NEW UNIT ROOT ASYMPTOTICS IN THE PRESENCE OF DETERMINISTIC TRENDS

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New unit root asymptotics in the presence of deterministic trends[☆]

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Abstract

Recent work by Phillips (Econometrica 66 (1998) 1299) has shown that stochastic trends can be validly represented in empirical regressions in terms of deterministic functions of time. These representations offer an alternative mechanism for modelling stochastic trends. It is shown here that the alternate representations affect the asymptotics of all commonly used unit root tests in the presence of trends. In particular, the critical values of unit root tests diverge when the number of deterministic regressors $K \to \infty$ as the sample size $n \to \infty$. When they are appropriately recentered and standardized, unit root limit distributions are shown to be normal as $K \to \infty$.

JEL classification: C22

Keywords: Deterministic trends; Divergent critical values; Large K asymptotics; Normal limit distribution; Unit root distribution

1. Introduction

Since Nelson and Plosser (1982) there has been a vast amount of empirical work concerned with the issue of testing difference stationarity against trend stationarity. In constructing such tests it is now common empirical practice to work with a general maintained hypothesis embodying alternative specifications to a unit root model that include a variety of deterministic trends and trend break functions. The latter offer

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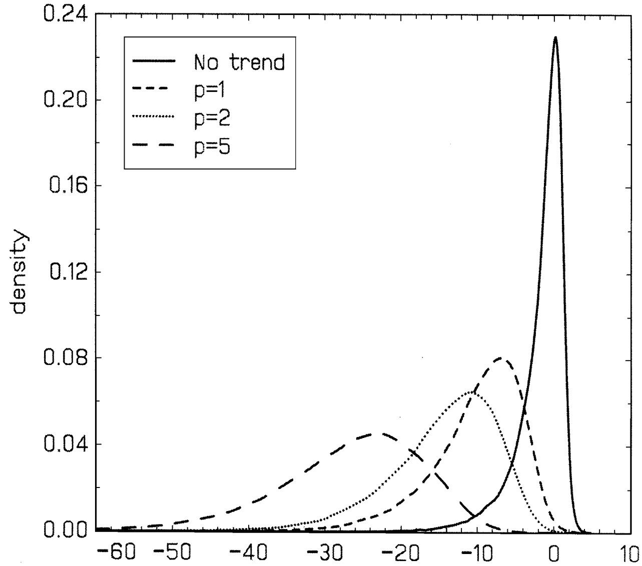

Fig. 1. Densities of $\int_0^1 W_X \, dW / \int_0^1 W_X^2$ for [1] $W_X = W$ and for $X' = (1, r, \ldots, r^p)$.

some interesting alternative explanations of data nonstationarity in terms of structural shifts. As is now well understood, the presence of such deterministic functions in the regression affects the asymptotic distribution of all the usual statistical tests for a unit root and does so under both null and local alternative hypotheses. This means, of course, that the critical values of the tests change with the specification of the deterministic trend functions, necessitating the use of different statistical tables according to the precise specification of the fitted model. Fig. 1 shows the asymptotic distributions of the coefficient-based test in regressions with no trend and with polynomial trends of degrees $p = 1, 2, 5$. [1] Clearly, there is substantial sensitivity in the distribution as the trend degree changes.

In a recent paper, Phillips (1998) has shown that deterministic trend specifications are not necessarily alternatives to a unit root model at all. More precisely, unit root processes have limiting forms in terms of Brownian motion and continuous stochastic processes such as Brownian motion have valid mathematical representations entirely in terms of deterministic functions. It is therefore possible to model a unit root process in the limit with an R^2 of unity by regression on deterministic trends. This result would appear to have certain implications for unit root modelling and testing. In particular, it indicates that one might mistakenly 'reject' a unit root model in favour of a trend 'alternative' when in fact the alternative model is nothing other than an alternate representation of the unit root process itself. Looking at Fig. 1, it is apparent that the limit distribution changes in shape (becoming more like a normal) and changes in location

[1] In this figure, W is standard Brownian motion and W_X is the L_2 projection residual of W on X.

Nomenclature

$\to_{\text{a.s.}}$	almost sure convergence
$\to_{\text{p}}$	convergence in probability
$=_{\text{d}}, \equiv$	distributional equivalence
$:=$	definitional equality
$W(r)$	standard Brownian motion
$BM(\sigma^2)$	Brownian motion with variance σ^2
$\Rightarrow, \to_{\text{d}}$	weak convergence
$[\cdot]$	integer part of
$r \wedge s$	$\min(r,s)$
$\sim$	asymptotic equivalence
$o_p(1)$	tends to zero in probability
$o_{a.s.}(1)$	tends to zero almost surely

(shifting into the left tail) as p increases. In consequence, if the approach to modelling the time series were such that one contemplated increasing p as the number (n) of sample observations increased, and to continue to do so as $n \to \infty$, then it would appear that a limit theory in which $p \to \infty$ when $n \to \infty$ may be more appropriate.

The purpose of the present paper is to make the heuristic discussion in the last paragraph a little more precise. The paper is organized as follows. Section 2 gives the background needed for the present development. Section 3 gives some preliminary theory and a main result for unit root asymptotics when the number of deterministic regressors (K) tends to infinity. Section 4 shows how to derive joint limit theory for a unit root autoregression when both the sample size (n) and K tend to infinity under the side condition that $K/n \to 0$. Section 5 concludes, offers some interpretations, and discusses some of the implications of the theory for applied work. Proofs are collected together in Section 6 and notational conventions are summarized in Section 7.

2. Background asymptotics

The development in this paper will concentrate on a unit root time series $y_t = \Sigma_1^t u_s$, whose increments u_t form a stationary time series with zero mean, finite absolute moments to order $p > 2$, and long run variance $\sigma^2 > 0$, and which satisfies the functional law

$$\frac{y_{[n\cdot]}}{\sqrt{n}} \Rightarrow B(\cdot) \equiv BM(\sigma^2), \tag{1}$$

for which primitive conditions are well known (e.g., see Phillips and Solo, 1992). It is convenient also to use the Hungarian strong approximation (e.g. Csörgõ and Horváth, 1993) to y_t, according to which we can construct an expanded probability space with a

Brownian motion $B(\cdot)$ for which

$$\sup_{0\leqslant k\leqslant n} |y_k - B(k)| = \mathrm{o}_{\mathrm{a.s.}}(n^{1/p})$$

or

$$\sup_{0\leqslant k\leqslant n} \left| \frac{y_k}{\sqrt{n}} - B\left(\frac{k}{n}\right) \right| = \mathrm{o}_{\mathrm{a.s.}}\left(\frac{1}{n^{1/2-1/p}}\right). \tag{2}$$

This gives the direct representation

$$\frac{y_{t-1}}{\sqrt{n}} = B\left(\frac{[nr]}{n}\right) + \mathrm{o}_{\mathrm{a.s.}}(1) \tag{3}$$

for $(t-1)/n \leqslant r < t/n,\ t > 1$. In what follows, we will assume that the space has been expanded as necessary in order for (3) to apply.

Phillips (1998) studied the asymptotic properties of regressions of y_t on deterministic regressors of the type

$$y_t = \sum_{k=1}^{K} \hat{b}_k \varphi_k\left(\frac{t}{n}\right) + \hat{u}_t, \tag{4}$$

or, equivalently (with $\hat{a}_k = n^{-1/2}\hat{b}_k$),

$$\frac{y_t}{\sqrt{n}} = \sum_{k=1}^{K} \hat{a}_k \varphi_k\left(\frac{t}{n}\right) + \frac{\hat{u}_t}{\sqrt{n}}, \tag{5}$$

when the regressors φ_k are the eigenfunctions of the covariance kernel, $\sigma^2 r \wedge s$, of the Brownian motion B. These functions have the form

$$\varphi_k(r) = \sqrt{2}\sin[(k-1/2)\pi r] \tag{6}$$

and constitute a complete orthonormal system in $L_2[0,1]$. When combined with the eigenvalues

$$\lambda_k = \frac{4}{(2k-1)^2\pi^2}$$

of the covariance kernel, these functions deliver an orthonormal representation of the Brownian motion B, viz.

$$B(r) = \sqrt{2}\sum_{k=1}^{\infty} \frac{\sin[(k-1/2)\pi r]}{(k-1/2)\pi}\xi_k = \sum_{k=1}^{\infty} \lambda_k^{1/2}\varphi_k(r)\xi_k, \tag{7}$$

where the components ξ_k are independently and identically distributed (iid) as $\mathrm{N}(0,\sigma^2)$. This series representation of $B(r)$ is convergent almost surely and uniformly in $r \in [0,1]$. Let ξ_K, and $\varphi_K(r)$ be K-vectors of the first K elements of $\{\xi_k\}$ and $\{\varphi_k(r)\}$, respectively, and $\xi_\perp$, and $\varphi_\perp(r)$ be vectors of the remaining elements of these sequences. Then, we may write (7) as

$$B(r) = \varphi_K(r)'\Lambda_K^{1/2}\xi_K + \varphi_\perp(r)'\Lambda_\perp^{1/2}\xi_\perp, \tag{8}$$

where $\Lambda_K = diag(\lambda_1, \ldots, \lambda_K)$ and $\Lambda_\perp = diag(\lambda_{K+1}, \lambda_{K+2}, \ldots)$. Note that the coefficient of the deterministic function $\varphi_k(r)$ in (7) is of order $O_p(1/k)$, so that the functions in the representation become less important as k gets large.

The reader is referred e.g. to Shorack and Wellner (1986) for more details on orthonormal representations of stochastic processes and to Phillips (1998) for further discussion of (7) and some related representations. One particular representation of interest has the form

$$B(r) = r\xi_0 + \sqrt{2}\sum_{k=1}^{\infty} \frac{\sin(k\pi r)}{k\pi}\xi_k \tag{9}$$

with

$$\xi_0 = B(1), \quad \xi_k = \sqrt{2}\int_0^1 \frac{\sin(k\pi s)}{k\pi}(B(s) - sB(1))\,\mathrm{d}s$$

and where, again, the ξ_k are iid $N(0, \sigma^2)$ and (9) converges almost surely and uniformly for $r \in [0,1]$. The representation (9) has a linear trend component with the random coefficient $\xi_0 = B(1)$ and shows that $B(r)$ can be written in terms of low-order polynomials and sinusoidal functions. Just as (4) is an empirical version of (7), the regression of y_t on a linear trend produces an empirical analogue of (9)—see Phillips (1996a).

Let $\hat{a}_K = (\hat{a}_k)$ be the coefficients and $\varphi_{Kt} = (\varphi_k(t/n))$ be the K-vector of regressors in (5). Let $c_K \in \mathbb{R}^K$ be any vector with $c_K' c_K = 1$, $t_{c_K'\hat{a}_K}$ be the usual least squares regression t-ratio for the linear combination of coefficients $c_K' a_K$, and let R^2 be the regression coefficient of determination. The following two results come in large part from Phillips (1998) and give the asymptotic properties of these statistics when K is fixed and when $K \to \infty$. Lemma 2.1 extends some of the early work on spurious regressions contained in Phillips (1986) and Durlauf and Phillips (1988). Lemma 2.2 deals with complete limit representations and shows that the empirical regression (5) succeeds in reproducing the entire L_2 orthonormal representation (i.e. (7) above) of $B(\cdot)$ when $K \to \infty$ as $n \to \infty$ provided that $K/n \to 0$.

