Lemma 6, and the last  $o_p(1)$  bound holds because

$$\left( B_{nT}(\hat{c}_q - c_0) - \frac{B_{nT}\mathcal{S}_{nT}}{\mathcal{H}_{nT}} \right)^2 - \left( B_{nT}(\hat{c} - c_0) - \frac{B_{nT}\mathcal{S}_{nT}}{\mathcal{H}_{nT}} \right)^2 \leq 0$$

by definition of  $B_{nT}(\hat{c}_q - c_0)$ . So,

$$(44) \quad \left| \left( B_{nT}(\hat{c}_q - c_0) - \frac{B_{nT}\mathcal{S}_{nT}}{\mathcal{H}_{nT}} \right)^2 - \left( B_{nT}(\hat{c} - c_0) - \frac{B_{nT}\mathcal{S}_{nT}}{\mathcal{H}_{nT}} \right)^2 \right| = o_p(1).$$

Now, for any given  $\delta > 0$ , set  $\varepsilon = \delta^2$ . Then, since  $B_{nT}(\hat{c}_q - c_0)$  achieves the minimum of the quadratic function  $f(\lambda) = (\lambda - (B_{nT}\mathcal{S}_{nT}/\mathcal{H}_{nT}))^2$  on the closed interval  $\{\lambda : B_{nT}(\bar{c} - c_0) \leq \lambda \leq -B_{nT}c_0\}$ , it follows that  $|B_{nT}(\hat{c} - c_0) - B_{nT}(\hat{c}_q - c_0)| > \delta$  implies

$$\left| \left( B_{nT}(\hat{c}_q - c_0) - \frac{B_{nT}\mathcal{S}_{nT}}{\mathcal{H}_{nT}} \right)^2 - \left( B_{nT}(\hat{c} - c_0) - \frac{B_{nT}\mathcal{S}_{nT}}{\mathcal{H}_{nT}} \right)^2 \right| > \varepsilon.$$

Therefore

$$\begin{aligned} & P\{|B_{nT}(\hat{c} - c_0) - B_{nT}(\hat{c}_q - c_0)| > \delta\} \\ & \leq P\left\{ \left| \left( B_{nT}(\hat{c}_q - c_0) - \frac{B_{nT}\mathcal{S}_{nT}}{\mathcal{H}_{nT}} \right)^2 - \left( B_{nT}(\hat{c} - c_0) - \frac{B_{nT}\mathcal{S}_{nT}}{\mathcal{H}_{nT}} \right)^2 \right| > \varepsilon \right\} \\ & \rightarrow 0, \end{aligned}$$

where the last convergence holds by (44), and we have completed the proof of Part (a).

Part (b). Recall that  $c_0 \in \mathbb{C}_0/\{0\}$ . For any  $\delta > 0$ ,

$$(45) \quad \begin{aligned} & P\{|B_{nT}(\hat{c}_q - c_0) - \hat{\phi}_{nT}| > \delta\} \\ & \leq P\left\{ \frac{B_{nT}\mathcal{S}_{nT}}{\mathcal{H}_{nT}} < B_{nT}(\underline{c} - c_0) \right\} + P\left\{ \frac{B_{nT}\mathcal{S}_{nT}}{\mathcal{H}_{nT}} > -B_{nT}c_0 \right\}. \end{aligned}$$

Since  $B_{nT}\mathcal{S}_{nT}/\mathcal{H}_{nT} = O_p(1)$ , for given  $\varepsilon > 0$ , we can choose  $\bar{K}$  and  $(n_0, T_0)$  such that

$$P\left\{ \left| \frac{B_{nT}\mathcal{S}_{nT}}{\mathcal{H}_{nT}} \right| > \bar{K} \right\} < \varepsilon$$

for all  $n \geq n_0$  and  $T \geq T_0$  such that  $\log n \leq 2 \log T + \log \varepsilon$ . Choose

$$n_1 = \max\left\{ \left( \frac{\bar{K}}{c_0 - \underline{c}} \right)^2, \left( \frac{\bar{K}}{c_0} \right)^2, n_0 \right\}.$$

Recall that  $B_{nT} = \sqrt{n}$ . Notice by definition that  $n \geq n_1$  implies that  $-B_{nT}c_0 \geq \bar{K}$  and  $B_{nT}(\underline{c} - c_0) \leq -\bar{K}$ . Set  $\log T_1 = \frac{1}{2} \log n_1 - \frac{1}{2} \log \varepsilon$ . Then, whenever  $n \geq n_1$  and  $T \geq T_1$  with  $\log n \leq 2 \log T + \log \varepsilon$ ,

$$(46) \quad \begin{aligned} & P\left\{ \frac{B_{nT}\mathcal{S}_{nT}}{\mathcal{H}_{nT}} < B_{nT}(\underline{c} - c_0) \right\} + P\left\{ \frac{B_{nT}\mathcal{S}_{nT}}{\mathcal{H}_{nT}} > -B_{nT}c_0 \right\} \\ & \leq 2P\left\{ \left| \frac{B_{nT}\mathcal{S}_{nT}}{\mathcal{H}_{nT}} \right| > \bar{K} \right\} \leq 2\varepsilon. \end{aligned}$$

In view of (45) and (46), for any given  $\delta, \varepsilon > 0$ ,

$$P\{|B_{nT}(\hat{c}_q - c_0) - \hat{\phi}_{nT}| > \delta\} \leq 2\varepsilon$$

if  $n \geq n_1$  and  $T \geq T_1$  with  $\log n \leq 2 \log T + \log \varepsilon$ , as required.

