92.3.4. *Generalized Inverses of Partitioned Matrices*, proposed by Peter C.B. Phillips. The covariance matrix Σ of the m-vector $x_t' = (x_{1t}', x_{2t}')$ is partitioned conformably with Σ as

$$\Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}.$$

Suppose Σ_{11} and $\Sigma_{11\cdot 2} = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{+}\Sigma_{21}$ are nonsingular where the affix "+" signifies the Moore Penrose inverse of a matrix. Define $\Sigma_{22\cdot 1} = \Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}$.

(i) Prove that

$$\Sigma_{22\cdot 1}^{+} = \Sigma_{22}^{+} + \Sigma_{22}^{+}\Sigma_{21}\Sigma_{11\cdot 2}^{-1}\Sigma_{12}\Sigma_{22}^{+}.$$

(ii) Show that

$$\Sigma^{+} = \begin{bmatrix} \Sigma_{11\cdot 2}^{-1} & -\Sigma_{11\cdot 2}^{-1}\Sigma_{12}\Sigma_{22}^{+} \\ -\Sigma_{22}^{+}\Sigma_{21}\Sigma_{11\cdot 2}^{-1} & \Sigma_{22\cdot 1}^{+} \end{bmatrix}.$$

Now suppose that $\Sigma_{11\cdot 2} = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{+}\Sigma_{21}$ and Σ_{11} may be singular. Define $\Sigma_{22\cdot 1} = \Sigma_{22} - \Sigma_{21}\Sigma_{11}^{+}\Sigma_{12}$.

(iii) Show that

$$\Sigma_g = \begin{bmatrix} \Sigma_{11\cdot 2}^{+} & -\Sigma_{11\cdot 2}^{+}\Sigma_{12}\Sigma_{22}^{+} \\ -\Sigma_{22}^{+}\Sigma_{21}\Sigma_{11\cdot 2}^{+} & \Sigma_{22}^{+} + \Sigma_{22}^{+}\Sigma_{21}\Sigma_{11\cdot 2}^{+}\Sigma_{12}\Sigma_{22}^{+} \end{bmatrix}$$

is a generalized inverse of Σ. Is Σ_g a Moore Penrose inverse?

(iv) Show that Σ_g is a generalized inverse of Σ if we replace the Moore Penrose inverses in its definition by arbitrary generalized inverses.

92.3.5. *Efficiency of Maximum Likelihood*, proposed by Peter C.B. Phillips. In the linear model

$$y_t = bx_t + u_t, \qquad (t = 1, \ldots, n) \tag{1}$$

the parameter has true value $b_0 \neq 0$ and $u_t \equiv$ i.i.d. $N(0, b_0^2)$. The x_t are nonrandom and $n^{-1}\Sigma_1^n x_t^2 \to m_x > 0$ as $n \to \infty$.

(i) Derive the asymptotic properties of the maximum likelihood estimator $\tilde{b}$ of b_0 in (1).

(ii) Compare the limit distribution of $\tilde{b}$ to that of the OLS estimator $\hat{b}$ of b_0 in (1). Is OLS asymptotically efficient?