

**THE BILL PHILLIPS LEGACY OF CONTINUOUS  
TIME MODELLING AND ECONOMETRIC MODEL DESIGN**

**AND**

**THE PUBLISHED PAPERS**

**BY**

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## 36 The Bill Phillips Legacy of Continuous Time Modelling and Econometric Model Design

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Bill Phillips' contributions to econometrics came at a time when the subject was almost exclusively concerned with applications that involved discrete time series data and models that were based on simultaneous equations. Not only was theoretical research in econometrics during the 1950s and 1960s, when Phillips did his work, dominated by the concerns of simultaneous equations systems, but empirical applications were also predominantly based on systems of this type. Since the available observations of economic variables were discrete time series data that were commonly measured at annual and quarterly intervals, it is logical that most econometric studies of the time were concerned with the development of statistical methods for simultaneous equations models that were fitted with discrete time series data.

One of Phillips' greatest contributions to econometrics is that he opened up a new field of research on continuous time econometric modelling and statistical inference. This field contrasted in important ways with the simultaneous equations paradigm that dominated the thinking of Phillips' contemporaries. In the first place, the probabilistic framework of continuous time stochastic models was necessarily more sophisticated than discrete time series in order to accommodate the function space realisations of random processes like Brownian motion. Secondly, the models themselves were formulated as recursive systems in terms of stochastic differential equations rather than as non-recursive systems like simultaneous equations. In consequence, the models were conceptually and causally quite different from simultaneous equations. These differences turn out to be pivotal in the construction of continuous systems of real-world economic processes, a fact that Phillips was acutely aware of in his own work. In several of the papers reprinted in this volume, Phillips remarks on these distinctions and points out their implications regarding matters of statistical estimation and identification. This is one of many examples in which Phillips' econometric work shows sensitivity to con-

cerns of model design, as well as matters of statistical estimation and inference.

Mathematicians since Wiener (1923) had been interested in the development of the probability space underpinnings of these models. And some statisticians, notably Bartlett (1946, 1955) Grenander (1950) and Bartlett and Rajalakshman (1953), had written on issues of statistical estimation. But continuous time statistical models had been almost totally ignored by econometricians. The sole exception was a short paper by Koopmans (1950b) that provided some early arguments for the merits of modelling in continuous time.

In consequence, when Phillips commenced his research in the 1950s on continuous time econometrics, it was a brand new field and his work represented a bold new departure from the prevailing tradition of Cowles-Commission-style econometrics. It was a courageous move. For, while there was some interest in the United Kingdom, the new line of research that Phillips pioneered attracted only a small following and it was all but ignored by the large and growing community of econometricians in North America. In part, this is explained by the on-going preoccupation with implementing simultaneous equations methods. But, it is also fair to say that continuous time econometrics was a subject that seemed technically forbidding to researchers whose training was limited to discrete time series and this undoubtedly limited entry to the new field.

Phillips' work proceeded on two fronts. The first of these was the development of dynamic macroeconomic models in terms of systems of differential equations whose purpose was to explain business cycle behaviour and to study control mechanisms within such systems. The main papers that grew out of this work are reprinted in chapters 40 and 41. These papers not only added to economic knowledge concerning business cycles and economic policy, they also made important methodological contributions to the design of economic models. It is in the latter respect that they have had the most enduring influence. The models Phillips constructed had much in common with engineering models of physical systems. Most of the differential equations were formulated as adjustment mechanisms in which variables like aggregate consumption and investment adjusted in a continuous way toward steady state values of these variables that were formulated so that the system was balanced in equilibrium or along equilibrium growth paths. These trade cycle and cyclical growth models were the harbingers of a broad group of econometric models that have since come to be known as error-correction mechanisms. Discrete time analogues of these models later appeared in Sargan's (1964) celebrated study of wages and prices in the United Kingdom, and they were subsequently used in the frequently cited empiri-

cal consumption function study of Davidson, Hendry, Srba and Yeo (1978). With the recent advent of the field of unit root econometrics and cointegration, the original work of Phillips in formulating models as adjustment mechanisms about equilibrium values has taken on a new significance in terms of its empirical implications. The historical importance and relevance of his work to the subject of error-correction modelling has still to be widely recognised in the literature and is, unfortunately, not cited in any of the recent textbooks, handbooks or overviews of the subject, although the priority and significance of his work on this topic have been pointed out (Phillips and Loretan 1991).

