88.1.2. *Asymptotic Properties of OLS and GLS*, proposed by P.C.B. Phillips.

A. Regression with Trend: Consider the regression equation

$$y_t = \beta t + u_t, \qquad t = 1, \ldots, T \tag{1}$$

where

$$u_t = \epsilon_t + \theta\epsilon_{t-1}, \qquad \epsilon_0 = 0, \qquad |\theta| < 1 \tag{2}$$

and $\{\epsilon_t\}$ is i.i.d. $(0, \sigma_\epsilon^2)$.

(i) Is generalized least squares (GLS) equivalent to ordinary least squares (OLS) in (1)?
(ii) Find the asymptotic distributions of the GLS and OLS estimators of β in (1) as $T \uparrow \infty$.
(iii) What do you conclude from your results in (i) and (ii)? Are your results a special case of a more general theorem?

B. Regression with Simultaneity and Trend: Consider the regression equation

$$y_t = \beta x_t + e_t, \qquad t = 1, \ldots, T \tag{3}$$

where

$$x_t = \alpha t + v_t, \tag{4}$$

$$e_t = u_t + \theta u_{t-1}, \qquad u_0 = 0, \qquad |\theta| < 1 \tag{5}$$

and $\{(u_t, v_t)\}$ is i.i.d. $(0, \Sigma)$ with

$$\Sigma = \begin{bmatrix} \sigma_u^2 & \sigma_{uv} \\ \sigma_{uv} & \sigma_v^2 \end{bmatrix}.$$

Let $\hat{\beta}$ be the OLS estimator of β in (3).

(iv) Prove that $\hat{\beta}$ is a consistent estimator of β.
(v) Find the asymptotic distribution of $\hat{\beta}$.
(vi) If $\tilde{\beta}$ is the GLS estimator of β (assuming that θ is known), show that $\tilde{\beta} \underset{p}{\to} \beta$ and that $\tilde{\beta}$ has the same asymptotic distribution as $\hat{\beta}$.