

90.4.6. *Testing Causality in an Autoregression with Cointegrated Regressors*, proposed by Peter C.B. Phillips and Hiro Toda. Consider the vector autoregression

$$\begin{bmatrix} y_{1t} \\ y_{2t} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} y_{1t-1} \\ y_{2t-1} \end{bmatrix} + \begin{bmatrix} \epsilon_{1t} \\ \epsilon_{2t} \end{bmatrix}$$

in which it is known that $a_{11} = 1$, $a_{22} = 0$, and where $\epsilon_t = (\epsilon_{1t}, \epsilon_{2t})'$ is i.i.d. $N(0, \Sigma)$ with $\Sigma = (\sigma_{ij})$. An investigator wishes to test the causality hypothesis

$$H_0: a_{12} = 0$$

and does so by running the single equation least-squares regression

$$y_{1t} = \hat{a}_{11} y_{1t-1} + \hat{a}_{12} y_{2t-1} + \hat{\epsilon}_{1t} \quad (1)$$

and by employing a conventional regression t -test.

Find the limit distribution of $t(\hat{a}_{12})$, the test statistic used by this investigator. Explain your result.