Lemma 2.1. *For fixed K, as $n \to \infty$ we have*:

(a) $c_K'\hat{a}_K \Rightarrow c_K'[\int_0^1 \varphi_K B] \overset{d}{=} N(0, c_K'\Lambda_K c_K)$,
(b) $n^{-2}\sum_{t=1}^n \hat{u}_t^2 \Rightarrow \int_0^1 B_{\varphi_K}^2$,
(c) $n^{-1/2} t_{c_K'\hat{a}_K} \Rightarrow c_K'[\int_0^1 \varphi_K B] / \left(\int_0^1 B_{\varphi_K}^2\right)^{1/2}$,
(d) $R^2 \Rightarrow 1 - \int_0^1 B_{\varphi_K}^2 / \int_0^1 B^2$,

where $B_{\varphi_K}(\cdot) = B(\cdot) - \left(\int_0^1 B\varphi_K'\right)\left(\int_0^1 \varphi_K\varphi_K'\right)^{-1}\varphi_K(\cdot)$ *is the* L_2 *-projection residual of* B *on* φ_K, $\Lambda_K = diag(\lambda_1, \ldots, \lambda_K)$, *and* λ_k *is the eigenvalue of the covariance function* $\sigma^2 r \wedge s$ *corresponding to* φ_k.

Lemma 2.2. *As $K \to \infty$, $c_K' \Lambda_K c_K$ tends to the positive constant $\sigma_c^2 = c' \Lambda c$, where $c = (c_k)$, $\Lambda = diag(\lambda_1, \lambda_2, \dots)$ and $c'c = 1$. Moreover, if $K \to \infty$ and $K/n \to 0$ as $n \to \infty$, we have*:

(a) $n^{-1} \sum_{t=1}^{n} \varphi_{Kt} \varphi_{Kt}' = I_K + \mathrm{O}(K/n)$;
(b) $\hat{a}_K = \Lambda_K^{1/2} \xi_K + \mathrm{O}_{\mathrm{a.s.}}(K/n + 1/n^{1/2-1/p})$;
(c) $c_K' \hat{a}_K \Rightarrow \mathrm{N}(0, \sigma_c^2)$;
(d) $n^{-2} \sum_{t=1}^{n} \hat{u}_t^2 \xrightarrow{\mathrm{p}} 0$,
(e) $n^{-1/2} t_{c_K' \hat{a}_K}$ *diverges*,
(f) $R^2 \xrightarrow{\mathrm{p}} 1$.

Remark 2.3. (a) In Lemma 2.2, the condition $K/n \to 0$ ensures that the matrix $n^{-1} \sum_{t=1}^{n} \varphi_{Kt} \varphi_{Kt}'$ is positive definite and that, as $n \to \infty$, it differs from the matrix

$$\int_0^1 \varphi_K(s) \varphi_K(s)' \mathrm{d}s = I_K,$$

where $\varphi_K(s) = (\varphi_k(s))$ by a term of $\mathrm{O}(K/n) = \mathrm{o}(1)$ as $n \to \infty$.

(b) Part (b) of Lemma 2.2 shows that the regression coefficients $\hat{a}_K$ in (5) differ from the variates $\eta_K = \Lambda_K^{1/2} \xi_K$ that appear in the orthonormal representation (8) by a term that is $\mathrm{o}_{\mathrm{a.s.}}(1)$ as $n \to \infty$. Hence, the regression coefficients $\hat{a}_K$ have components that are asymptotically independent as $n \to \infty$.

(c) As discussed in Phillips (1998), the divergence of the t-ratio $t_{c_K' \hat{a}_K}$ confirms that the coefficients of the deterministic regressors will inevitably be deemed significant as $n \to \infty$. This divergence also applies, but at a slightly reduced rate, when robust standard errors are used in the construction of the t-ratio (Phillips, 1998).

(d) Since $R^2 \xrightarrow{\mathrm{p}} 1$, the empirical regression successfully reproduces the full orthonormal representation of the limit Brownian motion corresponding to the dependent variable y_t. This outcome also applies to regressions on linearly independent deterministic functions other than the orthonormal set $\{\varphi_k\}$. Thus, modelling of stochastic trends by deterministic functions will inevitably be successful in large samples of data in the sense that the alternate representations in terms of these functions will be confirmed in statistical testing.

3. Main results

This section extends the analysis of regressions of form (5) by the inclusion of a lagged dependent variable in the regression. The model conforms to the usual setting for testing the presence of a unit root against trend stationarity. Thus, we consider the typical autoregression with trend equation

$$y_t = \hat{\rho} y_{t-1} + \sum_{k=1}^{K} \hat{b}_k \varphi_k \left(\frac{t}{n} \right) + \hat{u}_{t,K}, \tag{10}$$

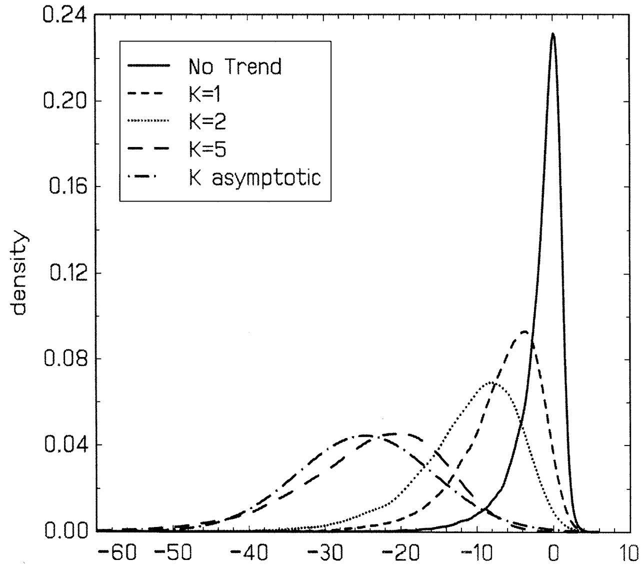

Fig. 2. Unit root densities $\int_0^1 W_{\varphi K}\, \mathrm{d}W / \int_0^1 W_{\varphi K}^2$ for $W_{\varphi K} = W$ and for $K = 1, 2, 5$.

which we write in observation vector form as

$$y = \hat{\rho} y_{-1} + \Phi_K \hat{b}_K + \hat{u}_K.$$

In the case of the augmented Dickey Fuller (*ADF*) test, the equation would be augmented with lagged differences. Our focus of interest will be the limit behavior of coefficient based and t-ratio based unit root tests. At a substantial level of generality regarding the increments u_t, semiparametric Z tests (Phillips, 1987; Phillips and Perron, 1988; Ouliaris et al., 1989) and *ADF* tests (Said and Dickey, 1984; Xiao and Phillips, 1998) have the same limit distributions. In particular, the coefficient tests behave as

$$Z_\rho, ADF_\rho \Rightarrow \frac{\int_0^1 W_{\varphi_K}\, \mathrm{d}W}{\int_0^1 W_{\varphi_K}^2} \tag{11}$$

and the t-ratio tests as

$$Z_t, ADF_t \Rightarrow \frac{\int_0^1 W_{\varphi_K}\, \mathrm{d}W}{\left(\int_0^1 W_{\varphi_K}^2\right)^{1/2}}, \tag{12}$$

where $W_{\varphi_K}(\cdot) = W(\cdot) - \left(\int_0^1 W\varphi_K'\right)\left(\int_0^1 \varphi_K \varphi_K'\right)^{-1} \varphi_K(\cdot)$ is the L_2-projection residual of W on φ_K and where W is standard Brownian motion.

The limit distribution (11) is shown in Fig. 2 for a selection of values of K. The situation is analogous to that of Fig. 1, which shows a corresponding selection for the case where φ_K is a polynomial of degree of K. In both cases, the limit distributions are highly sensitive to the inclusion of additional deterministic regressors. Interestingly, the distributions shown in Figs. 1 and 2 are very similar even though the deterministic regressors are quite different.

Our purpose now is to find the limit form of these distributions as $K \to \infty$. Our analysis will first use sequential limit theory, in which we consider the limit behavior of (11) and (12) as $K \to \infty$. This is equivalent to taking limits as $n \to \infty$, followed by $K \to \infty$, which we denote as $(n, K \to \infty)_{\text{seq}}$. Subsequently, we will show that the same results apply under the more general framework of joint limits whereby $n, K \to \infty$ simultaneously, which we denote as $(n, K \to \infty)$, under the condition that $K/n \to 0$. A general approach to multi-index asymptotics has been developed recently in Phillips and Moon (1999) which gives some useful background theory. In particular, this reference provides some conditions under which sequential and joint limit behavior is the same. Unfortunately, the theory in Phillips and Moon (1999) cannot be applied directly here because the multi-indexed random quantities are not constituted from panels with iid cross-section observations, as they are in that paper. Section 4 therefore provides some alternative limit theory that applies in the present case where the data is not multidimensional but involves two indexes which tend to infinity as $(n, K) \to \infty$.

The following two lemmas characterize the limit behavior, as $K \to \infty$, of the Brownian functionals that appear in the numerator and denominator of the unit root distributions (11) and (12).

Lemma 3.1. *As* $K \to \infty$:

(a) $\int_0^1 W_{\varphi_K}\, \mathrm{d}W \to_{\mathrm{p}} -\frac{1}{2}$;
(b) $K \int_0^1 W_{\varphi_K}^2 \to_{\mathrm{p}} \frac{1}{\pi^2}$.

Lemma 3.2. *As* $K \to \infty$:

(a) $\int_0^1 W_{\varphi_K}\, \mathrm{d}W + \dfrac{1}{2} = O_{\mathrm{p}}\left(\dfrac{\sqrt{\ln K}}{K}\right)$;
(b) $\sqrt{K}\left(K \int_0^1 W_{\varphi_K}^2 - \dfrac{1}{\pi^2}\right) \Rightarrow \frac{1}{\pi^2} \mathrm{N}\left(0, \dfrac{2}{3}\right)$.

The limit behavior of the unit root test statistics now follows directly and is given in the next result.

Theorem 3.3. *As* $K \to \infty$:

(a) $\dfrac{\int_0^1 W_{\varphi_K}\, \mathrm{d}W}{\int_0^1 W_{\varphi_K}^2} \sim -\dfrac{\pi^2 K}{2}, \quad \dfrac{\int_0^1 W_{\varphi_K}\, \mathrm{d}W}{\left(\int_0^1 W_{\varphi_K}^2\right)^{1/2}} \sim -\dfrac{\pi\sqrt{K}}{2}$;
(b) $\dfrac{1}{\sqrt{K}}\left(\dfrac{\int_0^1 W_{\varphi_K}\, \mathrm{d}W}{\int_0^1 W_{\varphi_K}^2} + \dfrac{\pi^2}{2}K\right) \Rightarrow \mathrm{N}(0, \frac{1}{6}\pi^4)$;
(c) $\left(\dfrac{\int_0^1 W_{\varphi_K}\, \mathrm{d}W}{\left(\int_0^1 W_{\varphi_K}^2\right)^{1/2}} + \dfrac{\pi}{2}\sqrt{K}\right) \Rightarrow \mathrm{N}\left(0, \dfrac{\pi^2}{24}\right)$.

