01.4.2. *Nonorthogonal Hilbert Projections in Trend Regression*, proposed by P.C.B. Phillips and Y. Sun. An observed continuous time process $X(t)$ is generated by the linear system

$$X(t) = \beta' Z(t) + W(t), \qquad t \in [0,1], \tag{1}$$

where $W(t)$ is an unobservable standard Brownian motion, $Z(t) = (t, \ldots, t^p)'$ is a time polynomial vector, and β is a parameter vector to be estimated.

The following two estimators of β are proposed:

$$\hat{\beta} = \left(\int_0^1 Z(t) Z(t)' \, dt \right)^{-1} \left(\int_0^1 Z(t) X(t) \, dt \right)$$

and

$$\tilde{\beta} = \left(\int_0^1 Z^{(1)}(t) Z^{(1)}(t)' \, dt \right)^{-1} \left(\int_0^1 Z^{(1)}(t) \, dX(t) \right),$$

where $Z^{(1)}$ is the vector of the first derivatives of Z.

Part A

1. Show that both $\hat{\beta}' Z(t)$ and $\tilde{\beta}' Z(t)$ are Hilbert projections in $L_2[0,1]$. How do these projections differ?
2. Find the distributions of $\hat{\beta}$ and $\tilde{\beta}$ and compare them in the case where $p = 1$. What do you conclude?

Part B

Suppose the system generating $X(t)$ is

$$X(t) = \beta' Z(t) + J_c(t), \qquad t \in [0,1], \tag{2}$$

where $J_c(t) = \int_0^t e^{(t-s)c} dW(s)$ for some known constant $c < 0$ is a linear diffusion or Orstein–Uhlenbeck process.

1. Are $\hat{\beta}' Z(t)$ and $\tilde{\beta}' Z(t)$ still Hilbert projections?
2. Calculate the distributions of $\hat{\beta}$ and $\tilde{\beta}$ and compare them in the case where $p = 1$. What do you conclude?
3. Taking c to be known, can you suggest an unbiased linear estimator of β that has smaller variance than $\hat{\beta}$ and $\tilde{\beta}$? Does it correspond to another Hilbert projection?