

93.4.4. *Reduced Rank Regression Asymptotics in Multivariate Regression*, proposed by Peter C.B. Phillips. In the multivariate regression model

$$y_t = \Pi x_t + u_t, \quad (t = 1, \dots, T), \quad (1)$$

y_t is an n -vector of dependent variables, x_t is an m -vector of nonrandom exogenous variables, and $u_t \equiv$ i.i.d. $N(0, \Omega)$ with $\Omega > 0$ (positive definite). The coefficient matrix Π in (1) has the form

$$\Pi = \alpha\beta', \quad \beta'\beta = 1, \quad (2)$$

where α ($n \times 1$) and β ($m \times 1$) are unknown vectors of parameters. The condition $\beta'\beta = 1$, together with the requirement that the first nonzero element of β is positive, identifies the vector β in Π . It is proposed to estimate (α, β, Ω) in (1) by maximum likelihood. You may assume that $X'X/T = I_m$ for all T where $X' = [x_1, \dots, x_T]$.

(a) Show that the MLE $\hat{\beta}$ of β satisfies

$$\hat{\beta} = \operatorname{argmin} |\beta'X'Q_Y X\beta| / |\beta'X'X\beta|,$$

where $Q_Y = I - Y(Y'Y)^{-1}Y'$. Find the corresponding MLE's of α and Ω .

(b) Prove that $\hat{\beta} \xrightarrow{P} \beta$.

(c) Find the limiting distribution of $\sqrt{T}(\hat{\beta} - \beta)$ and $\sqrt{T}(\hat{\alpha} - \alpha)$ and their joint limit distribution.

are estimated by maximum likelihood, giving the estimates $(\hat{\alpha}, \hat{\beta}) = \hat{\theta}$. The matrix $M_{xx} \rightarrow I_3$ as $T \rightarrow \infty$.

- (a) Find the asymptotic distribution of $\sqrt{T}(\hat{\theta} - \theta)$ as $T \rightarrow \infty$.
 (b) Construct a Wald statistic for testing the hypothesis

$$H_0: \alpha\beta = 0$$

and find its limit distribution as $T \rightarrow \infty$.

- (c) Given x_{iT+1} , equation (1) is used to produce the forecast

$$\hat{y}_{T+1} = \hat{\alpha}x_{1T+1} + \hat{\beta}x_{2T+1} + \hat{\alpha}\hat{\beta}x_{3T+1}.$$

Find the asymptotic variance of the forecast error $y_{T+1} - \hat{y}_{T+1}$.

- (d) If the true values of the parameters in equation (1) are $\alpha = \beta = 0$ (but this is not known to the econometrician), how are your answers to questions (a)–(c) affected?