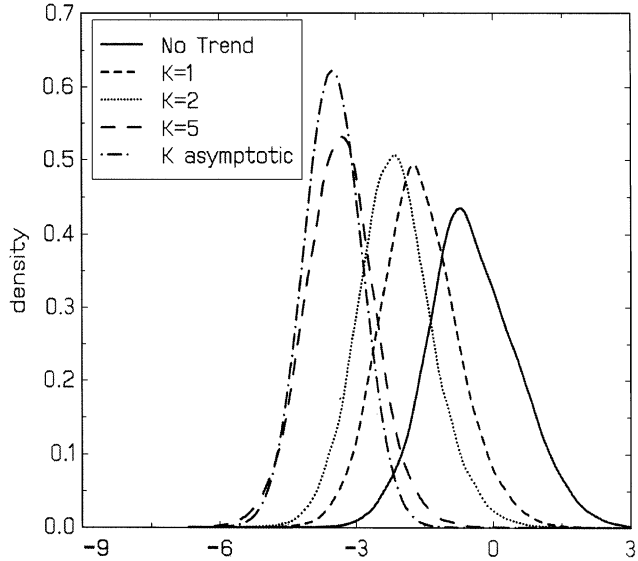

Fig. 3. Unit root t-ratio densities $\int_0^1 W_{\varphi K}\,\mathrm{d}W/(\int_0^1 W_{\varphi K}^2)^{1/2}$ for $W_{\varphi K} = W$ and for $K = 1, 2, 5$.

Discussion 3.4. (a) Both the coefficient and t-ratio forms of the unit root limit distributions diverge to $-\infty$ as $K \to \infty$. The critical values of these distributions that are used in statistical tests of unit root distributions also diverge.

(b) Apparently, the limiting forms of both the coefficient and t-ratio forms of the unit root distributions are normal as $K \to \infty$ when appropriately centered and scaled. The coefficient limit theory shown in part (b) of Theorem 3.3 indicates that scaling by $1/\sqrt{K}$ as well as recentering is required to achieve a well defined limit distribution. Part (c) indicates that only recentering of the t-ratio limit theory is required. Thus, the t-ratio test statistic is appropriately scaled, but diverges to minus infinity as $K \to \infty$.

(c) Both the limit normal distributions for the coefficient and t-ratio cases are corroborated in the numerical results shown in Figs. 2 and 3. Each of these figures shows the new limiting (large K asymptotic) normal approximation that applies for $K = 5$. In particular, the curve in Fig. 2 corresponding to 'K asymptotic' gives the Gaussian approximation $\mathrm{N}(-\pi^2 K, (\pi^4/6)K)$ with $K = 5$ for the coefficient test distribution; similarly, the curve in Fig. 3 corresponding to 'K asymptotic' gives the Gaussian approximation $\mathrm{N}(-(\pi/2)\sqrt{K}, \pi^2/24)$ with $K=5$ for the t-test distribution. The approximations are surprisingly good for such a small value of K.

4. Joint limit theory as (n, $K \to \infty$)

We now explore the conditions under which the main results above apply when $K \to \infty$ as $n \to \infty$. Our approach is to show that the sequential limit results as

$(n,K\to\infty)_{\text{seq}}$ that are used in the earlier derivations hold also for joint limits as $(n,K\to\infty)$ provided that the condition $K/n\to 0$ holds. We will confine our attention here to extending Lemma 3.1 and the divergence result given in Theorem 3.3(a).

We start by noting (see Phillips and Moon, 1999) that a multi-indexed sequence $X_{K,n}$ converges in probability jointly to X, written $X_{K,n}\to_{\text{p}}X$ as $(n,K\to\infty)$, if

$$\lim_{n,K\to\infty} P\{\|X_{K,n}-X\|>\varepsilon\}=0 \quad \forall\varepsilon>0. \tag{13}$$

To establish (13) it is sufficient to show that as $(n,K\to\infty)$

$$\text{E}\|X_{K,n}-X\|^2\to 0.$$

Using this approach, we can establish the joint limits in probability for the component statistics of unit root tests arising from the regression (10). The two statistics of primary interest are: (i) the residual moment matrix

$$y_{-1}'Q_K y_{-1}, \quad Q_K=I-\Phi_K(\Phi_K'\Phi_K)^{-1}\Phi_K',$$

where

$$\Phi_K'=(\varphi_{K1},\ldots,\varphi_{Kn}),$$

and $\varphi_{Kt}=(\varphi_k(t/n))$ as before; and (ii) the sample covariance $y_{-1}'Q_K u$.

We will look at the leading case where u_t is iid $\text{N}(0,\sigma^2)$ so that we need not have to be concerned with serial correlation corrections in the analysis that follows. The normality assumption simplifies the derivation but is not essential and could probably be replaced by a fourth-moment condition, although we have not done that analysis. The results for the more general case can be expected to follow in a similar way, albeit with much more complex derivations that allow for the form of the parametric or nonparametric serial correlation corrections used in the test statistics. Also, in Lemma 4.1, the rate condition $K^4/n\to 0$ is used in proving the lemma. This rate places a stronger restriction on the allowable expansion path for K than the requirement $K/n\to 0$ that is used elsewhere in the paper. It is mainly the result of the line of proof being used and it seems likely that it is stronger than is needed.

The limiting forms of the main statistics are given in the following result.

Lemma 4.1. *As* $(n,K\to\infty)$ *with* $K^4/n\to 0$, *we have*:

(a) $\text{E}(\frac{K}{n^2}y_{-1}'Q_K y_{-1})=\frac{\sigma^2}{\pi^2}+\text{O}(\frac{K^4}{n}+\frac{1}{K})$;
(b) $\text{E}(\frac{1}{n}y_{-1}'Q_K u)=-\frac{\sigma^2}{2}+\text{O}(\frac{K^3}{n}+\frac{1}{K})$;
(c) $\frac{K}{n^2}y_{-1}'Q_K y_{-1}\to_{\text{p}}\frac{\sigma^2}{\pi^2}$;
(d) $\frac{1}{n}y_{-1}'Q_K u\to_{\text{p}}-\frac{\sigma^2}{2}$.

The limit behavior of the unit root test statistics now follows directly and we give below the analogue of Theorem 3.3 (a).

Theorem 4.2. *If* $(n,K\to\infty)$ *and* $K^4/n\to 0$, *then*

$$Z_\rho, ADF_\rho\sim-\frac{\pi^2 K}{2}, \quad Z_t, ADF_t\sim-\frac{\pi\sqrt{K}}{2}.$$

Discussion 4.3. This limit theory is obviously very different from that of the conventional Z and ADF asymptotics for fixed K. Note, however, that if $\hat{\rho}$ is the fitted coefficient of the lagged dependent variable, as in the empirical regression (10), then the theorem implies that $\hat{\rho} = 1 - (\pi^2/2)(K/n) + o_p(\frac{K}{n}) \to_p 1$ as $(n, K \to \infty)$. Thus, $\hat{\rho}$ is still consistent for $\rho = 1$, but has a slower rate of approach than when K is fixed.

5. Conclusions

Earlier work by Phillips (1998) showed that a serious attempt to model a stochastic trend in terms of deterministic functions will always be successful, and is capable of producing an R^2 of unity in the limit. The present contribution shows that this outcome remains true when a lagged dependent variable is present in the regression. In consequence, deterministic functions and lagged variables are seen to jointly compete for the explanation of a stochastic trend in a time series. In such a competition, the results confirm that the deterministic functions will continue to be successful in modelling the trend even in the presence of an autoregression. The net effect of including K deterministic functions in the regression is that the rate of convergence to unity of the autoregressive coefficient $\hat{\rho}$ is slowed down. The explanation for the nonstationarity in the data is then shared between the deterministic regressors and the lagged dependent variable.

One way of interpreting these asymptotic results is as follows. The more serious is the attempt to model a stochastic trend by deterministic functions then the more successful it will be. It is important to recognize that careful design of a deterministic trend function, for any given realization of a time series, is certain to lead to a good deal of the low-frequency variation in the series being explained. Examples of such careful deterministic trend modelling abound in recent empirical work, especially in the application of models with breaking trends. When the data is inspected and a trend model is designed carefully to ensure a small number of fitted parameters yet still capture the essential features of the trending behavior, it may be viewed as a parsimonious form of a more general model for trends involving a larger number, K, of agnostic orthonormal regressors like φ_k in (10). In such cases, the actual K in the fitted regression may understate the number of regressors in the underlying maintained hypothesis from which the selected model is drawn. The appropriate form of asymptotic theory to use in such cases is far from being clear cut.

The fact that deterministic trend regressors and lagged regressors can both be used to model unit root processes raises some important modelling issues that have not been discussed here. Two of these issues are parsimony and forecasting. From both these perspectives, there may be good reasons for prefering the simplicity of lagged variable regressors to the complexity of deterministic trend/trend break representations. Criteria for choosing between such representations for trending time series have been explored recently in Phillips and Ploberger (1996) and Phillips (1996b). An analysis of how trending data affects the capacity to reproduce the properties of the optimal predictor is given in Ploberger and Phillips (2002, 2003). It is shown there that increasing the dimension of the parameter space carries a price in terms of the quantitative bound of how close we can come to the 'true' data generating process and, in consequence, how close we can reproduce the properties of the optimal predictor. It is further shown that this price goes up when we have trending data and use trending regressors and that the price is higher for deterministic (polynomial) trends than it is for stochastic trends. These are additional and rather complicated considerations affecting the choice between deterministic trend regressors and lagged variables in modelling trending data.

Appendix A. Technical points and proofs

Proof of Lemma 2.1. See Phillips (1998), Theorem 3.1. □

Proof of Lemma 2.2. Let $\Phi_{Kt} = \varphi_{Kt}\varphi_{Kt}'$ and note that $\Phi_{Kt} = \Phi_K(t/n)$ is a continuously differentiable matrix function with bounded derivatives of all orders in view of (6). Write, for $(t-1)/n \leqslant s < t/n$,

$$\begin{aligned}\Phi_{Kt} = \Phi_K\left(\frac{t}{n}\right) &= \Phi_K(s) + \Phi_K^{(1)}(s^*)\left(\frac{t}{n} - s\right)\\ &= \varphi_K(s)\varphi_K(s)' + \Phi_K^{(1)}(s^*)\left(\frac{t}{n} - s\right)\end{aligned}$$

with s^* on the line segment between t/n and s for each component of Φ_K. Then, since $K < n$, we have

$$\begin{aligned}n^{-1}\sum_{t=1}^{n}\varphi_{Kt}\varphi_{Kt}' &= \sum_{t=1}^{n}\int_{(t-1)/n}^{t/n}\varphi_K(s)\varphi_K(s)'\,\mathrm{d}s\\ &\quad + \sum_{t=1}^{n}\int_{(t-1)/n}^{t/n}\Phi_K^{(1)}(s^*)\left(\frac{t}{n} - s\right)\mathrm{d}s\\ &= \int_0^1\varphi_K(s)\varphi_K(s)'\,\mathrm{d}s\\ &\quad + \sum_{t=1}^{n}\int_{(t-1)/n}^{t/n}\Phi_K^{(1)}(s^*)\left(\frac{t}{n} - s\right)\mathrm{d}s\\ &= I_K + \sum_{t=1}^{n}\int_{(t-1)/n}^{t/n}\Phi_K^{(1)}(s^*)\left(\frac{t}{n} - s\right)\mathrm{d}s.\end{aligned}$$

The elements of $\Phi_K^{(1)}$ are uniformly bounded above, so that

$$\sup_{1\leqslant i,j\leqslant K}\ \sup_{1\leqslant t\leqslant n}\ \sup_{s\in[(t-1)/n,t/n]}|\Phi_{K,i,j}^{(1)}(s)| \leqslant M$$

for some finite $M>0$ that is independent of K. Also $|t/n-s|\leqslant 1/n$ uniformly for $s\in[(t-1)/n,t/n]$. Using the matrix norm $\|A\|=\max_i\sum_{j=1}^K|a_{ij}|$, we have

$$\left\|\sum_{t=1}^n\int_{t-1/n}^{t/n}\Phi_K^{(1)}(s^*)\left(\frac{t}{n}-s\right)\mathrm{d}s\right\|\leqslant\sum_{t=1}^n\int_{(t-1)/n}^{t/n}\|\Phi_K^{(1)}(s^*)\|\left|\frac{t}{n}-s\right|\mathrm{d}s$$

$$\leqslant\frac{K}{n}M\int_0^1\mathrm{d}s=\mathrm{O}\left(\frac{K}{n}\right)=\mathrm{o}(1).$$