*Q.E.D.*

## APPENDIX C: PROOFS OF SECTION 5

PROOF OF LEMMA 7: Part (a). Using (35) with  $c_0 = 0$ , one may write

$$\begin{aligned} & \sqrt{n}M_{1nT}(0) \\ &= \frac{1}{\sqrt{n}} \sum_{i=1}^n \left[ \frac{1}{T} \sum_{t=1}^T \varepsilon_{it} x_{it-1} - \frac{1}{T^2} \sum_{t=1}^T \sum_{s=1}^T \varepsilon_{it} x_{is-1} \tilde{h}_{1T}(t, s) + \sigma^2 \omega_{1T}(0) \right] \\ & \quad + \sqrt{n}(\hat{\sigma}^2 - \sigma^2) \omega_{1T}(0) \\ & \quad + \frac{1}{\sqrt{n}} \sum_{i=1}^n y_{i0} \left[ \left( \frac{1}{T} \sum_{t=1}^T \varepsilon_{it} \right) - \left( \frac{1}{T} \sum_{t=1}^T \varepsilon_{it} \frac{1}{T} \sum_{s=1}^T \tilde{h}_{1T}(t, s) \right) \right]. \end{aligned}$$

By Lemma 2,

$$\sqrt{n}(\hat{\sigma}^2 - \sigma^2) \omega_{1T}(0) = O_p\left(\frac{\sqrt{n}}{T}\right) + O_p\left(\frac{1}{\sqrt{T}}\right) = o_p(1),$$

where the last equality holds under Assumption 5. Also, using similar arguments that yield (42), we may have

$$\frac{1}{\sqrt{n}} \sum_{i=1}^n y_{i0} \left[ \left( \frac{1}{T} \sum_{t=1}^T \varepsilon_{it} \right) - \left( \frac{1}{T} \sum_{t=1}^T \varepsilon_{it} \frac{1}{T} \sum_{s=1}^T \tilde{h}_{1T}(t, s) \right) \right] = O_p\left(\frac{1}{\sqrt{T}}\right).$$

Since  $(n, T \rightarrow \infty)$  with  $(\sqrt{n}/T) \rightarrow 0$  under Assumption 5,

$$\begin{aligned} \sqrt{n}M_{1nT}(0) &= \frac{1}{\sqrt{n}} \sum_{i=1}^n \left[ \frac{1}{T} \sum_{t=1}^T \varepsilon_{it} x_{it-1} - \frac{1}{T^2} \sum_{t=1}^T \sum_{s=1}^T \varepsilon_{it} x_{is-1} \tilde{h}_{1T}(t, s) + \sigma^2 \omega_{1T}(0) \right] \\ & \quad + o_p(1). \end{aligned}$$

The required result follows by the limit of  $(1/\sqrt{n}) \sum_{i=1}^n (Q_{1iT} - Q_{2iT})$  in Lemma 12 with  $c_0 = 0$  and  $p = 1$ . *Q.E.D.*

Part (b). By Lemma 2, we may have

$$(47) \quad \begin{aligned} \sqrt{nd}M_{1nT}(0) &= -\frac{1}{\sqrt{n}} \sum_{i=1}^n \left[ \frac{1}{T^2} \sum_{t=1}^T y_{it-1}^2 - \frac{1}{T^3} \sum_{t=1}^T \sum_{s=1}^T y_{it-1} y_{is-1} \tilde{h}_{1T}(t, s) \right. \\ & \quad \left. - \sigma^2 \left( \frac{1}{T^2} \sum_{t=2}^T \sum_{s=1}^{t-1} \left( \frac{t-s-1}{T} \right) \tilde{h}_{1T}(t, s) \right) \right] \\ & \quad + O_p\left(\frac{\sqrt{n}}{T}\right) + O_p\left(\frac{1}{\sqrt{T}}\right). \end{aligned}$$

Using (35) with  $c_0 = 0$ , i.e.,  $y_{it-1} = x_{it-1} + y_{i0}$ , we write

$$\left[ \frac{1}{T^2} \sum_{t=1}^T y_{it-1}^2 - \frac{1}{T^3} \sum_{t=1}^T \sum_{s=1}^T y_{it-1} y_{is-1} \tilde{h}_{1T}(t, s) \right. \\ \left. - \sigma^2 \left( \frac{1}{T^2} \sum_{t=2}^T \sum_{s=1}^{t-1} \left( \frac{t-s-1}{T} \right) \tilde{h}_{1T}(t, s) \right) \right] = Q_{6iT} + R_{6iT},$$

where

$$\begin{aligned} Q_{6iT} &= \frac{1}{T^2} \sum_{t=1}^T x_{it-1}^2 - \frac{1}{T^3} \sum_{t=1}^T \sum_{s=1}^T x_{it-1} x_{is-1} \tilde{h}_{1T}(t, s) \\ & \quad - \sigma^2 \frac{1}{T^2} \sum_{t=2}^T \sum_{s=1}^{t-1} \left( \frac{t-s-1}{T} \right) \tilde{h}_{1T}(t, s) \end{aligned}$$

