94.5.1. *Fully Modified Least Squares in $I(2)$ Regression*, proposed by Peter C.B. Phillips and Yoosoon Chang. Consider the model

$$y_t = Bx_t + u_{0t} \tag{1}$$

$$\Delta^2 x_t = u_{xt} \tag{2}$$

where $u_t = (u_{0t}', u_{xt}')'$ is a stationary time series that satisfies the functional central limit theorem

$$n^{-1/2} \sum_{1}^{[n\cdot]} u_t \to_d B(\cdot) \equiv \mathrm{BM}(\Omega),$$

where $\Omega = \mathrm{lrvar}(u_t)$, the long-run variance matrix of $u_t$ and $\mathrm{BM}(\Omega)$ is vector Brownian motion with covariance matrix $\Omega$. Partition $B(\cdot)$ and $\Omega$ conformably with $u_t$ as $(B_0', B_x')'$ and

$$\Omega = \begin{bmatrix} \Omega_{00} & \Omega_{0x} \\ \Omega_{x0} & \Omega_{xx} \end{bmatrix}.$$

You may assume that $\Omega$ is positive definite.

(i) Find the limit distribution of the least-squares estimator of the matrix $B$ in (1).
(ii) Construct a *new*, fully modified least-squares estimator of $B$ by using the information in equation (2) and show that your estimator has a mixed normal limit distribution.
(iii) See if you can find the limit distribution of the usual Phillips–Hansen [1] fully modified estimator of $B$ in (1).

*REFERENCE*

1. Phillips, P.C.B & B.E. Hansen. Statistical inference in instrumental variable regressions with I(1) processes. *Review of Economic Studies* 57 (1990): 99–125.