Part (a) follows directly. To prove part (b), we use part (a), the strong approximation (2) and (8), giving

$$\begin{aligned}\hat{a}_K&=\left(n^{-1}\sum_{t=1}^n\varphi_{Kt}\varphi_{Kt}'\right)^{-1}\left(n^{-1}\sum_{t=1}^n\varphi_{Kt}\frac{y_t}{\sqrt{n}}\right)\\&=\left(I_K+\mathrm{O}\left(\frac{K}{n}\right)\right)^{-1}\left(n^{-1}\sum_{t=1}^n\varphi_K\left(\frac{t}{n}\right)\left[B\left(\frac{t}{n}\right)+\mathrm{o}_{\mathrm{a.s.}}\left(\frac{1}{n^{1/2-1/p}}\right)\right]\right)\\&=\left(I_K+\mathrm{O}\left(\frac{K}{n}\right)\right)\left(\int_0^1\varphi_K(r)B(r)\,\mathrm{d}r+\mathrm{o}_{\mathrm{a.s.}}\left(\frac{1}{n^{1/2-1/p}}\right)\right)\\&=\int_0^1\varphi_K(r)B(r)\,\mathrm{d}r+\mathrm{O}_{\mathrm{a.s.}}\left(\frac{K}{n}+\frac{1}{n^{1/2-1/p}}\right)\\&=\Lambda_K^{1/2}\xi_K+\mathrm{O}_{\mathrm{a.s.}}\left(\frac{K}{n}+\frac{1}{n^{1/2-1/p}}\right),\end{aligned}$$

as stated. Parts (c)–(f) are proved in Phillips (1998) Theorem 3.3. □

Proof of Lemmas 3.1 and 3.2. It is simplest to derive these two results together. To prove part (a) of Lemmas 3.1 and 3.2, we obtain an approximate representation of the stochastic integral $\int_0^1 W_{\varphi_K}\,\mathrm{d}W$ which reveals its limiting form. This form shows that as $K\to\infty$

$$\mathrm{E}\left(\int_0^1 W_{\varphi_K}\,\mathrm{d}W\right)\to-\frac{1}{2}\tag{A.1}$$

and

$$\mathrm{var}\left(\int_0^1 W_{\varphi_K}\,\mathrm{d}W\right)\to 0,\tag{A.2}$$

giving the result in Lemma 3.1(a). Analysis of the orders of magnitude give Lemma 3.2(a).

Start by writing

$$\int_0^1 W_{\varphi_K}\,\mathrm{d}W = \int_0^1 W\,\mathrm{d}W - \left(\int_0^1 W\varphi_K'\right)\left(\int_0^1 \varphi_K\varphi_K'\right)^{-1}\left(\int_0^1 \varphi_K\,\mathrm{d}W\right)$$

$$= \int_0^1 W\,\mathrm{d}W - \left(\int_0^1 W\varphi_K'\right)\left(\int_0^1 \varphi_K\,\mathrm{d}W\right), \tag{A.3}$$

whose expectation is

$$-\,\mathrm{E}\left[\left(\int_0^1 W\varphi_K'\right)\left(\int_0^1 \varphi_K\,\mathrm{d}W\right)\right]. \tag{A.4}$$

To evaluate (A.4), use the orthonormal representation (8) which we write as

$$W(r) = \varphi_K(r)'\Lambda_K^{1/2}\xi_K + \varphi_\perp(r)\Lambda_\perp^{1/2}\xi_\perp, \tag{A.5}$$

where $\xi_K, \xi_\perp$ are now vectors of independent standard normal variates, $\Lambda_K = diag(\lambda_1,\ldots,\lambda_K)$, $\Lambda_\perp = diag(\lambda_{K+1},\ldots)$, $\varphi_K' = (\varphi_1,\ldots,\varphi_K)$ and $\varphi_\perp' = (\varphi_{K+1},\ldots)$. Then, by the L_2 orthogonality of φ_K and $\varphi_\perp$, we have

$$\int_0^1 \varphi_K W = \Lambda_K^{1/2}\xi_K, \tag{A.6}$$

and since φ_K is continuous and of bounded variation we can apply integration by parts to $\int_0^1 \varphi_K\,\mathrm{d}W$ giving

$$\int_0^1 \varphi_K\,\mathrm{d}W = \varphi_K(1)W(1) - \int_0^1 \varphi_K^{(1)}W. \tag{A.7}$$

It follows that (A.4) is

$$-\mathrm{E}\left[\xi_K'\Lambda_K^{1/2}\left(\varphi_K(1)W(1) - \int_0^1 \varphi_K^{(1)}W\right)\right]$$

$$= -\,\mathrm{E}\left[\xi_K'\Lambda_K^{1/2}\varphi_K(1)\xi_K'\Lambda_K^{1/2}\varphi_K(1)\right] + \mathrm{E}\left[\xi_K'\Lambda_K^{1/2}\left(\int_0^1 \varphi_K^{(1)}(r)\varphi_K(r)'\right)\Lambda_K^{1/2}\xi_K\right]$$

$$= -\,tr\left[\mathrm{E}(\xi_K\xi_K')\Lambda_K^{1/2}\varphi_K(1)\varphi_K(1)'\Lambda_K^{1/2}\right]$$

$$+\,tr\left[\mathrm{E}(\xi_K\xi_K')\Lambda_K^{1/2}\left(\int_0^1 \varphi_K^{(1)}(r)\varphi_K(r)'\right)\Lambda_K^{1/2}\right]$$

$$= -\,tr[\Lambda_K\varphi_K(1)\varphi_K(1)'] + tr\left[\Lambda_K\left(\int_0^1 \varphi_K^{(1)}(r)\varphi_K(r)'\right)\right]$$

$$= -\varphi_K(1)'\Lambda_K\varphi_K(1) + \int_0^1 \varphi_K(r)'\Lambda_K\varphi_K^{(1)}(r). \tag{A.8}$$

As $K\to\infty$, (A.8) tends to

$$-\varphi(1)'\Lambda\varphi(1) + \int_0^1 \varphi(r)'\Lambda\varphi^{(1)}(r)$$
$$= -\sum_{k=1}^{\infty}\lambda_k\varphi_k(1)^2 + \sum_{k=1}^{\infty}\lambda_k\int_0^1 \varphi_k(r)\varphi_k^{(1)}(r).$$

Now

$$\int_0^1 \varphi_k(r)\varphi_k^{(1)}(r) = [\varphi_k(r)^2]_0^1 - \int_0^1 \varphi_k(r)\varphi_k^{(1)}(r),$$

so that

$$\int_0^1 \varphi_k(r)\varphi_k^{(1)}(r) = \frac{1}{2}\left[\varphi_k(1)^2 - \varphi_k(0)^2\right] = \frac{1}{2}\varphi_k(1)^2 = 1,$$

because

$$\varphi_k(1)^2 = (\sqrt{2}\,\sin\left(k-\frac{1}{2}\right)\pi)^2 = 2,$$

for all k. Hence,

$$\mathrm{E}\left(\int_0^1 W_{\varphi_K}\,\mathrm{d}W\right) = -2\sum_{k=1}^{\infty}\lambda_k + \sum_{k=1}^{\infty}\lambda_k = -\sum_{k=1}^{\infty}\lambda_k$$
$$= -\sum_{k=1}^{\infty}\frac{1}{(k-1/2)^2\pi^2} = -\frac{1}{2},$$

since (see also Gradshteyn and Ryzhik, 1994, formula 0.234-2)

$$\sum_{k=1}^{\infty}\frac{1}{(k-\frac{1}{2})^2} = 4\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2} = 4\left(\sum_{k=1}^{\infty}\frac{1}{k^2} - \sum_{k=1}^{\infty}\frac{1}{(2k)^2}\right)$$
$$= 4\left(1-\frac{1}{2^2}\right)\sum_{k=1}^{\infty}\frac{1}{k^2} = \frac{\pi^2}{2}, \tag{A.9}$$

which gives the result stated above in (A.1).

To prove (A.2), we start by writing (A.3) in the following form using (A.7) and (A.8)

$$\int_0^1 W_{\varphi_K}\,\mathrm{d}W = \frac{1}{2}(W(1)^2-1) - \left(\int_0^1 W\varphi_K'\right)\left(\int_0^1 \varphi_K\,\mathrm{d}W\right)$$
$$= \frac{1}{2}(W(1)^2-1) - (\xi_K'\Lambda_K^{1/2})\left(\varphi_K(1)W(1) - \int_0^1 \varphi_K^{(1)}W\right)$$

$$
\begin{aligned}
&= \frac{1}{2}(W(1)^2 - 1) \\
&\quad - (\xi_K' \Lambda_K^{1/2})(\varphi_K(1)\varphi_K(1)' \Lambda_K^{1/2} \xi_K + \varphi_K(1)\varphi_\perp(1)' \Lambda_\perp^{1/2} \xi_\perp) \\
&\quad + \xi_K' \Lambda_K^{1/2} \left(\int_0^1 \varphi_K^{(1)}(r)\varphi_K(r)' \right) \Lambda_K^{1/2} \xi_K \\
&\quad + \xi_K' \Lambda_K^{1/2} \left(\int_0^1 \varphi_K^{(1)}(r)\varphi_\perp(r)' \right) \Lambda_\perp^{1/2} \xi_\perp .
\end{aligned}
$$

Observe that for any K-vector b

$$
\begin{aligned}
& b' \left(\int_0^1 \varphi_K^{(1)}(r)\varphi_K(r)' \right) b \\
&= b' \frac{1}{2} \left(\int_0^1 \varphi_K^{(1)}(r)\varphi_K(r)' + \int_0^1 \varphi_K(r)\varphi_K^{(1)}(r)' \right) b \\
&= b' \frac{1}{2} \left([\varphi_K(r)\varphi_K(r)']_0^1 - \int_0^1 \varphi_K(r)\varphi_K^{(1)}(r)' + \int_0^1 \varphi_K(r)\varphi_K^{(1)}(r)' \right) b \\
&= \tfrac{1}{2} b' \varphi_K(1)\varphi_K(1)' b,
\end{aligned}
$$

so that

$$
\xi_K' \Lambda_K^{1/2} \left(\int_0^1 \varphi_K^{(1)}(r)\varphi_K(r)' \right) \Lambda_K^{1/2} \xi_K = \frac{1}{2} \xi_K' \Lambda_K^{1/2} \varphi_K(1)\varphi_K(1)' \Lambda_K^{1/2} \xi_K,
$$

and thus

$$
\begin{aligned}
\int_0^1 W_{\varphi_K} \, \mathrm{d}W &= \frac{1}{2}(W(1)^2 - 1) - \frac{1}{2} \xi_K' \Lambda_K^{1/2} \varphi_K(1)\varphi_K(1)' \Lambda_K^{1/2} \xi_K \\
&\quad - \xi_K' \Lambda_K^{1/2} \varphi_K(1)\varphi_\perp(1)' \Lambda_\perp^{1/2} \xi_\perp \\
&\quad + \xi_K' \Lambda_K^{1/2} \left(\int_0^1 \varphi_K^{(1)}(r)\varphi_\perp(r)' \right) \Lambda_\perp^{1/2} \xi_\perp \\
&= \frac{1}{2}(W(1)^2 - 1) - \frac{1}{2} \xi_K' \Lambda_K^{1/2} \varphi_K(1)\varphi_K(1)' \Lambda_K^{1/2} \xi_K \\
&\quad - \xi_K' \Lambda_K^{1/2} \left(\int_0^1 \varphi_K(r)\varphi_\perp^{(1)}(r)' \right) \Lambda_\perp^{1/2} \xi_\perp \\
&= -\frac{1}{2} + \xi_K' \Lambda_K^{1/2} \varphi_K(1)\varphi_\perp(1)' \Lambda_\perp^{1/2} \xi_\perp + \frac{1}{2} \xi_\perp' \Lambda_\perp^{1/2} \varphi_\perp(1)\varphi_\perp(1)' \Lambda_\perp^{1/2} \xi_\perp
\end{aligned}
$$

$$- \xi_K' \Lambda_K^{1/2} \left(\int_0^1 \varphi_K(r) \varphi_\perp^{(1)}(r)' \right) \Lambda_\perp^{1/2} \xi_\perp. \tag{A.10}$$