One might speculate that it was the engineering discipline of formulating models that had well-defined steady state solutions and stable adjustment mechanisms about those solutions that guided Bill Phillips naturally to a class of model that was capable of dealing in a coherent way with both stationary and non-stationary time series. Interestingly, this discipline also seems to have steered Phillips away from mechanistic forms of univariate time series modelling like the methods of Box and Jenkins (1976) which later grew into prominence in statistics. On the contrary, the error-correction mechanisms that dominated the early models of Phillips were essentially multivariate in character. It is this class of model that has survived and prospered in econometrics and now forms an essential part of the toolbox of empirical researchers in macroeconomics.

The second front for Bill Phillips' research was the direct empirical task of constructing and estimating continuous time econometric systems with discrete data. In tackling this problem, Phillips wrote the first scientific paper to deal exclusively with the problem of estimating the parameters in systems of stochastic differential equations with discrete time series data. That paper was published in the statistical journal *Biometrika* in 1959 and is reprinted here as chapter 42. Since Phillips' original work, a very complete statistical methodology of continuous time econometrics has been developed covering methods of estimation, inference, forecasting, policy analysis and control, diagnostic testing and numerical computation. Many of the articles that are central to these developments are contained in the two volumes by Bergstrom (1976, 1990), and the methods are now discussed at an introductory level in some texts, like that of Gandolfo (1981).

All of this work on econometric model design and statistical estimation can rightfully be seen as research that is part of Bill Phillips' legacy to econometrics. While none of his models or his methods of statistical estimation are actually used in empirical econometric work today, Phillips was responsible for opening up an approach to modelling and

an arena for research on continuous time econometrics that has proved to be of extraordinary consequence to present day research. Somewhat ironically, given the dominant preoccupation of his contemporaries in the 1950s and 1960s, Phillips' line of research has proved to be every bit as important to econometrics as the simultaneous equations model and it is potentially of much greater significance to modern time series econometric methods and applications. Moreover, in recent years, a much wider range of researchers have become interested in continuous time econometric systems and this has broadened the field in exciting new ways. There are several reasons for this widening of his legacy.

While it was macroeconomic applications of these methods that interested Bill Phillips, it is the large and growing subject area of finance that now offers fruitful applications of these methods and potential for a big empirical impact. In part, this is because data sources are much richer in finance than they are in macroeconomics, particularly so with regard to frequency of observation where we now have near continuous data recording of many financial variables like asset prices and exchange rates. In part too, financial econometric models have features that make them interesting potential candidates for continuous time econometric methodology. First, the economic theory models in finance on which they are based are most often themselves formulated in continuous time, prime examples being option price models in finance (for example, Merton 1990) and exchange rate target zone models in international finance (Krugman and Miller 1992). Also, many of these models involve non-linear rather than linear stochastic differential equations and offer an exciting area of development for researchers in continuous time econometrics, including methods of indirect inference (for example, Monfort 1996). Finally, stochastic calculus and the martingale structure of stochastic integrals have recently become firmly established as essential and revealing tools of analysis in models of asset pricing and exchange rate determination. In consequence, it has become natural to develop economic theory models in terms of stochastic differential equations (for example, Duffie 1988) and much more natural in turn to think of estimating such systems in empirical applications.

Knowing that stochastic differential equations generate data for which equispaced observations satisfy vector autoregressive moving average (VARMA) models in discrete time, Phillips recognised that the efficient estimation of continuous time models required algorithms for the estimation of VARMA models. Since none was available, Phillips devised one. His algorithm was presented in his Walras-Bowley lecture to the North American Meetings of the Econometric Society held in San Francisco in 1966. It was published posthumously in 1978 and is reprinted in chapter

