

92.2.7. *Simultaneous Equations Bias in Level VAR Estimation*, proposed by Peter C.B. Phillips. Consider the bivariate cointegrated system

$$\begin{bmatrix} y_{1t} \\ y_{2t} \end{bmatrix} = \begin{bmatrix} 0 & a_{12} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y_{1t-1} \\ y_{2t-1} \end{bmatrix} + \begin{bmatrix} u_{1t} \\ u_{2t} \end{bmatrix}$$

where  $u_t = (u_{1t}, u_{2t})'$  is i.i.d.  $(0, \Sigma = (\sigma_{ij}))$ . Data generated from this system is estimated using a levels vector autoregression with one lag, that is, a VAR(1). Denote the estimates of the coefficients in the first equation obtained in this way by  $\hat{a}_{11}$  and  $\hat{a}_{12}$ .

- Find the asymptotic distribution of  $(\hat{a}_{11}, \hat{a}_{12})$ .
- Show that the asymptotic distribution of  $\hat{a}_{12}$  suffers from “simultaneous equations bias.”
- Show that the asymptotic distribution of  $\hat{a}_{12}$  is identical to that of the OLS estimate of  $a_{12}$  in the model

$$y_{1t} = a_{12}y_{2t-1} + u_{1t}.$$

Use this to explain the presence of “simultaneous equations bias” in the limit distribution.