Note that

$$\varphi_\perp(1)' \Lambda_\perp^{1/2} \xi_\perp = \sum_{k=K+1}^{\infty} \lambda_k^{1/2} \varphi_k(1) \xi_k =_{\mathrm{d}} \mathrm{N}(0, \varphi_\perp(1)' \Lambda_\perp \varphi_\perp(1)),$$

and

$$\varphi_\perp(1)' \Lambda_\perp \varphi_\perp(1) = \sum_{k=K+1}^{\infty} \lambda_k \varphi_k(1)^2 = \sum_{k=K+1}^{\infty} \frac{2}{(k-\frac{1}{2})^2 \pi^2} = \mathrm{O}\left(\frac{1}{K}\right).$$

It follows that $\varphi_\perp(1)' \Lambda_\perp^{1/2} \xi_\perp = \mathrm{O_p}(1/\sqrt{K})$ and so

$$\xi_\perp' \Lambda_\perp^{1/2} \varphi_\perp(1) \varphi_\perp(1)' \Lambda_\perp^{1/2} \xi_\perp = \mathrm{O_p}\left(\frac{1}{K}\right).$$

Hence

$$\int_0^1 W_{\varphi_K} \, \mathrm{d}W = -\frac{1}{2} + \xi_K' \Lambda_K^{1/2} \left(\int_0^1 \varphi_K^{(1)}(r) \varphi_\perp(r)' \right) \Lambda_\perp^{1/2} \xi_\perp + \mathrm{O_p}(1/K), \tag{A.11}$$

and

$$\mathrm{var}\left(\int_0^1 W_{\varphi_K} \, \mathrm{d}W \right) = tr\left[\Lambda_K \left(\int_0^1 \varphi_K^{(1)}(r) \varphi_\perp(r)' \right) \Lambda_\perp \left(\int_0^1 \varphi_\perp(r) \varphi_K^{(1)}(r)' \right) \right] + \mathrm{O}\left(\frac{1}{K}\right).$$

The second term on the right-hand side of (A.11), $\xi_K' \Lambda_K^{1/2} (\int_0^1 \varphi_K^{(1)}(r) \varphi_\perp(r)') \Lambda_\perp^{1/2} \xi_\perp$, has mean zero and variance

$$\mathrm{var}\left[\xi_K' \Lambda_K^{1/2} \left(\int_0^1 \varphi_K^{(1)}(r) \varphi_\perp(r)' \right) \Lambda_\perp^{1/2} \xi_\perp \right] = tr\left[\Lambda_K \left(\int_0^1 \varphi_K^{(1)}(r) \varphi_\perp(r)' \right) \Lambda_\perp \left(\int_0^1 \varphi_\perp(r) \varphi_K^{(1)}(r)' \right) \right], \tag{A.12}$$

which we now evaluate. Since $\varphi_k(r) = \sqrt{2} \sin[(k-1/2)\pi r]$ and $\varphi_k'(r) = \sqrt{2}(k-1/2)\pi \cos[(k-1/2)\pi r]$ we have

$$\int_0^1 \varphi_\ell'(r) \varphi_k(r) \, \mathrm{d}r = 2\left(\ell - \frac{1}{2}\right) \pi \int_0^1 \cos[(\ell - 1/2)\pi r] \sin[(k-1/2)\pi r] \, \mathrm{d}r$$

$$= \left(\ell - \frac{1}{2}\right) \pi \int_0^1 \{\sin[(\ell + k - 1)\pi r] + \sin[(k-\ell)\pi r]\} \, \mathrm{d}r$$

$$= \left(\ell - \frac{1}{2}\right)\left\{\frac{1-\cos[(\ell+k-1)\pi]}{(\ell+k-1)} + \frac{1-\cos[(k-\ell)\pi]}{(k-\ell)}\right\}$$

$$= \begin{cases} \dfrac{2(\ell-\frac{1}{2})}{\ell+k-1} & \text{if } k-\ell \text{ is even,} \\ \dfrac{2(\ell-\frac{1}{2})}{k-\ell} & \text{if } k-\ell \text{ is odd.} \end{cases}$$

It follows that

$$tr\left[\Lambda_K\left(\int_0^1 \varphi_K^{(1)}(r)\varphi_\perp(r)'\right)\Lambda_\perp\left(\int_0^1 \varphi_\perp(r)\varphi_K^{(1)}(r)'\right)\right]$$

$$= \sum_{k=K+1}^{\infty}\sum_{\ell=1}^{K}\lambda_\ell\left[\left(\frac{2\ell-1}{\ell+k-1}\right)^2\mathbf{1}\,[k-\ell=\text{even}] + \left(\frac{(2\ell-1)}{k-\ell}\right)^2\mathbf{1}\,[k-\ell=\text{odd}]\right]\lambda_k$$

$$=\Sigma_A + \Sigma_B. \tag{A.13}$$

We now proceed to evaluate each of these terms in turn. First, consider Σ_A. Using the formula $\lambda_\ell = 4/(2\ell-1)^2\pi^2$, we find

$$\Sigma_A = \left(\frac{4}{\pi^2}\right)^2\sum_{k=K+1}^{\infty}\sum_{\ell=1}^{K}\frac{1}{(2k-1)^2(\ell+k-1)^2}\mathbf{1}[k-\ell=\text{even}]$$

$$= \frac{4}{\pi^4}\sum_{k=K+1}^{\infty}\frac{1}{(2k-1)^2}\sum_{n=1}^{[(K-1)/2]}\frac{1}{(k-n-\frac{1}{2})^2}$$

$$= \frac{4}{\pi^4}\sum_{k=K+1}^{\infty}\frac{1}{(2k-1)^2k^2}\sum_{n=1}^{[(K-1)/2]}\left(\frac{1}{1-(n+\frac{1}{2})/k}\right)^2$$

$$= \frac{4}{\pi^4}\sum_{k=K+1}^{\infty}\frac{1}{(2k-1)^2k}\left[\int_0^{K/2k}\left(\frac{1}{1-x}\right)^2\mathrm{d}x + \mathrm{O}\left(\frac{1}{k}\right)\right]$$

$$= \frac{4}{\pi^4}\sum_{k=K+1}^{\infty}\frac{1}{(2k-1)^2k}\left\{\left[\frac{1}{1-x}\right]_0^{K/2k} + \mathrm{O}\left(\frac{1}{k}\right)\right\}$$

$$= \frac{4}{\pi^4}\sum_{k=K+1}^{\infty}\frac{1}{(2k-1)^2k}\left\{\frac{K}{2k-K} + \mathrm{O}\left(\frac{1}{k}\right)\right\}$$

$$= \frac{4K}{\pi^4}\int_{K+1}^{\infty}\frac{\mathrm{d}y}{y(2y-1)^2(2y-K)} + \mathrm{O}\left(\frac{1}{K^3}\right)$$

$$= \frac{K}{\pi^4 2} \int_{K+1}^{\infty} \frac{\mathrm{d}y}{y^3(y-K/2)} + \mathrm{O}\left(\frac{1}{K^3}\right)$$

$$= \frac{K}{\pi^4 2} \frac{1}{(K+1)^3} \int_{1}^{\infty} \frac{dt}{t^3(t-\frac{1}{2})} + \mathrm{O}\left(\frac{1}{K^3}\right)$$

$$= \frac{1}{2\pi^4 K^2} \int_{1}^{\infty} \frac{\mathrm{d}t}{t^3(t-\frac{1}{2})} + \mathrm{O}\left(\frac{1}{K^3}\right).$$

Next, Σ_B is

$$\Sigma_B = \left(\frac{4}{\pi^2}\right)^2 \sum_{k=K+1}^{\infty} \sum_{\ell=1}^{K} \frac{1}{(2k-1)^2(k-\ell)^2} \mathbf{1}[k-\ell = \mathrm{odd}]$$

$$= \left(\frac{4}{\pi^2}\right)^2 \sum_{k=K+1}^{\infty} \frac{1}{(2k-1)^2} \frac{1}{k^2} \sum_{n=[(k-K-1)/2]}^{[(k-2)/2]} \frac{1}{((2n+1)/k)^2}$$

$$= \left(\frac{4}{\pi^2}\right)^2 \sum_{k=K+1}^{\infty} \frac{1}{(2k-1)^2} \left[\frac{1}{k} \int_{1/2-(K+1)/2k}^{1/2-1/k} \frac{\mathrm{d}x}{(2x+1/k)^2} + \mathrm{O}\left(\frac{1}{k^2}\right)\right]$$

$$= \frac{1}{2} \left(\frac{4}{\pi^2}\right)^2 \sum_{k=K+1}^{\infty} \frac{1}{(2k-1)^2} \frac{1}{k} \left[-\frac{1}{(2x+1/k)}\right]_{1/2-(K+1)/2k}^{1/2-1/k} + \mathrm{O}\left(\frac{1}{K^3}\right)$$

$$= \frac{1}{2} \left(\frac{4}{\pi^2}\right)^2 \sum_{k=K+1}^{\infty} \frac{1}{(2k-1)^2} \left[\frac{K-1}{(k-1)(k-K)}\right] + \mathrm{O}\left(\frac{1}{K^3}\right)$$

$$= \frac{1}{2} \left(\frac{4}{\pi^2}\right)^2 (K-1) \int_{K+1}^{\infty} \frac{\mathrm{d}y}{(2y-1)^2(y-1)(y-K)} + \mathrm{O}\left(\frac{1}{K^3}\right)$$

$$= \frac{2K}{\pi^4} \int_{K+1}^{\infty} \frac{\mathrm{d}y}{y^3(y-K)} \left[1 + \mathrm{O}\left(\frac{1}{K}\right)\right] + \mathrm{O}\left(\frac{1}{K^3}\right)$$

$$= \frac{2}{\pi^4} \frac{\ln(K)}{K^2}[1 + \mathrm{o}(1)].$$

Hence, Σ_B dominates Σ_A and we deduce from (A.12) and (A.13) that

$$\mathrm{var}\left(\xi_K' \Lambda_K^{1/2} \left(\int_0^1 \varphi_K^{(1)}(r)\varphi_\perp(r)'\right) \Lambda_\perp^{1/2} \xi_\perp\right) = \frac{2}{\pi^4} \frac{\ln(K)}{K^2}[1 + \mathrm{o}(1)],$$

whence

$$\xi_K' \Lambda_K^{1/2} \left(\int_0^1 \varphi_K^{(1)}(r)\varphi_\perp(r)'\right) \Lambda_\perp^{1/2} \xi_\perp = \mathrm{O_p}\left(\frac{\sqrt{\ln K}}{K}\right), \tag{A.14}$$

whose order was originally given by Seiji Nabeya in a personal communication. It follows from (A.11) and (A.14) that

$$\int_0^1 W_{\varphi_K}\,\mathrm{d}W = -\frac{1}{2} + \mathrm{O_p}\left(\frac{\sqrt{\ln K}}{K}\right),$$

giving Lemma 3.2(a).

