90.3.5. *Optimal Structural Estimation of Triangular Systems: II. The Nonstationary Case*, proposed by Peter C.B. Phillips.

This Problem is a continuation of Problem 90.2.5. Consider the structural model

$$y_{1t} = \beta y_{2t} + u_{1t} \tag{1}$$

$$y_{2t} = \gamma x_t + u_{2t} \tag{2}$$

where $t = 1, \ldots, n$, $u_t = (u_{1t}, u_{2t})'$ is i.i.d. $N(0, \Sigma)$ with covariance matrix $\Sigma = \sigma^2 \Sigma_0$ and

$$\Sigma_0 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

is a known matrix. The regressor x_t is predetermined and will be defined below.

An econometrician (A) wishes to obtain asymptotically efficient estimates of the parameter β in equation (1). Since (2) is a reduced form equation, he argues that the single equation method two-stage least-squares (2SLS) is optimal. A colleague (B) points out that Σ_0 is known and the model can therefore be reduced to a truly triangular system (i.e., with triangular structural coefficient matrix and diagonal error covariance matrix) by subtracting equation (2) from equation (1) leading to a revised model whose first equation is now

$$y_{1t} = (1 + \beta) y_{2t} - \gamma x_t + v_{1t}. \tag{1$'$}$$

Econometrician B argues that $v_{1t} = u_{1t} - u_{2t}$ is independent of u_{2t}, (2) is still a reduced form, and therefore ordinary least squares (OLS) on (1)$'$ is asymptotically efficient.

1. *The case of deterministic trends.* Let the regressor in equation (2) be the deterministic trend $x_t = t$. Find the limit distributions of estimates of β obtained by 2SLS on (1), OLS on (1)′, and full system maximum likelihood. Which econometrician is right or are both of them wrong?

Set $x_t = t$ in the structural model (1) and (2) of the previous problem. How do the results differ from before?

2. *The case of stochastic trends.* Set $\gamma = 1$ and $x_t = y_{2t-1}$ in equation (2) of the structural model, leading to the random walk

$$y_{2t} = y_{2t-1} + u_{2t}. \tag{2$'$}$$

Econometrician (A) persists in believing that 2SLS applied to equation (1) using y_{2t-1} as an instrument leads to optimal estimates of β. Econometrician B now asserts that his approach is even more compelling than before. Not only is Σ_0 known but so also is the coefficient γ in equation (2). This means that the residual u_{2t} is known and subtracting (2)′ from (1) we get

$$y_{1t} - \Delta y_{2t} = \beta y_{2t} + v_{1t}. \tag{1$''$}$$

Again he suggests that OLS will be asymptotically efficient and to use all the information available he recommends equation (1)″ for the regression.

Which econometrician is right this time?

3. Now suppose that both econometricians believe the model with deterministic trends to be correct, whereas in fact the model has a stochastic trend as in (2)′. Thus, the true model is (1) and (2)′ but the model used for estimation is (1) and (2). Again, econometrician A recommends 2SLS on equation (1) and econometrician B argues for the use of OLS on (1)′. In both cases, $x_t = t$.

Which estimation procedure is preferable this time?

SOLUTIONS

89.3.2. *Simultaneous Confidence Ellipsoids*—Solution, proposed by Ali S. Hadi and Martin T. Wells, Cornell University.

Since $\sigma^{-1}\hat{\beta} \sim N_p(\beta, V)$ and $VH(H'VH)^{-1}H'VH(H'VH)^{-1}H'V = VH(H'VH)^{-1}H'V$, then $Q \equiv \sigma^{-2}(\beta - \hat{\beta})'H(H'VH)^{-1}H'(\beta - \hat{\beta})$ has a χ^2 distribution with rank $(H(H'VH)^{-1}H'V)$ degrees of freedom (see, e.g., [1]). Note that the matrix $H(H'VH)^{-1}H'V$ is idempotent, thus

$$\begin{aligned}\operatorname{rank}(H(H'VH)^{-1}H'V) &= \operatorname{trace}(H(H'VH)^{-1}H'V)\\ &= \operatorname{trace}(H'VH(H'VH)^{-1}) = \operatorname{trace}(I_q) = q.\end{aligned}$$

Since $\hat{\sigma}^2$ is independent of $\hat{\beta}$, $\hat{\sigma}^2$ is independent of Q, therefore