93.4.5. *Nonlinear Testing and Forecasting Asymptotics with Potential Rank Failure*, proposed by Peter C.B. Phillips. In the model

$$y_t = \alpha x_{1t} + \beta x_{2t} + \alpha\beta x_{3t} + u_t, \qquad (t = 1, \ldots, T),$$

the $x_{it} (i = 1,2,3)$ are nonrandom with positive definite sample second moment matrix M_{xx} and $u_t \equiv$ i.i.d. $N(0, \sigma^2)$. The parameters $(\alpha, \beta) = \theta$ in (1)

are estimated by maximum likelihood, giving the estimates $(\hat{\alpha}, \hat{\beta}) = \hat{\theta}$. The matrix $M_{xx} \to I_3$ as $T \to \infty$.

(a) Find the asymptotic distribution of $\sqrt{T}(\hat{\theta} - \theta)$ as $T \to \infty$.

(b) Construct a Wald statistic for testing the hypothesis

$$H_0: \alpha\beta = 0$$

and find its limit distribution as $T \to \infty$.

(c) Given x_{iT+1}, equation (1) is used to produce the forecast

$$\hat{y}_{T+1} = \hat{\alpha}x_{1T+1} + \hat{\beta}x_{2T+1} + \hat{\alpha}\hat{\beta}x_{3T+1}.$$

Find the asymptotic variance of the forecast error $y_{T+1} - \hat{y}_{T+1}$.

(d) If the true values of the parameters in equation (1) are $\alpha = \beta = 0$ (but this is not known to the econometrician), how are your answers to questions (a)-(c) affected?