For Part (b) of Lemma 3.1, observe that

$$\int_0^1 W_{\varphi_K}^2 = \xi_\perp' \Lambda_\perp \xi_\perp = \sum_{k=K+1}^{\infty} \lambda_k \xi_k^2,$$

and

$$\mathrm{E}(\xi_\perp' \Lambda_\perp \xi_\perp) = \sum_{k=K+1}^{\infty} \lambda_k = \sum_{k=K+1}^{\infty} \frac{1}{(k-1/2)^2\pi^2}.$$

As $K \to \infty$, we find

$$K\mathrm{E}(\xi_\perp' \Lambda_\perp \xi_\perp) \sim \frac{K}{\pi^2}\int_{K+1}^{\infty} \frac{\mathrm{d}x}{(x-1/2)^2} \sim \frac{K}{\pi^2(K+1/2)} \to \frac{1}{\pi^2}, \tag{A.15}$$

and

$$\mathrm{var}\,(\xi_\perp' \Lambda_\perp \xi_\perp) = \sum_{k=K+1}^{\infty} \lambda_k^2 = \mathrm{O}\left(\frac{1}{K^3}\right),$$

so that

$$K\xi_\perp' \Lambda_\perp \xi_\perp \to_{\mathrm{p}} \frac{1}{\pi^2}. \tag{A.16}$$

Thus,

$$K\int_0^1 W_{\varphi_K}^2 = K\xi_\perp' \Lambda_\perp \xi_\perp \to_{\mathrm{p}} \frac{1}{\pi^2}, \tag{A.17}$$

as required.

Next, consider

$$\begin{aligned} K\int_0^1 W_{\varphi_K}^2 - \frac{1}{\pi^2} &= K\xi_\perp' \Lambda_\perp \xi_\perp - \frac{1}{\pi^2} \\ &= K\sum_{k=K+1}^{\infty} \lambda_k(\xi_k^2 - 1) + K\sum_{k=K+1}^{\infty} \lambda_k - \frac{1}{\pi^2} \\ &= K\sum_{k=K+1}^{\infty} \lambda_k(\xi_k^2 - 1) + \mathrm{O}\left(\frac{1}{K}\right), \end{aligned}$$

since

$$K\sum_{k=K+1}^{\infty}\lambda_k-\frac{1}{\pi^2}=\frac{K}{\pi^2}\left(\sum_{k=K+1}^{\infty}\frac{1}{(k-1/2)^2}-\int_{K+1}^{\infty}\frac{\mathrm{d}x}{(x-1/2)^2}\right)$$

$$+\frac{1}{\pi^2}\left(K\int_{K+1}^{\infty}\frac{\mathrm{d}x}{(x-1/2)^2}-1\right)$$

$$=\frac{K}{\pi^2}\left(\sum_{k=K+1}^{\infty}\left[\frac{1}{(k-1/2)^2}-\left(\frac{1}{k-1/2}-\frac{1}{k+\frac{1}{2}}\right)\right]\right)$$

$$+\frac{1}{\pi^2}\left(\frac{K}{K+\frac{1}{2}}-1\right)$$

$$=\frac{K}{\pi^2}\left(\sum_{k=K+1}^{\infty}\frac{1}{(k-1/2)^2(k+1/2)}\right)-\frac{1}{\pi^2}\left(\frac{1/2}{K+1/2}\right)$$

$$=\mathrm{O}\left(\frac{1}{K}\right).$$

It follows that

$$\sqrt{K}(K\int_0^1 W_{\varphi_K}^2-\frac{1}{\pi^2})=K^{3/2}\sum_{k=K+1}^{\infty}\lambda_k(\xi_k^2-1)+\mathrm{O_p}\left(\frac{1}{\sqrt{K}}\right)$$

$$=\frac{1}{\pi^2}\frac{1}{\sqrt{K}}\sum_{k=K+1}^{\infty}\frac{K^2}{(k-1/2)^2}(\xi_k^2-1)+\mathrm{O_p}\left(\frac{1}{\sqrt{K}}\right). \quad \text{(A.18)}$$

Now the variates

$$\frac{K^2}{(k-1/2)^2}(\xi_k^2-1)$$

are independent with uniformly bounded moments of all orders for all $k>K$ and

$$\mathrm{var}\left(\sum_{k=K+1}^{\infty}\frac{K^2}{(k-1/2)^2}(\xi_k^2-1)\right)=2\sum_{k=K+1}^{\infty}\left(\frac{K^2}{(k-1/2)^2}\right)^2$$

$$=2K^4\sum_{k=K+1}^{\infty}\left(\frac{1}{(k-1/2)^4}\right)$$

$$\sim 2K^4 \int_{K+1}^{\infty} \frac{\mathrm{d}x}{(x-1/2)^4}$$
$$= \frac{2}{3} K^4 \frac{1}{(K+1/2)^3}$$
$$= \mathrm{O}(K)$$

as $K \to \infty$. It follows by the martingale central limit theorem that

$$\frac{1}{\sqrt{K}} \sum_{k=K+1}^{\infty} \frac{K^2}{(k-1/2)^2} (\xi_k^2 - 1) \Rightarrow \mathrm{N}\left(0, \frac{2}{3}\right).$$

Hence,

$$\sqrt{K} \left(K \int_0^1 W_{\varphi_K}^2 - \frac{1}{\pi^2} \right) \Rightarrow \frac{1}{\pi^2} \mathrm{N}\left(0, \frac{2}{3}\right),$$

as required for Lemma 3.2(b). □

Proof of Theorem 3.3. Part (a) is an immediate consequence of Lemma 3.2. In particular, we have

$$\frac{\int_0^1 W_{\varphi_K}\, \mathrm{d}W}{\int_0^1 W_{\varphi_K}^2} \sim \frac{-1/2K}{1/\pi^2} + \mathrm{o_p}(1) = \mathrm{O_p}(K)$$

and

$$\frac{\int_0^1 W_{\varphi_K}\, \mathrm{d}W}{(\int_0^1 W_{\varphi_K}^2)^{1/2}} \sim \frac{-1/2\sqrt{K}}{1/\pi} + \mathrm{o_p}(1) = \mathrm{O_p}(\sqrt{K}).$$

For Part (b), write

$$\frac{1}{\sqrt{K}} \left(\frac{\int_0^1 W_{\varphi_K}\, \mathrm{d}W}{\int_0^1 W_{\varphi_K}^2} + \frac{\pi^2}{2} K \right)$$
$$= \frac{1}{\sqrt{K}} \frac{\left(\int_0^1 W_{\varphi_K}\, \mathrm{d}W + 1/2\right) + \pi^2/2(K \int_0^1 W_{\varphi_K}^2 - 1/\pi^2)}{\int_0^1 W_{\varphi_K}^2}$$
$$= \frac{\sqrt{K} \left(\int_0^1 W_{\varphi_K}\, \mathrm{d}W + 1/2\right) + \pi^2/2\sqrt{K}(K \int_0^1 W_{\varphi_K}^2 - 1/\pi^2)}{K \int_0^1 W_{\varphi_K}^2}$$
$$\Rightarrow \frac{(\pi^2/2)(1/\pi^2)\mathrm{N}(0, 2/2)}{1\{\pi^2\}}$$
$$\equiv \mathrm{N}(0, \tfrac{1}{6}\pi^4),$$

which is the stated result. For the t-ratio limit distribution

$$
\begin{aligned}
&\left(\frac{\int_0^1 W_{\varphi_K}\,\mathrm{d}W}{\left(\int_0^1 W_{\varphi_K}^2\right)^{1/2}}+\frac{\pi}{2}\sqrt{K}\right)\\
&=\frac{(\int_0^1 W_{\varphi_K}\,\mathrm{d}W+1/2)+\pi/2(\sqrt{K}(\int_0^1 W_{\varphi_K}^2)^{1/2}-1/\pi)}{(\int_0^1 W_{\varphi_K}^2)^{1/2}}\\
&=\frac{\sqrt{K}(\int_0^1 W_{\varphi_K}\,\mathrm{d}W+1/2)+\pi/2\sqrt{K}(\sqrt{K}(\int_0^1 W_{\varphi_K}^2)^{1/2}-1/\pi)}{(K\int_0^1 W_{\varphi_K}^2)^{1/2}}\\
&\Rightarrow\frac{(\pi/2)(1/2\pi)\mathrm{N}(0,2/3)}{1/\pi}\\
&\equiv\frac{\pi}{4}\mathrm{N}\left(0,\frac{2}{3}\right)\equiv\mathrm{N}\left(0,\frac{\pi^2}{24}\right),
\end{aligned}
$$

where the third line holds by virtue of the following delta method calculation:

$$
\begin{aligned}
\sqrt{K}(\sqrt{K}\left(\int_0^1 W_{\varphi_K}^2\right)^{1/2}-\frac{1}{\pi})&\sim\frac{\pi}{2}\sqrt{K}(K\int_0^1 W_{\varphi_K}^2-\frac{1}{\pi^2})\\
&\Rightarrow\frac{\pi}{2}\frac{1}{\pi^2}\mathrm{N}\left(0,\frac{2}{3}\right). \quad\square
\end{aligned}
$$

Proof of Lemma 4.1. To prove part (a) we need to show that, as $(n,K\to\infty)$ with $K^4/n\to 0$,

$$
\mathrm{E}\left(\frac{K}{n^2}y_{-1}'Q_K y_{-1}\right)=\frac{\sigma^2}{\pi^2}+\mathrm{O}\left(\frac{K^4}{n}+\frac{1}{K}\right).
$$

Note that $\mathrm{E}(y_{-1}y_{-1}')=\sigma^2LL'=\sigma^2\Omega$, say, where L is a lower triangular matrix with unity in all elements in and below the main diagonal. From Lemma 2.2(a)

$$
(\Phi_K'\Phi_K)^{-1}=\frac{1}{n}I_K+\mathrm{O}\left(\frac{K}{n^2}\right). \tag{A.19}
$$

Then,

$$
\begin{aligned}
&\mathrm{E}\left(\frac{K}{n^2}y_{-1}'Q_K y_{-1}\right)\\
&=\frac{K}{n^2}tr\{\Omega(I-\Phi_K(\Phi_K'\Phi_K)^{-1}\Phi_K')\}\\
&=\frac{K}{n^2}tr\Omega-\frac{K}{n^2}\sum_{k=1}^{K}\frac{1}{n}\varphi_k'\Omega\varphi_k\left[1+\mathrm{O}\left(\frac{K^2}{n}\right)\right]
\end{aligned}
$$

$$=\sigma^2\frac{K}{n^2}\frac{n(n+1)}{2}-\sigma^2K\sum_{k=1}^{K}\frac{1}{n^2}\sum_{t,s=1}^{n}\varphi_k\left(\frac{t}{n}\right)\left(\frac{t\wedge s}{n}\right)\varphi_k\left(\frac{s}{n}\right)\left[1+\mathrm{O}\left(\frac{K^2}{n}\right)\right]$$

$$=\sigma^2K\frac{1}{2}-\sigma^2K\sum_{k=1}^{K}\left\{\int_0^1\int_0^1\varphi_k(r)(r\wedge p)\varphi_k(p)\,\mathrm{d}r\,\mathrm{d}p\left[1+\mathrm{O}\left(\frac{K^2}{n}\right)\right]\right.$$

$$\left.+\mathrm{O}\left(\frac{1}{n}\right)\right\}$$

$$=\sigma^2K\left(\frac{1}{2}-\sum_{k=1}^{K}\lambda_k\right)+\mathrm{O}\left(\frac{K^4}{n}\right)$$

$$=\sigma^2K\left(\frac{1}{2}-\sum_{k=1}^{\infty}\lambda_k+\sum_{k=K+1}^{\infty}\lambda_k\right)+\mathrm{O}\left(\frac{K^4}{n}\right)$$

$$=\sigma^2K\left(\frac{1}{2}-\frac{1}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{(k-1/2)^2}+\sum_{k=K+1}^{\infty}\lambda_k\right)+\mathrm{O}\left(\frac{K^4}{n}\right)$$

$$=\sigma^2K\sum_{k=K+1}^{\infty}\lambda_k+\mathrm{O}\left(\frac{K^4}{n}\right),$$

using (A.9). Also, as in (A.15), we find that as $K\to\infty$

$$K\sum_{k=K+1}^{\infty}\lambda_k=\frac{1}{\pi^2}+\mathrm{O}\left(\frac{1}{K}\right), \tag{A.20}$$

and the stated result follows immediately.

