90.4.7. *Geometry of the Equivalence of OLS and GLS in the Linear Model*—Solution, proposed by Peter C.B. Phillips.

Proof of the Lemma

Part (a): $[\mathcal{M}_\Omega^c = \Omega(\mathcal{M}_\Omega^\perp)]$. $(\Rightarrow)$ If $\eta \in \mathcal{M}_\Omega^c$ then by definition we have $\eta'\Omega^+\epsilon = 0\ \forall\epsilon \in \mathcal{M}$. Thus $a'\epsilon = 0$ where $a = \Omega^+\eta$ and since $\mathcal{R}(\Omega) = \mathcal{R}(\Omega^+)$ we have $a \in \mathcal{M}_\Omega^\perp$. It follows that $\Omega a = \Omega\Omega^+\eta$. But $\eta \in \mathcal{R}(\Omega)$ and $\Omega\Omega^+$ is the identity on $\mathcal{R}(\Omega)$ so that $\eta = \Omega a$. Thus $\eta \in \Omega(\mathcal{M}_\Omega^\perp)$.

$(\Leftarrow)$ Now take $\eta = \Omega a$ with $a \in \mathcal{M}_\Omega^\perp$. Then for all ϵ $(= \Omega b$, say) in $\mathcal{M}$ we have $\eta'\Omega^+\epsilon = a'\Omega b = a'\epsilon = 0$. It follows by definition that $\eta \in \mathcal{M}_\Omega^c$. ■

Part (b): $[[\Omega(\mathcal{M}_\Omega^\perp)]_\Omega^\perp = \Omega^+\mathcal{M}]$. By part (a), we have

$$\Omega(\mathcal{M}_\Omega^\perp) = \{\eta = \Omega x \mid \eta'\Omega^+\epsilon = 0\ \forall\epsilon \in \mathcal{M}\} = \{\Omega x \mid x'\Omega\Omega^+\epsilon = 0\ \forall\epsilon \in \mathcal{M}\}.$$

Hence

$$(\Omega(\mathcal{M}_\Omega^\perp))_\Omega^\perp = \{\Omega^+\epsilon \mid \epsilon \in \mathcal{M}\} = \Omega^+\mathcal{M},$$

as required. ■

Proof of the Theorem: [OLS = GLS iff $\Omega^+\mathcal{M} = \mathcal{M}$ iff $\Omega\mathcal{M} = \mathcal{M}$].

We write $\mathcal{R}(\Omega) = \mathcal{M} \oplus \mathcal{M}_\Omega^\perp = \mathcal{M} \oplus \Omega(\mathcal{M}_\Omega^\perp)$. Then OLS and GLS estimates of $X\beta$ in (1) are obtained by projections on $\mathcal{M}$ along $\mathcal{M}_\Omega^\perp$ and $\Omega(\mathcal{M}_\Omega^\perp)$, respectively. Thus, OLS = GLS iff $\Omega(\mathcal{M}_\Omega^\perp) = \mathcal{M}_\Omega^\perp$ iff $(\Omega(\mathcal{M}_\Omega^\perp))_\Omega^\perp = \mathcal{M}$ iff $\Omega^+\mathcal{M} = \mathcal{M}$ iff $\Omega\mathcal{M} = \mathcal{M}$. The final "iff" follows because: $(\Rightarrow)$ $\Omega^+\mathcal{M} = \mathcal{M}$ implies $\Omega\Omega^+\mathcal{M} = \Omega\mathcal{M} = \mathcal{M}$ since $\Omega\Omega^+$ is the identity on $\mathcal{R}(\Omega)$ and hence $\mathcal{M}$; and $(\Leftarrow)$ $\Omega\mathcal{M} = \mathcal{M}$ implies $\Omega^+\Omega\mathcal{M} = \Omega^+\mathcal{M} = \mathcal{M}$ since $\Omega^+\Omega$ also keeps elements of $\mathcal{M}$ invariant. ■

Discussion

The innovation in the above treatment is part (a) of the lemma. This characterizes the Ω-conjugate subspace $\mathcal{M}_\Omega^c$ as the image of $\mathcal{M}_\Omega^\perp$ under the mapping Ω. OLS and GLS are equivalent when projections onto $\mathcal{M}$ along $\mathcal{M}_\Omega^c$ are orthogonal projections. This will occur when and only when the orthogonal complement of $\mathcal{M}_\Omega^c$ in $\mathcal{R}(\Omega)$ is $\mathcal{M}$. Part (b) of the lemma tells us that this occurs when $\Omega^+\mathcal{M} = \mathcal{M}$ and the Kruskal [1] condition $\Omega\mathcal{M} = \mathcal{M}$ then follows directly.

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2. Malinvaud, E. *Statistical Methods of Econometrics*. Amsterdam: North Holland, 1980.