92.2.8. *Partitioned Regression with Rank-Deficient Regressions*, proposed by Peter C.B. Phillips. Let the m-vectors x_t $(t = 1, \ldots, n)$ be i.i.d. $N(0, \Sigma)$ and suppose that x_t is partitioned into subvectors of dimension m_0, m_1, and m_2 with $m = m_0 + m_1 + m_2$ as $x_t' = (x_{0t}', x_{1t}', x_{2t}')$. The covariance matrix Σ is partitioned into blocks as $\Sigma = (\Sigma_{ij})(i, j = 0, 1, 2)$, conformably with x_t. The matrices $\Sigma, \Sigma_{ii}(i = 0, 1, 2)$ are all nonsingular.

(a) Show that the regression of x_{0t} on x_{1t} and x_{2t} can be written in the form

$$x_{0t} = \Sigma_{01}\Sigma_{11}^{-1}x_{1t} + \Sigma_{02\cdot 1}\Sigma_{22\cdot 1}^{-1}x_{2\cdot 1t} + x_{0\cdot 12t}, \qquad \textbf{(1)}$$

where we use the notation,

$$\Sigma_{ij\cdot k} = \Sigma_{ij} - \Sigma_{ik}\Sigma_{kk}^{-1}\Sigma_{kj},$$

$$x_{i\cdot jt} = x_{it} - \Sigma_{ij}\Sigma_{jj}^{-1}x_{jt}.$$

The residual $x_{0\cdot 12t}$ in (1) is independent of x_{1t} and x_{2t} and

$$x_{0\cdot 12t} \equiv N(0, \Sigma_{00\cdot 12}),$$

where

$$\Sigma_{00\cdot 12} = \Sigma_{00} - \Sigma_{01}\Sigma_{11}^{-1}\Sigma_{10} - \Sigma_{02\cdot 1}\Sigma_{22\cdot 1}^{-1}\Sigma_{20\cdot 1},$$

and "$\equiv$" signifies equivalence in distribution.

(b) Write the OLS regression equation of x_{0t} on x_{1t} and x_{2t} as

$$x_{0t} = \hat{B}_1 x_{1t} + \hat{B}_2 x_{2t} + \hat{x}_{0\cdot 12t}.$$

(i) Show that as $n \to \infty$

$$\hat{B}_1 \to_p B_1 = \Sigma_{01\cdot 2}\Sigma_{11\cdot 2}^{-1} = \Sigma_{01}\Sigma_{11}^{-1} - \Sigma_{02\cdot 1}\Sigma_{22\cdot 1}^{-1}\Sigma_{21}\Sigma_{11}^{-1},$$
$$\hat{B}_2 \to_p B_2 = \Sigma_{02\cdot 1}\Sigma_{22\cdot 1}^{-1} = \Sigma_{02}\Sigma_{22}^{-1} - \Sigma_{01\cdot 2}\Sigma_{11\cdot 2}^{-1}\Sigma_{12}\Sigma_{22}^{-1}.$$

(ii) Find the limit distribution as $n \to \infty$ of

$$\sqrt{n}[\hat{B}_1 - B_1, \hat{B}_2 - B_2]$$

and show that the limiting covariance between the matrices in the partition above is

$$-\Sigma_{00\cdot 12} \otimes \Sigma_{11\cdot 2}^{-1}\Sigma_{12}\Sigma_{22}^{-1} = -\Sigma_{00\cdot 12} \otimes \Sigma_{11}^{-1}\Sigma_{12}\Sigma_{22\cdot 1}^{-1}.$$

Verify the equality above.

(c) Now suppose that the matrices Σ_{11} and Σ_{22} are singular.

(i) Show that the regression of x_{0t} on x_{1t} and x_{2t} can be written in the form

$$x_{0t} = B_1 x_{1t} + B_2 x_{2t} + x_{0\cdot 12t}, \qquad (1)'$$

where $B_1 = \Sigma_{01}\Sigma_{11}^{-} - \Sigma_{02\cdot 1}\Sigma_{22\cdot 1}^{-}\Sigma_{21}\Sigma_{11}^{-}$, $B_2 = \Sigma_{02\cdot 1}\Sigma_{22\cdot 1}^{-}$, $\Sigma_{22\cdot 1} = \Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-}\Sigma_{12}$, $\Sigma_{02\cdot 1} = \Sigma_{02} - \Sigma_{01}\Sigma_{11}^{-}\Sigma_{12}$ and any choice of generalized inverses may be employed in these formulae. In (1)′ $x_{0\cdot 12t} \equiv$ i.i.d. $N(0, \Sigma_{00\cdot 12})$ and $\Sigma_{00\cdot 12} = \Sigma_{00} - \Sigma_{01}\Sigma_{11}^{-}\Sigma_{10} - \Sigma_{02\cdot 1}\Sigma_{22\cdot 1}^{-}\Sigma_{20\cdot 1}$, again for any choice of generalized inverse.

(ii) Are the coefficient matrices

$$B_1 = \Sigma_{01}\Sigma_{11}^{-} - \Sigma_{02\cdot 1}\Sigma_{22\cdot 1}^{-}\Sigma_{21}\Sigma_{11}^{-},$$

and

$$B_2 = \Sigma_{02\cdot 1}\Sigma_{22\cdot 1}^{-},$$

in (1)′ identified? If not, why not? Find matrices that are observationally equivalent to B_1 and B_2. What asymptotically estimable functions of these coefficient matrices exist (i.e., what functions of the coefficient matrices B_1 and B_2 can be consistently estimated)?

(iii) Show that least-squares regression of x_{0t} on x_{1t} and x_{2t} leads to coefficient matrices $\hat{B}_1$ and $\hat{B}_2$ whose form is as follows:

$$\hat{B}_1 = X_0'X_1(X_1'X_1)^{-} - X_0'Q_1X_2(X_2'Q_1X_2)^{-}X_2'X_1(X_1'X_1)^{-},$$

and

$$\hat{B}_2 = X_0' Q_1 X_2 (X_2' Q_1 X_2)^-,$$

where $Q_1 = I - X_1(X_1'X_1)^- X_1'$ and any choice of generalized inverse is used. In these formulae the matrices $X_i' = [x_{i1}, \ldots, x_{in}]$ are the data matrices of the variables x_{it} $(i = 0, 1, 2)$.

(iv) Find consistent estimators of the asymptotically estimable functions mentioned in part (ii) that are based on $\hat{B}_1$ and $\hat{B}_2$ as given in part (iii).