For part (b) we need to show that

$$\mathrm{E}\left(\frac{1}{n}y_{-1}'Q_Ku\right)\to-\frac{\sigma^2}{2}.$$

Note that

$$\mathrm{E}\left(\frac{1}{n}y_{-1}'Q_Ku\right)$$

$$=\sigma^2\frac{1}{n}tr\{(I-\Phi_K(\Phi_K'\Phi_K)^{-1}\Phi_K')(L-I)\}$$

$$=\frac{\sigma^2}{n}tr\{\Phi_K(\Phi_K'\Phi_K)^{-1}\Phi_K'(I-L)\}$$

$$=\sigma^2\frac{K}{n}-\frac{\sigma^2}{n^2}\sum_{k=1}^{K}\varphi_k'L\varphi_k\left[1+\mathrm{O}\left(\frac{K^2}{n}\right)\right]$$

$$=\mathrm{O}\left(\frac{K}{n}\right)-\sigma^2\sum_{k=1}^{K}\frac{1}{n^2}\sum_{s=1}^{n}\sum_{t=s}^{n}\varphi_k\left(\frac{t}{n}\right)\varphi_k\left(\frac{s}{n}\right)\left[1+\mathrm{O}\left(\frac{K^2}{n}\right)\right]$$

$$=\mathrm{O}\left(\frac{K}{n}\right)-\sigma^2\sum_{k=1}^{K}\int_0^1\varphi_k(r)\int_0^r\varphi_k(p)\,\mathrm{d}p\,\mathrm{d}r+\mathrm{O}\left(\frac{K^3}{n}\right). \tag{A.21}$$

The double integral in the second member of (A.21) can be calculated by standard methods giving the result

$$\int_0^1\varphi_k(r)\int_0^r\varphi_k(p)\,\mathrm{d}p\,\mathrm{d}r=2\int_0^1\sin\left(\left(k-\frac{1}{2}\right)\pi r\right)\int_0^r\sin((k-1/2)\pi p)\,\mathrm{d}p\,\mathrm{d}r$$

$$=\frac{1}{(k-1/2)^2\pi^2}. \tag{A.22}$$

It follows from (A.21), (A.22) and (A.9) that

$$\mathrm{E}\left(\frac{1}{n}y_{-1}'Q_Ku\right)=-\frac{\sigma^2}{\pi^2}\sum_{k=1}^{K}\frac{1}{(k-1/2)^2}+\mathrm{O}\left(\frac{K^3}{n}\right)$$

$$=-\frac{\sigma^2}{\pi^2}\left[\frac{\pi^2}{2}+\mathrm{O}\left(\frac{1}{K}\right)\right]+\mathrm{O}\left(\frac{K^3}{n}\right)$$

$$=-\frac{\sigma^2}{2}+\mathrm{O}\left(\frac{K^3}{n}+\frac{1}{K}\right),$$

giving the stated result.

For part (c) it is sufficient to show that under the stated conditions

$$\mathrm{var}\left(\frac{K}{n^2}y_{-1}'Q_Ky_{-1}\right)\to 0.$$

Let $y_{-1}=L\varepsilon$, where $\varepsilon\equiv \mathrm{N}(0,I_n)$. Then, $\Omega=\sigma^2LL'$ and

$$y_{-1}'Q_Ky_{-1}=\varepsilon'A_K\varepsilon,\quad \text{with } A_K=L'[I-\Phi_K(\Phi_K'\Phi_K)^{-1}\Phi_K']L.$$

Then

$$\mathrm{var}\left(\frac{K}{n^2}y_{-1}'Q_Ky_{-1}\right)=\left(\frac{K}{n^2}\right)^2\mathrm{var}(u'A_Ku)$$

$$=\frac{K^2}{n^4}2tr(A_K^2). \tag{A.23}$$

Evaluating $tr(A_K^2)$ we find

$$tr(A_K^2) = tr\left\{\Omega\left(I - \frac{1}{n}\Phi_K\Phi_K' + O\left(\frac{K}{n^2}\right)\right)\Omega\left(I - \frac{1}{n}\Phi_K\Phi_K' + O\left(\frac{K}{n^2}\right)\right)\right\}$$

$$= tr(\Omega^2) - \frac{2}{n}tr(\Phi_K'\Omega^2\Phi_K) + \frac{1}{n^2}tr(\Phi_K'\Omega\Phi_K)^2 + O\left(\frac{K^2}{n^2}\right). \tag{A.24}$$

Clearly

$$(tr\Omega)^2 = \sigma^4\left(\sum_{j=1}^{n} j\right)^2 = \sigma^4\left(\frac{n(n+1)}{2}\right)^2,$$

and the k'th diagonal element of Ω^2 is $\sigma^4[\sum_{j=1}^{k-1} j^2 + k^2(n-k+1)]$, so that

$$tr(\Omega^2) = \sigma^4\sum_{k=1}^{n}\left\{\sum_{j=1}^{k-1} j^2 + k^2(n-k+1)\right\}$$

$$= \sigma^4\sum_{k=1}^{n}\left\{\frac{(k-1)k(2k-1)}{6} - k^3 + k^2 + k^2 n\right\}$$

$$= \sigma^4\sum_{k=1}^{n}\left\{\left(-\frac{2}{3}k^3 + \frac{1}{6}k\right) + k^2\left(n + \frac{1}{2}\right)\right\}$$

$$= \left\{-\frac{2}{3}\left(\frac{n(n+1)}{2}\right)^2 + \frac{1}{6}\frac{n(n+1)}{2} + \frac{n(n+1)(2n+1)}{6}\left(n + \frac{1}{2}\right)\right\}\sigma^4$$

$$\sim \tfrac{1}{6}n^4\sigma^4. \tag{A.25}$$

Moreover,

$$\frac{1}{n}tr(\Phi_K'\Omega^2\Phi_K) = \frac{1}{n}\sum_{k=1}^{K}\sum_{t,s=1}^{n}\varphi_k\left(\frac{t}{n}\right)[\Omega^2]_{t,s}\varphi_k\left(\frac{s}{n}\right)$$

$$= n^3\sum_{k=1}^{K}\frac{1}{n^3}\sum_{t,s=1}^{n}\varphi_k\left(\frac{t}{n}\right)\frac{1}{n}[\Omega^2]_{t,s}\varphi_k\left(\frac{s}{n}\right)$$

$$= \sigma^4 n^4\sum_{k=1}^{K}\frac{1}{n^3}\sum_{t,s,q=1}^{n}\varphi_k\left(\frac{t}{n}\right)\frac{t\wedge q}{n}\frac{q\wedge s}{n}\varphi_k\left(\frac{s}{n}\right)$$

$$= \sigma^4 n^4\sum_{k=1}^{K}\int_0^1\int_0^1\int_0^1 \varphi_k(r)(r\wedge p)(p\wedge u)\varphi_k(u)\,\mathrm{d}r\,\mathrm{d}p\,\mathrm{d}u + O\left(n^4\frac{K}{n}\right)$$

$$=\sigma^4 n^4 \sum_{k=1}^{K} \lambda_k \int_0^1 \int_0^1 \varphi_k(r)(r \wedge p)\varphi_k(p)\,\mathrm{d}r\,\mathrm{d}p + \mathrm{O}(n^3 K)$$

$$=\sigma^4 n^4 \sum_{k=1}^{K} \lambda_k^2 \int_0^1 \varphi_k(r)^2\,\mathrm{d}r + \mathrm{O}(n^3 K)$$

$$=\sigma^4 n^4 \sum_{k=1}^{K} \lambda_k^2 + \mathrm{O}(n^3 K). \tag{A.26}$$

By a similar calculation we find

$$\frac{1}{n^2} tr(\Phi_K' \Omega \Phi_K)^2 = \sigma^4 n^4 \sum_{k=1}^{K} \lambda_k^2 + \mathrm{O}(n^3 K^2). \tag{A.27}$$

Hence, combining (A.24)–(A.27) we get

$$tr(A_K^2) = \frac{1}{6} n^4 \sigma^4 - n^4 \sigma^4 \sum_{k=1}^{K} \lambda_k^2 + \mathrm{O}(n^3 K^2)$$

$$= \frac{1}{6} n^4 \sigma^4 - n^4 \sigma^4 \frac{1}{\pi^4} \sum_{k=1}^{K} \frac{1}{(k-1/2)^4} + \mathrm{O}(n^3 K^2). \tag{A.28}$$

Next

$$\sum_{k=1}^{K} \frac{1}{(k-1/2)^4} = 2^4 \left[\sum_{k=1}^{\infty} \frac{1}{(2k-1)^4} - \mathrm{O}\left(\frac{1}{K^3}\right) \right]$$

$$= 2^4 \frac{\pi^4}{96} + \mathrm{O}\left(\frac{1}{K^3}\right) = \frac{\pi^4}{6} + \mathrm{O}\left(\frac{1}{K^3}\right), \tag{A.29}$$

where the formula for the infinite sum is given, for example, in Gradshteyn and Ryzhik (1994, formula 0.234-5).

It follows from (A.28) and (A.29) that

$$tr(A_K^2) = \mathrm{O}(n^3 K^2) + \mathrm{O}\left(\frac{n^4}{K^3}\right).$$

Thus, from (A.23)

$$\mathrm{var}\left(\frac{K}{n^2} y_{-1}' Q_K y_{-1}\right) = \frac{K^2}{n^4} 2tr(A_K^2) = \mathrm{O}\left(\frac{K^4}{n} + \frac{1}{K}\right) = \mathrm{o}(1),$$

and the stated result (c) follows immediately.

