94.3.5. *Some Exponential Martingales*, proposed by Peter C.B. Phillips and Douglas J. Hodgson. Let Z_n be the random walk

$$Z_n = Z_{n-1} + u_n$$

where $Z_0 = 0$ and $u_n \equiv$ i.i.d. (0,1) with $\varphi(\lambda) = E\{\exp(\lambda u_i)\}$.

(a) Show that $M_n = \exp(\lambda Z_n)/\varphi(\lambda)^n$ is a martingale with respect to the filtration $\mathcal{F}_n = \sigma(u_n, u_{n-1}, \ldots, u_1)$.

(b) Show that if $u_i \equiv$ i.i.d. $N(0,1)$, then

$$M_n = n^{-1/2} \exp\{(1/2n) Z_n^2\}$$

satisfies the martingale property

$$E(M_n \mid \mathcal{F}_{n-1}) = M_{n-1} \quad \text{for } n = 2,3,\ldots.$$

Is M_n a martingale? If not, show how to construct a martingale from M_n.