86.3.4. *An Integral Over a Matrix Space*, proposed by Peter C. B. Phillips. Prove that

$$\int\limits_{S>0} \operatorname{etr}(-S)(\det S)^{q-(n+1)/2}(a'S^{-1}a)^{p}(a'S^{-2}a)^{p/2}\,dS$$
$$= (a'a)^{p/2}\frac{\Gamma_n(q)\Gamma(q+\frac{1}{2})\Gamma(q+p/2+1-n/2)}{\Gamma(q+1-n/2)\Gamma(q+p/2+1/2)}$$

where the integral is over the domain of all positive definite $n \times n$ matrices $S > 0$, $\Gamma_n(q) = \pi^{n(n-1)/4}\prod_{i=1}^{n}\Gamma(q-(i-1)/2)$ is the multivariate gamma function, a is a constant $n \times 1$ vector and $\mathrm{Re}(q) > (n-1)/2$, $\mathrm{Re}(p) \geqslant -1$. The operator $\mathrm{etr}(\ ) = \exp\{\mathrm{trace}(\ )\}$.

The integral arose recently in some exact distribution theory for the seemingly unrelated regression model [1].

REFERENCE

1. Phillips, P.C.B. The exact distribution of the SUR estimator. *Econometrica* 53 (1985): 745–756.