and

$$\begin{aligned} R_{6iT} &= \frac{2y_{i0}}{\sqrt{T}} \left( \frac{1}{T\sqrt{T}} \sum_{i=1}^T x_{it-1} \right) + \left( \frac{y_{i0}}{\sqrt{T}} \right)^2 - \frac{2y_{i0}}{\sqrt{T}} \left( \frac{1}{T\sqrt{T}} \sum_{i=1}^T x_{it-1} \left( \frac{1}{T} \sum_{s=1}^T \tilde{h}_{1T}(t, s) \right) \right) \\ &\quad - \left( \frac{y_{i0}}{\sqrt{T}} \right)^2 \left( \frac{1}{T} \sum_{i=1}^T \frac{1}{T} \sum_{s=1}^T \tilde{h}_{1T}(t, s) \right) \\ &= R_{61iT} + R_{62iT} + R_{63iT} + R_{64iT}, \quad \text{say.} \end{aligned}$$

Notice that  $Q_{6iT}$  is mean zero and independent across  $i$  with finite asymptotic variance  $\text{var}(Q_{6iT}) \rightarrow \sigma^4(11/6300)$ . Then, by Theorem 8 with  $C_i = 1$ , we have

$$(48) \quad \frac{1}{\sqrt{n}} \sum_{i=1}^n Q_{6iT} \Rightarrow N\left(0, \sigma^4 \frac{11}{6300}\right).$$

Also, using similar arguments that yield (42), we can show that

$$(49) \quad \frac{1}{\sqrt{n}} \sum_{i=1}^n R_{61iT}, \frac{1}{\sqrt{n}} \sum_{i=1}^n R_{63iT} = O_p\left(\frac{1}{\sqrt{T}}\right),$$

$$(50) \quad \frac{1}{\sqrt{n}} \sum_{i=1}^n R_{62iT}, \frac{1}{\sqrt{n}} \sum_{i=1}^n R_{64iT} = O_p\left(\frac{\sqrt{n}}{T}\right).$$

Then, in view of (47)–(50), we deduce that

$$(51) \quad \sqrt{n}dM_{1nT}(0) \Rightarrow N\left(0, \sigma^4 \frac{11}{6300}\right)$$

as  $(n, T \rightarrow \infty)$  following Assumption 5. *Q.E.D.*

Part (c). Notice that

$$\begin{aligned} \sqrt{n}(d^2 M_{1nT}(0)) &= \hat{\sigma}^2 d^2 \omega_{1T}(0) \\ &= \hat{\sigma}^2 \left( \frac{1}{T^2} \sum_{t=3}^T \sum_{s=1}^{t-2} \left( \frac{t-s-1}{T} \right) \left( \frac{t-s-2}{T} \right) \tilde{h}_{1T}(t, s) \right). \end{aligned}$$

From

$$\sup_{1 \leq t \leq T} \sup_{\frac{t-1}{T} \leq r \leq \frac{t}{T}} \left| \left( \frac{t}{T} \right)^k - r^k \right| = \frac{1}{T} O(1) \quad \text{for all finite } k,$$

we have

$$\frac{1}{T^2} \sum_{t=2}^T \sum_{s=1}^{t-1} \left( \frac{t-s-1}{T} \right) \left( \frac{t-s-2}{T} \right) \tilde{h}_{1T}(t, s) \rightarrow \int_0^1 \int_0^r (r-s)^2 \tilde{h}_1(r, s) ds dr + \frac{1}{T} O(1).$$

Also, a direct calculation shows that

$$\int_0^1 \int_0^r (r-s)^2 \tilde{h}_1(r, s) ds dr = 0.$$

Therefore, since  $\hat{\sigma}^2 \rightarrow_p \sigma^2$  by Lemma 2 and  $(\sqrt{n}/T) \rightarrow 0$  under Assumption 5, we have

$$\sqrt{n}(d^2 M_{1nT}(0)) = O_p\left(\frac{\sqrt{n}}{T}\right) = o_p(1),$$

as required.

*Q.E.D.*

Part (d). By definition,

$$d^3 M_{1nT}(c) = \hat{\sigma}^2 d^3 \omega_{1T}(c),$$

where

$$d^3 \omega_{1T}(c) = \frac{1}{T^2} \sum_{t=4}^T \sum_{s=1}^{t-3} \left(1 + \frac{c}{T}\right)^{t-s-4} \left(\frac{t-s-1}{T}\right) \left(\frac{t-s-2}{T}\right) \left(\frac{t-s-3}{T}\right) \tilde{h}_{1T}(t, s).$$

Then, since  $\hat{\sigma}^2 \rightarrow_p \sigma^2$  and by Lemma 9,

$$d^3 M_{1nT}(c) \rightarrow_p \sigma^2 d^3 M_1(c) = \sigma^2 \int_0^1 \int_0^r e^{c(r-s)} (r-s)^3 \tilde{h}_1(r, s) ds dr$$

uniformly in  $c \in \mathbb{C}$ . Also, a direct calculation shows that

$$d^3 M_1(0) = \int_0^1 \int_0^r (r-s)^3 \tilde{h}_1(r, s) ds dr = -\frac{1}{70},$$

and we have the required result.

