89.3.5. *The Limit Distribution of the Generalized Inverse of a Singular Covariance Matrix Estimate*; proposed by Peter C.B. Phillips.

1. Suppose $\{X_i\} \equiv$ i.i.d. $N(0,\Sigma)$. Let $M_{XX} = n^{-1}\Sigma_1^n X_i X_i'$. Prove that

   $\sqrt{n}(M_{XX} - \Sigma) \Rightarrow N(0, 2P_D(\Sigma \otimes \Sigma))$,

   where D is the duplication matrix and P_D is the orthogonal projection matrix onto the range of D.
2. Assuming Σ to be positive definite, find the limit distribution of

   $\sqrt{n}(M_{XX}^{-1} - \Sigma^{-1})$.
3. *If* Σ is singular in part (1), show that $M_{XX}^{+} \xrightarrow[p]{} \Sigma^{+}$ and find the limit distribution of

   $\sqrt{n}(M_{XX}^{+} - \Sigma^{+})$,

   where the superscript "+" signifies the Moore Penrose inverse.
4. Let R be an $n \times r$ matrix of rank r whose columns span the null space of Σ. Define the nonsingular generalized inverses

   $M_{XX}^{-} = (M_{XX} + RR')^{-1}, \qquad \Sigma^{-} = (\Sigma + RR')^{-1}$.

(i) Show that

$$M_{XX}^{-}\Sigma M_{MM}^{-} \underset{p}{\rightarrow} \Sigma^{+}.$$

(ii) Find the limit distribution of

$$\sqrt{n}(M_{XX}^{-} - \Sigma^{-})$$

and compare your result with that of part (3).