45. Like his earlier paper in *Biometrika*, this paper broke new ground by being the first to tackle the general problem of estimating VARMA models and it predated the interest in these systems in the statistical literature of the 1970s and 1980s (Hannan and Dreistler 1988). Characteristically, the Phillips algorithm was one of creative improvisation. It used an iterated linearized least squares approach to overcome the central difficulty presented by the VARMA likelihood of being a very complicated rational function of the parameters. While the Phillips iteration is not used in VARMA estimation today, related recursive techniques (primarily those of Hannan and Rissanen (1982) and Hannan and Kavalieris (1984)) have indeed subsequently proved themselves to be a favoured practical method of estimating models with moving-average errors and of empirically determining the appropriate lag dimensions of these models. Interestingly, the latter problem is not addressed in Phillips' paper. One reason for this is that when the underlying system is a  $p$ th-order stochastic differential equation system the corresponding exact discrete model is a VARMA with known autoregressive order  $p$  and moving-average order of  $p - 1$ . Thus, in this special case as Phillips was well aware, order estimation is unnecessary because the correct orders are determined by the original specification of the continuous time system. There is now a huge literature on the subject of VARMA modelling, and many aspects of it are reviewed by Hannan and Deistler (1988), although curiously they do not reference Phillips' paper. Phillips' work on this topic illustrates his capacity to take a neglected subject and make an important original contribution long before it evolves into a rich field of research.

A final fascinating example of Phillips' uncanny ability to home in early on subjects of great importance is given by Robin Court in his essay on the Lucas Critique introducing chapters 50, 51 and 52. Court reveals that the 1968 Phillips paper (chapter 50) demonstrates a fundamental lack of identifiability of econometric models in the presence of endogenous policy rules and that this lack of identifiability affects reduced forms. As Court's essay documents, this problem is closely related to the Lucas Critique that policy changes affect the structure of econometric models and thereby frustratingly discombobulate econometric evidence. One can carry Robin Court's interesting discussion a little further. The usual antidote to the Lucas Critique is to set up models in terms of so-called deep structural parameters (for example, the parameters of an economic agent's preferences) that are invariant to policy changes. However, Phillips' point that the effect of endogenous policy rules may be to lose the identifiability of the reduced form means that even deep structural parameters may be unrecoverable when the reduced-

form coefficients are themselves unidentified. One can further speculate on the potential effects of unidentifiable reduced forms on the validity of econometric tests of the Lucas critique. To this extent at least, the 1968 Phillips paper (chapter 50) and his later unpublished work from 1972 (chapter 52) that is discussed by Court, may yet have an influence on subsequent research, irrespective of the historical issue of his work on this topic pre-dating that of Lucas (1976).

In conclusion, Bill Phillips' legacy to econometrics came primarily and most obviously from his direct scientific contributions to continuous time modelling and its associated statistical methods. However, his influence on the subject has been more pervasive than the immediate progeny of his scientific work on continuous time econometrics. Of great significance is his influence on matters of econometric model design, a subtle and vital subject on which there is no ultimate authority and on which there has been much debate and discussion in recent years. Phillips' work in the 1950s pointed a clear path to error correction mechanisms as a foundation for theoretical and empirical econometric models, a tradition that has been faithfully upheld in the continuous time literature ever since and that has recently gained strong support in discrete time series modelling. As discussed above, this approach is now very relevant to modern multivariate time series methods for modelling stationary and non-stationary processes. Moreover, Phillips' thinking on the effects of integrating endogenous policy into econometric analysis connects in surprisingly close ways with that of Lucas and seems to offer insights that are still worthy of exploration. Overall, his research is marked by a talent for creative thinking, improvisation and the courage to forge new ways forward rather than follow established scientific paradigms. And his papers demonstrate a concern for fundamentals and an ability to explain essentials in a simple and readable manner that makes for a wide readership. In all of this, Bill Phillips' work on econometrics continues to define its own unique paradigm and to be a fine example to the econometrics community.

## 37 The Published Papers

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*Peter C.B. Phillips*

Chapters 40, 41, 42 and 43 comprise Bill Phillips' published works on the subject of formulating dynamic continuous time econometric models, estimating these models with discrete time series data, and using them in prediction and for economic policy. The papers represent a progressive series of attempts to tackle this subject and they show a growing recognition of its many different practical and conceptual aspects.

Chapter 42 on 'The Estimation of the Parameters in Systems of Stochastic Differential Equations' offers the most general treatment of the subject and it is the most cited of these papers. Historically, it is a landmark contribution, being the first paper in the literature of statistics and economics to deal exclusively with the problem of estimating continuous time systems with discrete observations. It is also the first paper in what has subsequently become a large econometric literature on this general subject. It can rightfully be thought of as the pioneering work in this research area.

