

94.2.6. *Convergence of a Nonlinear Time Series Model*, proposed by Peter C.B. Phillips. In the model

$$X_t = \left(\frac{1}{2} + \varepsilon_t\right)X_{t-1}, \quad t = 1, 2, \dots$$

the shocks ε_t are independent and identically distributed with mean zero and variance $\frac{1}{4}$, and X_0 is a random variable with zero mean and finite variance $\sigma^2 > 0$.

Show that $Z_t = 2^{t/2}X_t$ converges almost surely as $t \rightarrow \infty$. (Hint: Use the martingale convergence theorem for a suitable function of Z_t .) Hence, show that $X_t \rightarrow_{\text{a.s.}} 0$ as $t \rightarrow \infty$.