*Q.E.D.*

PROOF OF LEMMA 8: Part (a). By definition, we can write

$$\begin{aligned} M_{2nT}(0) &= \frac{1}{n} \sum_{i=1}^n \left( \frac{1}{T} \sum_{t=1}^T \varepsilon_{it} y_{it-1} \right) - \frac{1}{n} \sum_{i=1}^n \left( \frac{1}{\sqrt{T}} \sum_{t=1}^T \varepsilon_{it} \right) \left( \frac{1}{T\sqrt{T}} \sum_{t=1}^T y_{it-1} \right) \\ &\quad - \frac{1}{n} \sum_{i=1}^n \left( \frac{1}{\sqrt{T}} \sum_{t=1}^T \varepsilon_{it} \right) \left( \frac{1}{\sqrt{T}} \sum_{t=1}^T \frac{t-1}{T} \varepsilon_{it} \right) \\ &\quad + \frac{1}{n} \sum_{i=1}^n \left( \frac{1}{\sqrt{T}} \sum_{t=1}^T \varepsilon_{it} \right)^2 \left( \frac{1}{T} \sum_{t=1}^T \frac{t-1}{T} \right) + \hat{\sigma}^2 \lambda_{1T}(0). \end{aligned}$$

Noticing that

$$(52) \quad \frac{1}{T} \sum_{t=1}^T y_{it-1} \varepsilon_{it} = \frac{1}{2T} (y_{iT}^2 - y_{i0}^2) - \frac{1}{2T} \sum_{t=1}^T \varepsilon_{it}^2,$$

$$(53) \quad \frac{1}{\sqrt{T}} \sum_{t=1}^T \frac{t-1}{T} \varepsilon_{it} = \frac{T-1}{T} \frac{y_{iT}}{\sqrt{T}} - \frac{1}{T\sqrt{T}} \sum_{t=1}^T y_{it-1} + \frac{1}{T\sqrt{T}} y_{i0},$$

and

$$(54) \quad \frac{1}{\sqrt{T}} \sum_{t=1}^T \varepsilon_{it} = \frac{y_{iT}}{\sqrt{T}} - \frac{y_{i0}}{\sqrt{T}},$$

one may rearrange

$$\begin{aligned} M_{2nT}(0) &= \frac{1}{2n} \sum_{i=1}^n \left( \frac{y_{iT}}{\sqrt{T}} \right)^2 - \frac{1}{2nT} \sum_{i=1}^n y_{i0}^2 - \frac{1}{2nT} \sum_{i=1}^n \sum_{t=1}^T \varepsilon_{it}^2 \\ &\quad - \left( \frac{T-1}{T} \right) \frac{1}{n} \sum_{i=1}^n \left( \frac{y_{iT}}{\sqrt{T}} \right)^2 + \left( \frac{T-1}{T} \right) \frac{1}{n\sqrt{T}} \sum_{i=1}^n \left( \frac{y_{iT}}{\sqrt{T}} \right) y_{i0} \end{aligned}$$



$$\begin{aligned}
& -\frac{1}{nT\sqrt{T}} \sum_{i=1}^n \left( \frac{y_{iT}}{\sqrt{T}} \right) y_{i0} + \frac{1}{nT^2} \sum_{i=1}^n y_{i0}^2 \\
& + \left( \frac{1}{T} \sum_{t=1}^T \frac{t-1}{T} \right) \frac{1}{n} \sum_{i=1}^n \left( \frac{y_{iT}}{\sqrt{T}} \right)^2 - 2 \left( \frac{1}{T} \sum_{t=1}^T \frac{t-1}{T} \right) \frac{1}{n\sqrt{T}} \sum_{i=1}^n \left( \frac{y_{iT}}{\sqrt{T}} \right) y_{i0} \\
& + \left( \frac{1}{T} \sum_{t=1}^T \frac{t-1}{T} \right) \frac{1}{nT} \sum_{i=1}^n y_{i0}^2 + \hat{\sigma}^2 \lambda_{1T}(0) \\
& = I_1 + \cdots + I_{12}, \quad \text{say.}
\end{aligned}$$

Using Assumption 2 and the results in Lemma 11 and modifying its proof, it is possible to show that

$$\begin{aligned}
I_1 + I_4 + I_8 &= -\frac{1}{T} \left( \frac{1}{n} \sum_{i=1}^n \left( \frac{y_{iT}}{\sqrt{T}} \right)^2 \right) = O_p \left( \frac{1}{T} \right), \\
I_2, I_{10} &= O_p \left( \frac{1}{T} \right), \quad I_5 + I_9 = 0, \\
I_6 &= O_p \left( \frac{1}{T\sqrt{T}} \right), \quad I_7 = O_p \left( \frac{1}{T^2} \right).
\end{aligned}$$

