94.5.2. *Spurious Regression and Generalized Least Squares*, proposed by Peter C.B. Phillips and Douglas J. Hodgson. Let X_t be an $I(1)$ n-vector process generated by

$$\Delta X_t = \nu_t = C(L)e_t, \qquad C(L) = \sum_{j=1}^{\infty} C_j L^j, \qquad \sum_{0}^{\infty} j^{1/2} |C_j| < \infty,$$

where $e_t \equiv$ i.i.d.$(0, \Sigma_e)$ with finite fourth cumulants. Suppose that the matrix $C(1)\Sigma_e C(1)'$ is positive definite. Partition X_t as $X_t' = (X_{1t}', X_{2t}')$ into subvectors of dimension n_1 and n_2. Consider the following two regressions:

$$X_{1t} = \hat{B}X_{2t} + \hat{u}_t, \tag{P1}$$

$$\Delta X_{1t} = \tilde{B}\Delta X_{2t} + \tilde{u}_t. \tag{P2}$$

(a) Determine the asymptotic behavior of $\hat{B}$ and $\tilde{B}$ and give a limit theory for these estimators appropriately scaled and centered.

(b) Let $\hat{P}$ be the estimated autoregressive coefficient in the following regression of the residuals from (P1):

$$\hat{u}_t = \hat{P}\hat{u}_{t-1} + \text{error}.$$

Determine the asymptotic behavior of $\hat{P}$ and give a limit theory for an appropriately scaled and centered statistic based on $\hat{P}$.

(c) Use $\hat{P}$ to apply a matrix Cochrane–Orcutt transformation to the data and thereby estimate the coefficient matrix from the linear regression

$$\tilde{X}_{1t} = Z_t \operatorname{vec}(B^*) + \text{error}, \tag{P3}$$

where

$$\tilde{X}_{1t} = X_{1t} - \hat{P}X_{1t-1},$$

and

$$Z_t = I \otimes X_{2t}' - \hat{P} \otimes X_{2t-1}'.$$

Compare the estimates B^* and $\tilde{B}$. Is there any asymptotic relation between them?