89.3.4. *Estimation and Testing in Linear Models with Singular Covariance Matrices*, proposed by Peter C.B. Phillips. Consider the model

$$y_t = Z_t\beta + u_t, \qquad t = 1, \ldots, T \tag{1}$$

where $\{u_t\} \equiv$ i.i.d. $N(0,\Sigma)$, y_t is an n vector of endogenous variables, Z_t is an $n \times k$ matrix of exogenous variables, and β is an unknown parameter vector. The covariance matrix Σ is known to be singular and the data are known to satisfy

$$R'y_t = 0, \qquad R'Z_t = 0 \text{ for all } t$$

so that $R'u_t = 0$ with probability one for a certain (known) matrix R ($n \times r$) of rank r.

1. Find the likelihood function for the model (1) and by using the first-order conditions write down the estimating equations for the maximum-likelihood estimates of (β,Σ).
2. Let $(\hat\beta,\hat\Sigma)$ denote the least-squares estimates of (β,Σ). Find the limiting distribution of $(\sqrt{T}(\hat\beta - \beta), \sqrt{T}(\hat\Sigma - \Sigma))$, stating any further conditions that you need to derive it.
3. Suppose $k = n$. Construct a Wald test of the null hypothesis

   $$H_0 : \beta'\Sigma\beta = 0$$

   using the least-squares estimates $(\hat\beta, \hat\Sigma)$ and give its limit distribution.

(i) Show that

$$M_{XX}^{-}\Sigma M_{MM}^{-} \underset{p}{\rightarrow} \Sigma^{+}.$$

(ii) Find the limit distribution of

$$\sqrt{n}(M_{XX}^{-} - \Sigma^{-})$$

and compare your result with that of part (3).