Thus,

$$\begin{aligned}
\sqrt{n}M_{2nT}(0) &= \sqrt{n} \left( -\frac{1}{2nT} \sum_{i=1}^n \sum_{t=1}^T \varepsilon_{it}^2 + \hat{\sigma}^2 \lambda_{1T}(0) \right) \\
&= \sqrt{n} \left( -\frac{1}{2} \tilde{\sigma}^2 + \hat{\sigma}^2 \lambda_{1T}(0) \right) \\
&= -\frac{\sqrt{n}}{2} (\tilde{\sigma}^2 - \hat{\sigma}^2) + \hat{\sigma}^2 \sqrt{n} \left( \lambda_{1T}(0) - \frac{1}{2} \right).
\end{aligned}$$

By Lemma 2,  $-\frac{\sqrt{n}}{2} (\tilde{\sigma}^2 - \hat{\sigma}^2) = O_p(\sqrt{n}/T)$ . Also,  $\sqrt{n}(\lambda_{1T}(0) - \frac{1}{2}) = O(\sqrt{n}/T)$ . Therefore,

$$\sqrt{n}M_{2nT}(0) = O_p \left( \frac{\sqrt{n}}{T} \right) = o_p(1)$$

as  $(n, T \rightarrow \infty)$  following Assumption 5.

*Q.E.D.*

Next, we sketch proofs for Parts (b)–(d). The details of the proofs for Part (b), (c), and (d) are similar to those of Part (b) of Lemma 7, Part (a), and Lemma 3, respectively, and we omit the details.<sup>12</sup>

Parts (b)–(d). Take the first derivative of  $M_{2nT}(c)$  with respect to the parameter  $c$  and evaluating it at  $c = 0$  with  $c_0 = 0$ , apply Lemma 2, and use the relations of (53) and (54). Then, one may

<sup>12</sup>One can obtain detailed derivations of  $dM_{2nT}(0)$ ,  $d^2M_{2nT}(0)$  from the first author upon request.

find that

$$\begin{aligned} & \sqrt{nd}M_{2nT}(0) \\ &= \frac{1}{\sqrt{n}} \sum_{i=1}^n \left[ \left( +2 \frac{x_{iT}}{\sqrt{T}} \left( \frac{1}{T\sqrt{T}} \sum_{t=1}^T \frac{t-1}{T} x_{it-1} \right) - 2\sigma^2 \frac{1}{T} \sum_{t=1}^T \left( \frac{t-1}{T} \right)^2 \right. \right. \\ & \quad \left. \left. - \frac{1}{3} \left( \frac{x_{iT}}{\sqrt{T}} \right)^2 + \frac{1}{3} \sigma^2 \right) \right] + o_p \left( \frac{\sqrt{n}}{T} \right) \\ &= \frac{1}{\sqrt{n}} \sum_{i=1}^n Q_{7iT} + o_p(1), \quad \text{say,} \end{aligned}$$

where  $x_{it}$  is defined in (34) and the  $o_p(1)$  term holds since  $(n, T \rightarrow \infty)$  with  $(\sqrt{n}/T) \rightarrow 0$  under Assumption 5. Direct calculations show that  $E Q_{7iT} = 0$  and  $\text{var}(Q_{7iT}) \rightarrow \sigma^4/45$ . By applying Theorem 8 with  $C_i = 1$ , then one may derive

$$(55) \quad \sqrt{nd}M_{2nT}(0) \Rightarrow N \left( 0, \frac{\sigma^4}{45} \right),$$

as required.

The proof of Part (c) is similar to that of Part (b). Taking the second order derivative of  $M_{2nT}(c)$  with respect to the parameter  $c$  with  $c_0 = 0$ , considering Lemma 2, and rearranging terms using the relations of (52) and (53), it is possible to show that as  $(n, T \rightarrow \infty)$  following Assumption 5,

$$\sqrt{nd}^2 M_{2nT}(0) = O_p \left( \frac{\sqrt{n}}{T} \right) = o_p(1).$$

The proof of Part (d) is similar to the proof of Lemma 3. After taking the third order derivative of  $M_{2nT}(c)$  with respect to  $c$  and using the results in Lemma 11, it is possible to show the required result. *Q.E.D.*

**PROOF OF THEOREM 4:** Define  $\hat{\theta}_{nT} = n^{1/6} \hat{c}$ . First, we consider the case where  $\{|\hat{\theta}_{nT}| > 1\}$ . By the definition of the GMM estimator, we have

$$\begin{aligned} o_p(1) &\leq n(Z_{nT}(0) - Z_{nT}(\hat{c})) \\ &= - \sum_{k=1}^6 (n^{(1-k/6)} \mathcal{A}_{k,nT}) \hat{\theta}_{nT}^k - \sum_{k=3}^6 \hat{\theta}_{nT}^k (n^{(1-k/6)} \mathcal{N}_{k,nT}(\hat{c}, 0)). \end{aligned}$$

In view of (24)–(31) and from Assumption 6,  $\hat{\theta}_{nT}$  satisfies

$$(56) \quad \begin{aligned} o_p(1) &\leq -|\hat{\theta}_{nT}|^6 + |\hat{\theta}_{nT}|^5 o_p(1) + |\hat{\theta}_{nT}|^4 o_p(1) \\ &\quad + 2O_p(1)|\hat{\theta}_{nT}|^3 + |\hat{\theta}_{nT}|^2 o_p(1) + |\hat{\theta}_{nT}| o_p(1). \end{aligned}$$