The model formulated by Phillips in this paper is the linear system of lagged dependencies

$$y_i(t) = \sum_{j=1, j \neq i}^n \int_{-\infty}^t w_{ij}(t-s)y_j(s)ds + \sum_{j=1}^n \int_{-\infty}^t r_{ij}(t-s)\zeta_j(s)ds \quad (1)$$

relating the observable variables  $y_i(t)$  for  $i = 1, \dots, n$  and the unobservable disturbances  $\zeta_j(t)$ , all measured in continuous time. If the weighting functions  $w_{ij}(\cdot)$  and  $r_{ij}(\cdot)$  are restricted to the class of square integrable functions, whose Laplace transforms are proper rational functions, then (1) can be regarded as the solution of the stochastic differential equation system

$$F(D)y(t) = G(D)\zeta(T), \quad (2)$$



where  $F$  and  $G$  are matrix polynomials in the stochastic differential operator  $D = d/dt$ , with the degree of  $G$  usually being lower than the degree of  $F$ .

As it stands, the system (2) is rather general and we can expect that many different models of economic behaviour can usefully be represented by such a system. Chapter 40, 'Some Notes on the Time-Forms of Reactions in Interdependent Dynamic Systems' discusses some explicit versions of this type of model, including the following consumption function which relates aggregate consumption,  $C$ , to past income,  $Y$ , according to the relation

$$C(t) = \int_0^t f(\tau)Y(t - \tau)d\tau + \kappa + \varepsilon(t). \quad (3)$$

Here, the time form of the response expires after a finite time period  $l$ , whereas in the general system (1) the initial conditions are effectively set in the infinite past. Much of chapter 40 (first published in 1956) is spent studying time forms of reactions of one economic variable to another in equations like (3) and interdependent systems involving several such stochastic integral equations.

In (3) the variables are quantified as instantaneous rates of consumption and instantaneous rates of income generation, whereas econometric measurements of them are time averages given by integrals over unit time periods such as  $C_t = \int_{t-1}^t C(s)ds$ , and  $Y_t = \int_{t-1}^t Y(s)ds$ . Such time averaging involves an information loss and hence discrete time dynamic relations between  $C_t$  and  $Y_{t-k}$ , for  $k \geq 0$ , will only approximately represent the true time form of the dynamic response in the original continuous time relation (3). Phillips studies this phenomena and illustrates how the original dynamic reaction function can be quite poorly represented in discrete time. Fifteen years later, Sims (1971a, 1971b) made precisely the same point and gave a formal mathematical analysis of discrete distributed lag approximations to continuously distributed lags. Curiously, Sims did not refer to Phillips' earlier 1956 contribution on this subject and must simply have been unaware of it.

In addition to studying the time forms of lagged responses, chapter 40 also makes some suggestions about econometric estimation of the discrete dynamic model approximation. This part of the paper has an interesting concluding section that surveys several practical possibilities which range from estimating unrestricted dynamic systems, through to estimating specific lag relationships (recognised by Phillips to be inevitably misspecified). An intriguing feature of this final section is the proposal by Phillips to use an analogue computing device (or 'simulator', as Phillips

called it) to compute the integrated 'fitted' values of the dependent variable which could be compared with the observed discrete data on an oscilloscope. Phillips (chapter 40) goes on to propose

that the time-forms of the reaction functions set on the separate response simulators would be adjusted by a trial and error process to obtain the minimum integral of the square of the residual.

Interestingly, this proposal is very similar to modern methods of dealing with non-linear stochastic differential equations using the so-called method of indirect inference.<sup>1</sup>

In chapter 40, Phillips develops an estimation procedure for the general system of linear stochastic differential equations given in (2). His idea is to expand the Laplace transform of the auto-covariance matrix function of the system variates  $y(t)$  in partial fractions and use sample auto-covariance matrices from discrete data to estimate the unknown elements in this expansion and recover the coefficients in the lag polynomials in (2) from this expansion.

Ingenious though this procedure was, it has several major shortcomings and appears never to have been applied in practice either by Phillips himself or anyone else, although the paper ends with a comment that simulation experiments were being conducted. One disadvantage is that the procedure makes no allowance for potential *a priori* restrictions on the parameters of the system (2). Conceivably, this difficulty could be dealt with by applying a generalised method of moment approach instead of Phillips' moment matching method. A more serious disadvantage is that the method does not take into account the aliasing problem in fitting continuous systems with discrete data, which was discussed later in papers by Telser (1967) and myself (Phillips 1973). Both of these points can be illustrated by reference to the special first-order differential equation case of (2) in which

$$F(D) = I - AD, \quad \text{and} \quad G(D) = I;$$

where the  $n \times n$  matrix  $A$  has distinct eigenvalues in the left half plane, so the system is stationary. The method proposed by Phillips leads in this case to the estimate  $A^*$  satisfying

$$\exp(A^*) = C_1 C_0^{-1}, \quad (4)$$

where

$$C_1 = T^{-1} \sum_{t=1}^T y(t)y(t-1)', \quad C_0 = T^{-1} \sum_{t=1}^T y(t)y(t)'$$

(In fact, somewhat earlier than Phillips, Quenouille (1957) had suggested this estimator for the first-order stochastic differential equation case.)