For part (d) it is sufficient to show that under the stated conditions

$$\operatorname{var}\left(\frac{1}{n}y_{-1}'Q_K u\right) \to 0. \tag{A.30}$$

Let $y_{-1} = Ju$, where $u \equiv \mathrm{N}(0, I_n)$ and J is a lower triangular matrix with unity in every position below the main diagonal and zeros elsewhere. Then

$$\frac{1}{n}y_{-1}'Q_K u = \frac{1}{n}u'J'Q_K u = \frac{1}{n}u'B_K u,$$

where

$$B_K = \tfrac{1}{2}[J'Q_K + Q_K J] = \tfrac{1}{2}[J' + J - J'\Phi_K(\Phi_K'\Phi_K)^{-1}\Phi_K' - \Phi_K(\Phi_K'\Phi_K)^{-1}\Phi_K'J].$$

Now

$$\operatorname{var}\left[\frac{1}{n}u'B_K u\right] = \frac{1}{n^2}tr(B_K^2) \tag{A.31}$$

and evaluating $tr(B_K^2)$ we find

$$\begin{aligned} tr(B_K^2) &= \tfrac{1}{4}tr(J'Q_K J + 2Q_K J Q_K J + Q_K J J' Q_K) \\ &= \tfrac{1}{4}(S_1 + S_2 + S_3), \text{ say.} \end{aligned} \tag{A.32}$$

Start with the first term

$$\begin{aligned} S_1 &= tr(J'Q_K J) \\ &= tr(J'J) - \frac{1}{n}tr(J'\Phi_K\Phi_K'J) + \mathrm{O}\left(\frac{K^2}{n^2}n^2\right) \\ &= \frac{n(n-1)}{2} - \frac{1}{n}tr(J'\Phi_K\Phi_K'J) + \mathrm{O}(K^2). \end{aligned}$$

Now

$$\begin{aligned} tr(J'\Phi_K\Phi_K'J) &= tr(\Phi_K'JJ'\Phi_K) \\ &= n^3\sum_{k=1}^{K}\frac{1}{n^2}\sum_{t,s=1}^{n}\varphi_k\left(\frac{t}{n}\right)\left(\frac{(t-1)\wedge(s-1)}{n}\right)\varphi_k\left(\frac{s}{n}\right) \\ &= n^3\sum_{k=1}^{K}\left[\int_0^1\int_0^1 \varphi_k(r)(r\wedge p)\varphi_k(p)\,\mathrm{d}r\,\mathrm{d}p + \mathrm{O}\left(\frac{1}{n}\right)\right] \\ &= n^3\sum_{k=1}^{K}\lambda_k + \mathrm{O}\left(n^3\frac{K}{n}\right) = \frac{n^3}{\pi^2}\sum_{k=1}^{K}\frac{1}{(k-\frac{1}{2})^2} + \mathrm{O}(n^2K) \\ &= \frac{n^3}{\pi^2}\left[\frac{\pi^2}{2} + \mathrm{o}(1)\right] + \mathrm{O}(n^2K) = \frac{n^3}{2} + \mathrm{o}(n^3). \end{aligned}$$

Thus,

$$\frac{1}{n^2}S_1 = \frac{1}{2} - \frac{1}{2} + o(1) = o(1), \tag{A.33}$$

as $n \to \infty$. Next consider S_3 in (A.32). We have

$$\begin{aligned} S_3 &= tr(Q_K JJ' Q_K) = S_1 - tr(\Phi_K(\Phi_K'\Phi_K)^{-1}\Phi_K' JJ' Q_K) \\ &= S_1 = o(n^2), \end{aligned}$$

from (A.33). Finally, S_2 in (A.32) is

$$\begin{aligned} S_2 &= tr(Q_K J Q_K J) = tr(J^2 Q_K) - tr(\Phi_K(\Phi_K'\Phi_K)^{-1}\Phi_K' J Q_K J) \\ &= tr(J^2) - 2tr(J^2\Phi_K(\Phi_K'\Phi_K)^{-1}\Phi_K') \\ &\quad + tr(\Phi_K(\Phi_K'\Phi_K)^{-1}\Phi_K' J \Phi_K(\Phi_K'\Phi_K)^{-1}\Phi_K' J). \end{aligned} \tag{A.34}$$

Now

$$tr(J^2) = 0 \tag{A.35}$$

and

$$\begin{aligned} &tr(J^2\Phi_K(\Phi_K'\Phi_K)^{-1}\Phi_K') \\ &= \frac{1}{n} tr(\Phi_K' J^2 \Phi_K) + O\left(\frac{K^2}{n^2} n^2\right) \end{aligned} \tag{A.36}$$

$$\begin{aligned} &= n^2 \sum_{k=1}^{K} \frac{1}{n^2} \sum_{t=3}^{n} \sum_{s=1}^{t-2} \varphi_k\left(\frac{t}{n}\right)\left(\frac{(t-s-1)}{n}\right)\varphi_k\left(\frac{s}{n}\right) + O(K^2) \\ &= n^2 \sum_{k=1}^{K}\left[\int_0^1 \varphi_k(r)\int_0^r (r-p)\varphi_k(p)\,\mathrm{d}p\,\mathrm{d}r + O\left(\frac{1}{n}\right)\right] + O(K^2). \end{aligned} \tag{A.37}$$

Upon calculation, the double integral in the summation in (A.37) is

$$\begin{aligned} &\int_0^1 \varphi_k(r)\int_0^r (r-p)\varphi_k(p)\,\mathrm{d}p\,\mathrm{d}r \\ &= 2\int_0^1 \sin[(k-1/2)\pi r]\int_0^r (r-p)\sin[(k-1/2)\pi p]\,\mathrm{d}p\,\mathrm{d}r \\ &= \frac{1}{(k-1/2)^2\pi^2}. \end{aligned}$$

It follows that

$$tr(J^2\Phi_K(\Phi_K'\Phi_K)^{-1}\Phi_K') = n^2\sum_{k=1}^{K}\frac{1}{(k-1/2)^2\pi^2} + \mathrm{o}(n^2)$$

$$= \frac{n^2}{\pi^2}\left[\frac{\pi^2}{2} + \mathrm{o}(1)\right] + \mathrm{o}(n^2) = \frac{n^2}{2} + \mathrm{o}(n^2). \tag{A.38}$$

Next, we evaluate

$$tr(\Phi_K(\Phi_K'\Phi_K)^{-1}\Phi_K'J\Phi_K(\Phi_K'\Phi_K)^{-1}\Phi_K'J)$$

$$= \frac{1}{n^2}tr([\Phi_K'J\Phi_K]^2) + \mathrm{O}\left(\frac{K^2}{n^2}n^2\right)$$

$$= n^2\sum_{k,\ell=1}^{K}\left[\frac{1}{n^2}\sum_{t=3}^{n}\sum_{s=1}^{t-2}\varphi_k\left(\frac{t}{n}\right)\varphi_\ell\left(\frac{s}{n}\right)\right]\left[\frac{1}{n^2}\sum_{t=3}^{n}\sum_{s=1}^{t-2}\varphi_\ell\left(\frac{t}{n}\right)\varphi_k\left(\frac{s}{n}\right)\right] + \mathrm{O}(K^2)$$

$$= n^2\sum_{k,\ell=1}^{K}\left[\int_0^1\varphi_k(r)\int_0^r\varphi_\ell(p)\,\mathrm{d}p\,\mathrm{d}r\right]\left[\int_0^1\varphi_\ell(r)\int_0^r\varphi_k(p)\,\mathrm{d}p\,\mathrm{d}r\right]$$

$$+ \mathrm{O}(K^2). \tag{A.39}$$

The integrals appearing in this expression reduce to

$$\int_0^1\varphi_k(r)\int_0^r\varphi_\ell(p)\,\mathrm{d}p\,\mathrm{d}r = \frac{2}{(\ell-1/2)\pi}\frac{1}{(k-1/2)\pi}.$$

Then

$$tr(\Phi_K(\Phi_K'\Phi_K)^{-1}\Phi_K'J\Phi_K(\Phi_K'\Phi_K)^{-1}\Phi_K'J)$$

$$= n^2\sum_{k,\ell=1}^{K}\left[\frac{2}{(\ell-1/2)\pi}\frac{1}{(k-1/2)\pi}\right]^2 + \mathrm{o}(n^2)$$

$$= 4n^2\left(\sum_{k=1}^{K}\left[\frac{1}{(\ell-1/2)\pi}\right]^2\right)^2 + \mathrm{o}(n^2)$$

$$= \frac{4n^2}{\pi^4}\left(\left[\frac{\pi^2}{2} + \mathrm{o}(1)\right]\right)^2 + \mathrm{o}(n^2) = n^2 + \mathrm{o}(n^2). \tag{A.40}$$

Adding the components (A.35), (A.38) and (A.40) of (A.34), we get:

$$S_2 = 0 - 2\frac{n^2}{2} + n^2 + \mathrm{o}(n^2) = \mathrm{o}(n^2),$$

and therefore $(1/n^2)S_2 = o(1)$ and so from (A.31) and (A.32)

$$\operatorname{var}\left[\frac{1}{n}u'B_K u\right] = \frac{1}{n^2}tr(B_K^2) = o(1),$$

as required for (A.30). □

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References

Csörgő, M., Horváth, L., 1993. Weighted Approximations in Probability and Statistics. Wiley, New York.

Durlauf, S.N., Phillips, P.C.P., 1988. Trends versus random walks in time series analysis. Econometrica 56, 1333–1354.

Gradshteyn, I.S., Ryzhik, I.M., 1994. Tables of Integrals, Series and Products. Academic Press, London, UK.

Nelson, C.R., Plosser, C., 1982. Trends and random walks in macroeconomic time series: some evidence and implications. Journal of Monetary Economics 10, 139–162.

Ouliaris, S., Park, J.Y., Phillips, P.C.B., 1989. Testing for a unit root in the presence of a maintained trend. In: Raj, B. (Ed.), Advances in Econometrics and Modelling. Kluwer Academic Publishers (Chapter 1), Norwell, MA.

Phillips, P.C.B., 1986. Understanding spurious regressions in econometrics. Journal of Econometrics 33, 311–340.

Phillips, P.C.B., 1987. Time series regression with a unit root. Econometrica 55, 277–301.

Phillips, P.C.B., 1996a. Spurious Regression Unmasked. Cowles Foundation Discussion Paper, #1135.

Phillips, P.C.B., 1996b. Econometric model determination. Econometrica 64, 763–812.

Phillips, P.C.B., 1998. New tools for understanding spurious regressions. Econometrica 66, 1299–1326.

Phillips, P.C.B., Moon, H.R., 1999. Linear regression limit theory for nonstationary panel data. Econometrica 67, 1057–1111.

Phillips, P.C.B., Perron, P., 1988. Testing for a unit root in time series regression. Biometrika 75, 335–346.

Phillips, P.C.B., Ploberger, W., 1996. An asymptotic theory of Bayesian inference for time series. Econometrica 64, 381–413.

Phillips, P.C.B., Solo, V., 1992. Asymptotics for linear processes. Annals of Statistics 20, 971–1001.

Ploberger, W., Phillips, P.C.B., 2002. Rissanen's theorem and econometric time series. In: Hugo, A. Keuzenkamp, Michael McAleer, Arnold Zeller (Eds.), Simplicity, Inference and Econometric Modelling. Cambridge University Press, Cambridge.

Ploberger, W., Phillips, P.C.B., 2003. Empirical limits for time series econometric models, Econometrica, forthcoming.

Said, S.E., Dickey, D.A., 1984. Testing for unit roots in autoregessive-moving average models of unknown order. Biometrika 71, 599–608.

Shorack, G.R., Wellner, J.A., 1986. Empirical Processes with Applications to Statistics. Wiley, New York.

Xiao, Z., Phillips, P.C.B., 1998. An ADF coefficient test for a unit root in ARMA models of unknown order with empirical applications to the US economy. The Econometrics Journal 1, 27–43.