On the set  $\{|\hat{\theta}_{nT}| > 1\}$

$$\text{RHS of (56)} \leq -|\hat{\theta}_{nT}|^6 (1 + o_p(1)) + 2O_p(1)|\hat{\theta}_{nT}|^3.$$

Then,

$$|\hat{\theta}_{nT}|^6 \leq 2O_p(1)|\hat{\theta}_{nT}|^3 + o_p(1).$$

Following relation (7.4) in Andrews (1999, p. 1377), we can deduce that

$$|\hat{\theta}_{nT}|^3 \leq O_p(1) + o_p(1).$$

Therefore, when  $\{|\hat{\theta}_{nT}| > 1\}$ ,

$$(57) \quad |\hat{\theta}_{nT}| \leq O_p(1).$$

Finally, let the  $O_p(1)$  random variable in (57) be  $\xi_{nT}$ . Then,

$$\begin{aligned} |\hat{\theta}_{nT}| &= |\hat{\theta}_{nT} \mathbf{1}\{|\hat{\theta}_{nT}| \leq 1\} + |\hat{\theta}_{nT} \mathbf{1}\{|\hat{\theta}_{nT}| > 1\}| \\ &\leq |\hat{\theta}_{nT} \mathbf{1}\{|\hat{\theta}_{nT}| \leq 1\} + \xi_{nT} \leq 1 + \xi_{nT} = O_p(1). \end{aligned} \quad Q.E.D.$$

PROOF OF THEOREM 5: The proof of the theorem is similar to that of Theorem 3 and is omitted.

APPENDIX D: NUMERICAL VALIDATION OF THE IDENTIFICATION CONDITION OF  $m(c)$ <sup>13</sup>

This section provides a numerical confirmation that the uniform limit of the moment condition function  $m(c) = (m_1(c), m_2(c))'$  has a zero only at the true parameter  $c = c_0$ . We restrict the parameter set to  $\mathbb{C} = [-10, 0]$  in this numerical exercise. The choice of the lower limit  $\bar{c} = -10$  is made for computational convenience, and the results hold for all finite values of  $\bar{c} < 0$ . All the numerical analysis in this section is done with Mathematica and with Maple using Scientific Workplace Version 3.0.

When  $g_{1t} = t$ .

The procedure we apply is to find all the roots of  $m_2(c) = 0$  and verify whether these roots are also the roots of  $m_1(c) = 0$ . We first notice that for given  $c_0$ , the function  $m_2(c)$  is simply the ratio of two polynomials—the denominator and the numerator of  $m_2(c)$ , say  $m_{d2}(c)$  and  $m_{n2}(c)$ , respectively, are a fourth degree polynomial and a fifth degree polynomial in  $c$ , respectively.<sup>14</sup>

Case A: When  $c_0 \neq 0$ .

Step 1: Numerical Calculation of the roots of  $m_2(c) = 0$ .

By a direct calculation, we find that the denominator of  $m_2(c)$ ,  $m_{d2}(c)$ , equals  $4c_0^5(c^2 - 3c + 3)^2$  when  $c_0 \neq 0$ . Since  $c^2 - 3c + 3 = (c - (3/2))^2 + (3/4) > 0$ , the denominator of  $m_2(c)$  has no real zeros for all  $c_0 \neq 0$ . Thus, if we are concerned with the zeros of  $m_2(c)$ , it suffices to consider only the numerator of  $m_2(c)$ ,  $m_{n2}(c)$ . By definition of  $m_2(c)$ , we find that the true value  $c = c_0$  is always a zero of  $m_{n2}(c)$ . Also, by inspection, we find that  $c = 0$  is always a zero of  $m_{n2}(c)$ . Thus, we can write

$$m_{n2}(c) = c(c - c_0)\tilde{m}_{n2}(c),$$

where  $\tilde{m}_{n2}(c)$  is a third degree polynomial. Using Mathematica, we solve the third degree polynomial  $\tilde{m}_{n2}(c)$  and find three roots of  $\tilde{m}_{n2}(c)$  as a function of the true parameter  $c_0$ . For the numerical calculation we choose  $\bar{c} = -10$ , and so we assume that the parameter set  $\mathbb{C} = [-10, 0]$ .

<sup>13</sup>We are indebted to John Owens for the numerical analysis in this section.

<sup>14</sup>This is verified easily by noticing that  $\dot{g}_{pc}(r)$ ,  $A_p(c)$ ,  $B_p(c)$  are polynomials of  $c$ , except for the last term in  $m_2(c)$ . However, a direct calculation shows that the last term is also a ratio of two polynomials, viz.,

$$\begin{aligned} &\int_0^1 \int_0^r e^{c(r-s)} \dot{g}_{pc}(s)' A_p(c)^{-1} \dot{g}_{pc}(r) ds dr \\ &= \int_0^1 \int_0^r e^{c(r-s)} (1 - cs) \left(1 - c + \frac{1}{3}c^2\right)^{-1} (1 - cr) ds dr = -\frac{1}{2} \frac{2c - 3}{3 - 3c + c^2}. \end{aligned}$$