First, note that if there are restrictions on the coefficient matrix  $A$ , then the estimator  $A^*$  will not necessarily satisfy these restrictions. However, a GMM estimator could be designed to overcome this difficulty by writing down the moment matching equations, functionalising  $A = A(\theta)$  in terms of the unrestricted parameters and then estimating  $\theta$  by minimising an appropriately weighted quadratic form constructed from the differences  $\exp(A(\theta)) - C_1 C_0^{-1}$ . This type of GMM approach to the estimation of stochastic differential equations does not yet seem to have been tried, and may be useful here and in incomplete systems.

Second, the equation system (4) cannot be solved uniquely without imposing conditions on the imaginary parts of the complex eigenvalues of the matrix  $B^* = C_1 C_0^{-1}$ . This is because the matrix function  $A^* = \log(B^*)$  is a multi-valued function when  $B^*$  has complex latent roots. This is the manifestation of the aliasing problem in econometric estimation of a continuous system coefficient matrix  $A$  with discrete data. If the latent roots of  $B^*$  are all real and positive then the problem does not arise, but complex roots can be expected in all but the smallest systems in practical applications. A final practical difficulty with this approach is that when negative real roots of  $B^*$  arise, the system does not admit a solution.

While all of these problems can potentially be overcome, the Phillips approach was never used. A short time after Phillips' work, Bergstrom (1966a) suggested a highly practical approach for econometric estimation that utilised standard simultaneous equations methodology. In 1972, I gave a non-linear regression approach that was based on the exact reduced-form model satisfied by the data in discrete time and this method yielded consistent and asymptotically efficient estimates of the parameters of first-order stochastic differential equations (Phillips 1972). These two methods and their various extensions have formed the basis of most empirical work in the field ever since.<sup>2</sup>

Chapters 42 and 43 are short studies that focus on certain issues of specification, prediction and regulation of continuous time econometric systems. The approach in both papers is to proceed by way of illustrative examples and no general treatment is attempted. The joint paper with Quenouille (chapter 43) discusses the specification of discrete time dynamic models – vector autoregressions (VARs) and structural VARs – and differential equations. Here, and in chapter 42, it is argued that recursive formulations provide more realistic models in continuous time than non-recursive systems basically because behavioural units like economic agents do not respond to change instantaneously. In terms of the formulation of systems like (1) and (3) this means that the response

functions do not contain any delta function impulses, and then in (2) the numerator polynomials in the differential operator are of lower degree than the denominator polynomials. Such restrictions aid in the identification of these systems. To wit, in the authors' words in chapter 43 (although we may note that these ignore aliasing difficulties):

In both continuous and discrete systems the problem of identification will often be overcome if careful attention is given to the formulation of the basic model as a system of dynamic behaviour relationships.

Both these papers discuss the regulation of economic systems using feedback control mechanisms. Part of the discussion is conducted in terms of non-stochastic systems, but part also deals with stochastic optimum control in the sense of Wiener (1949), where the object is to minimise a certain mean squared error criterion function involving the variable that is the object of the stabilisation policy. Chapter 41, on 'Cybernetics and the Regulation of Economic Systems' makes the point particularly strongly that excessive feedback can easily be the cause of instability in economic systems, just as it is in mechanical systems. In stochastic systems 'the optimum strength of the stabilisation policy depends on the quantitative values of all the relationships in the system and also on the auto-correlation function of the disturbances'.

Phillips was cautious in drawing conclusions from the illustrative examples in these papers. He saw that they pointed out a clear need to know much more about the quantitative relations between economic variables before sensible and helpful economic policy could be formulated on a scientific econometric basis, a conclusion that was echoed years later by Christ (1975) in his empirical evaluation of the performance of econometric models of the USA.

## Notes

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1 See Monfort (1996).

2 See Bergstrom (1988) for an historical review.