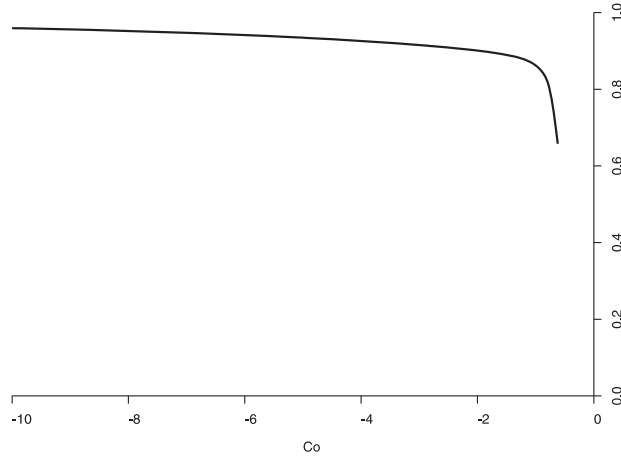


FIGURE A.1.—Graph of roots of  $\tilde{m}_{n2}(c)$ .

Figures A.1 and A.2 plot the graphs of these roots on  $\mathbb{C}$  only when the roots are real numbers (Figure A.2 shows the graph on a finer scale to the left of the origin). As we see from the graphs, for  $c_0 < 0$ , the roots of  $\tilde{m}_{n2}(c)$  are all positive, and so  $\tilde{m}_{n2}(c)$  does not have a root in the parameter set  $\mathbb{C}$ .

*Step 2:* Plug the root  $c = 0$  of  $m_2(c)$  in  $m_1(c)$ . We now investigate, for given  $c_0 \in \mathbb{C} \setminus \{0\}$ , whether  $m_1(c) = 0$  when  $c = 0$ . By matching the given true parameter  $c_0$  with  $m_1(0)$ , we can define the function  $m_{1\_0}(c_0)$  of  $c_0$ . Using Maple, we calculate

$$m_{1\_0}(c_0) = \frac{1}{4c^4} \left( \begin{array}{l} -c_0^3 + 48e^{c_0} - 8e^{c_0}c_0^2 - 8c_0^2 - 24 \\ +c_0^3e^{2c_0} - 8e^{2c_0}c_0^2 + 24ce^{2c_0} - 24e^{2c_0} - 24c_0 \end{array} \right),$$

and plot the graph of  $m_{1\_0}(c_0)$ . Figure A.3 plots  $m_{1\_0}(c_0)$  over the domain  $c_0 \in [-10, .4]$  and

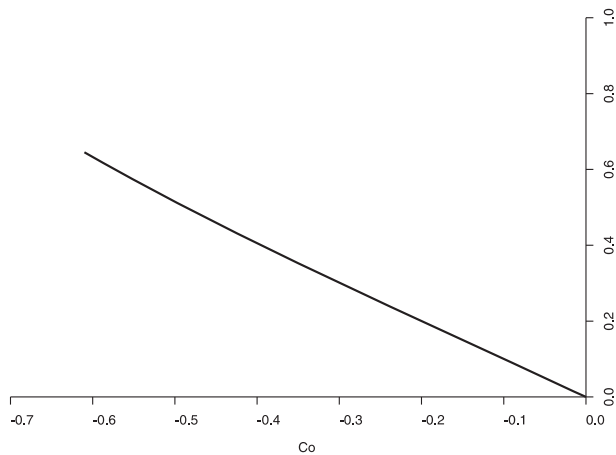


FIGURE A.2.—Graph of roots of  $\tilde{m}_{n2}(c)$ .

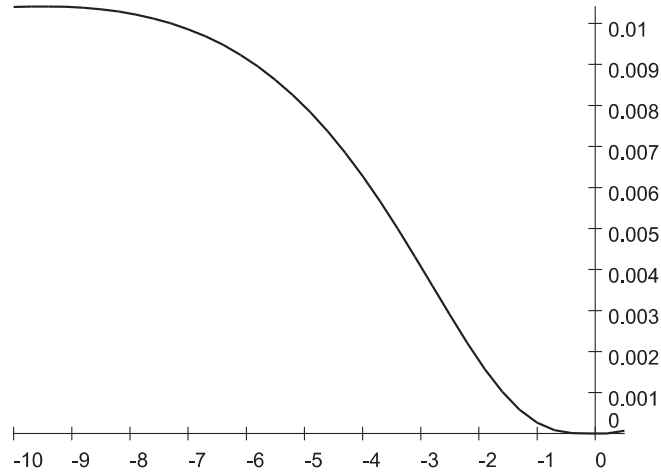


FIGURE A.3.—Graph of  $m_{1_0}(c_0)$ .

Figure A.4 plots the same function on the domain  $c_0 \in [.4, 0]$ . Through these graphs, we can confirm that  $m_{1_0}(c_0)$  is positive but very close to zero when the true value  $c_0$  is close to zero.

To investigate further the behavior of  $m_{1_0}(c_0)$  around  $c_0 = 0$ , in Figure A.5 we plot the graph of the first derivative of the numerator of  $m_{1_0}(c_0)$  over  $c_0 \in [-.05, 0]$ .

The graph shows that the first derivative of the numerator of  $m_{1_0}(c_0)$  is negative around zero, and so  $m_{1_0}(c_0)$  is strictly decreasing. Therefore, we conclude that  $m_{1_0}(c_0)$  is not zero for all  $c_0 \in \mathbb{C}_0$ .

*Case B:* When  $c_0 = 0$ . Using Maple, we calculate  $m_2(c)$  when  $c_0 = 0$ , and plot the graph in Figures A.6 and A.7. These figures confirm that  $m_2(c) = 0$  only when  $c = c_0 = 0$ .

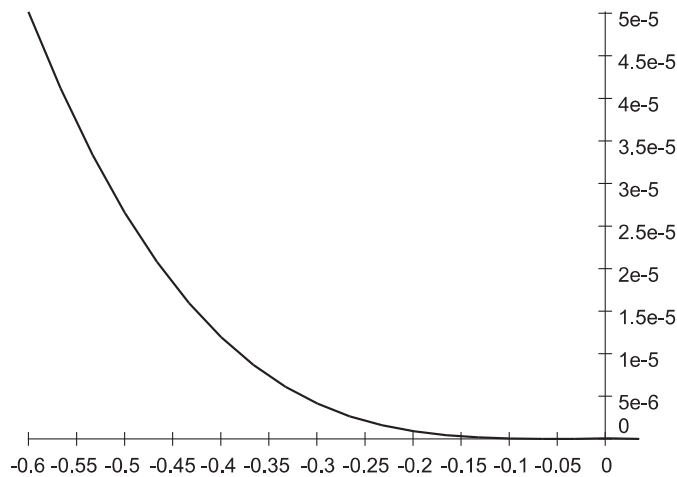


FIGURE A.4.—Graph of  $m_{1_0}(c_0)$ .

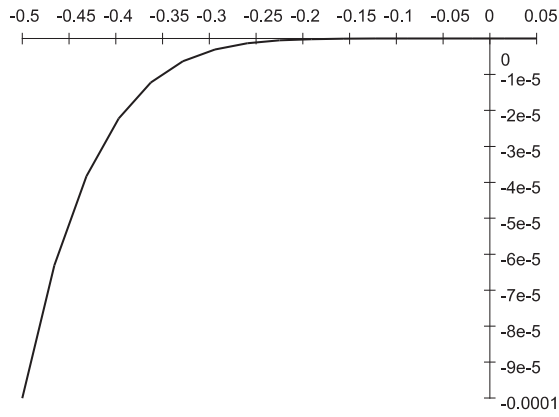


FIGURE A.5.—Graph of the first derivative of the numerator of  $m_{1_0}(c_0)$ .

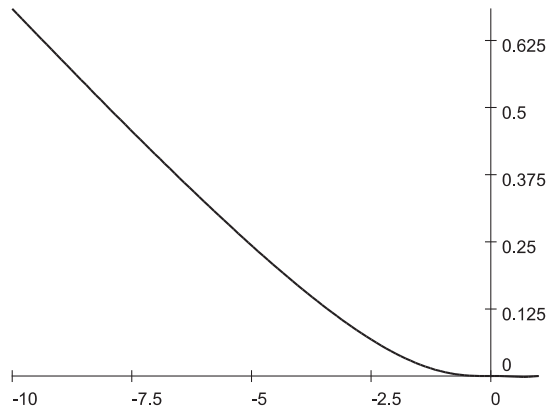


FIGURE A.6.—Graph of  $m_2(c_0)$  when  $c_0 = 0$ .

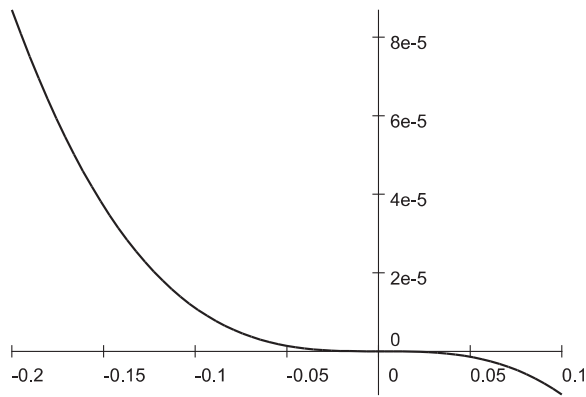


FIGURE A.7.—Graph of  $m_2(c_0)$  when  $c_0 = 0$ .

When  $g_{2t} = (t, t^2)$ .

Although the expressions involved in  $m_2(c)$  in this case are far more complex, the analysis is simpler. Like the case of  $g_{1t} = t$ , we find that the denominator of  $m_2(c)$  does not change sign over  $\mathbb{C} = [-10, 0]$ , and so we focus on the numerator of  $m_2(c)$ . Similarly to the case of  $g_{1t} = t$ , we numerically calculate the real roots of the numerator of  $m_2(c)$  for  $c_0 \in \mathbb{C} = [-10, 0]$ , and we find that there exists only one root in the range of  $c_0$ , which implies that  $m_2(c) = 0$  only at the true  $c_0$ . Therefore, when  $g_{2t} = (t, t^2)$ , the limit of the moment condition  $m(c)$  identifies the true parameter  $c_0$  in  $\mathbb{C}$ .

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