Problem 85.2.2. *Distribution of the F-Ratio*, solution proposed by A. Ullah and P.C.B. Phillips, University of Western Ontario and Yale University. Consider the linear model

$$y = X\beta + u \tag{1}$$

where y is a $T \times 1$ vector of observations on the dependent variable, X is a $T \times k$ matrix of observations on k nonstochastic variables and u is a $T \times 1$ vector of disturbances which follow the multivariate $t(Mt)$ distribution as

$$u \sim Mt\left(0, \sigma^2 \frac{\gamma}{\gamma - 2} I_T\right), \tag{2}$$

where γ is the d.f. parameter. Our aim is to analyse the distribution of the F-ratio

$$F = \frac{(Rb - r)'[R(X'X)^{-1}R']^{-1}(Rb - r)/m}{(y - b)'(y - Xb)/(T - k)} \tag{3}$$

for testing H_0: $R\beta = r$ against H_1: $R\beta \neq r$, where R is a full rank $m \times k$ matrix of known constants, r is an $m \times 1$ vector of known constants and $b = (X'X)^{-1}X'y$ is the least squares estimator of the parameter vector β.

The result can be stated as follows:

THEOREM. *Under assumption* (2), *the non-null density function of* F *in equation* (3) *is given by*

$$f(F) = \left[\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{\gamma}{2}\right)\right]^{-1} \sum_{j=0}^{\infty} \frac{\Gamma\left(j + \frac{m+n}{2}\right)}{\Gamma\left(j + \frac{m}{2}\right)} \frac{\Gamma\left(j + \frac{\gamma}{2}\right)}{\Gamma(j+1)} \left(\frac{m}{n}\right)^{m/2+j}$$
$$\times \frac{F^{m/2+j-1}}{\left(1 + \frac{m}{n}F\right)^{(m+n)/2+j}} \frac{\theta^j}{(1+\theta)^{j+\gamma/2}} \tag{4}$$

where

$$\theta = \frac{(R\beta - r)'(R(X'X)^{-1}R')^{-1}(R\beta - r)}{\gamma\sigma^2} \tag{5}$$

Proof. First, we note that $Rb - r = R\beta - r + R(X'X)^{-1}X'u = R(\beta - \beta_o) + R(X'X)^{-1}X'u$ where $\beta_o = R^{-}r$ is any solution of $R\beta_o = r$. Next we introduce

$$B = [R(X'X)^{-1}R']^{-1}, \qquad A = R(X'X)^{-1}X', \qquad C = A'BA = C^2,$$
$$M = I - X(X'X)^{-1}X', \qquad n = T - k, \qquad \delta = X(\beta - \beta_o). \tag{6}$$

Observe that the idempotent matrices C and M, respectively, are of rank m and n, and

$$AX = R \quad \text{and} \quad MC = 0 \tag{7}$$

Now using equations (6) and (7) in equation (3) we can write

$$F = \frac{(u + \delta)'C(u + \delta)/m}{u'Mu/n}. \tag{8}$$

But, according to equation (2), u is distributed as Mt. Thus,

$$u = \sqrt{\gamma}\,\frac{v}{q} \tag{9}$$

where v and q^2 are independently distributed as

$$v \sim \mathrm{MN}(0, \sigma^2 I) \quad \text{and} \quad q^2 \sim \chi^2_\gamma \tag{10}$$

MN stands for multivariate normal. Substituting equation (9) in equation (8) we can, therefore, write

$$F = \frac{w'Cw/m}{w'Mw/n}; \tag{11}$$

where

$$w = v + \delta_1, \qquad \delta_1 = q\frac{\delta}{\sqrt{\gamma}} \quad \text{and} \quad w'Mw = v'Mv. \tag{12}$$

To obtain the density function of F in equation (11) we observe that, given q,

$$w \sim \mathrm{MN}(\delta_1, \sigma^2 I_T), \qquad \frac{w'Mw}{\sigma^2} \sim \chi^2_n \quad \text{and} \quad \frac{w'Cw}{\sigma^2} \sim \chi^2_m(\lambda^2) \tag{13}$$

where

$$\lambda^2 = \frac{\delta' C\delta_1}{\sigma^2} = q^2\,\frac{(R\beta - r)'B(R\beta - r)}{\gamma\sigma^2} = q^2\theta. \tag{14}$$

Thus, given q, F in equation (11) is a noncentral $F(m, n, \lambda^2)$ and its exact density is well known as

$$f(F\,|\,q^2) = e^{-\lambda^2/2}\sum_{j=0}^{\infty}\frac{\Gamma\left(j + \frac{m+n}{2}\right)}{\Gamma\left(j + \frac{m}{2}\right)\Gamma\left(\frac{n}{2}\right)}\left(\frac{m}{n}\right)^{m/2+j}\frac{F^{m/2+j-1}}{\left(1 + \frac{m}{n}F\right)^{(m+n)/2+j}}\frac{\left(\frac{\lambda^2}{2}\right)^j}{j!} \tag{15}$$

The unconditional density of F can then be obtained by noting that

$$f(F) = \int_0^\infty f(F\,|\,q^2)f(q^2)\,dq^2, \tag{16}$$

where $f(q^2) = 2^{-\gamma/2}(\Gamma(\gamma/2))^{-1}e^{-q^2/2}(q^2)^{\gamma/2-1}$, Using

$$\int_0^\infty e^{-\lambda^2/2}(\lambda^2)^j e^{-(1/2)q^2}(q^2)^{\gamma/2-1}\,dq^2 = \left(\frac{2}{1+\theta}\right)^{j+\gamma/2}\Gamma\left(j+\frac{\gamma}{2}\right)\theta^j \tag{17}$$

the result stated in equation (4) is obtained upon integration.

Remarks. (i) When $\gamma \to \infty$, $u \sim \text{MN}(0, \sigma^2 I)$. For this case $f(F)$ in equation (4) reduces to the noncentral F density under normality as given in equation (15). (ii) Under $H_0: R\beta = r, \theta$ in equation (5) is zero. Thus substituting $\theta = 0$ in equation (4), the null distribution of F under Mt is the central F distribution with m and n d.f. It is

$$f(F) = \frac{\Gamma\left(\frac{m+n}{2}\right)}{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{m}{2}\right)}\left(\frac{m}{n}\right)^{m/2}\frac{F^{m/2}}{\left(1+\frac{m}{n}F\right)^{(m+n)/2}}.$$

This proves the first part of the problem. Note that the null distribution under Mt is the same as that under MN. (iii) Note that the Mt density of u in equation (2) is

$$f(u) = \int_0^\infty f(u\,|\,q^2) f(q^2)\,dq^2$$

where $f(u\,|\,q^2) \sim \text{MN}$ and $f(q^2)$ is χ^2_γ. The result in equation (4), $f(F)$, can be generalized for $f(u)$ generated by other choices of $f(q^2)$. Under H_1, the $f(F)$ will be sensitive to the choice of $f(q^2)$. However, under H_0 the $f(F)$ will be invariant to the choice of $f(q^2)$. This is because $f(F\,|\,q^2)$ does not depend on q^2 under H_0 (substitute $\lambda^2 = 0$ in equation (15)). Thus, for any choice of $f(q^2)$, $f(F) = \int_0^\infty f(F\,|\,q^2) f(q^2)\,dq^2 = f(F\,|\,q^2)\int_0^\infty f(q^2)\,dq^2) = f(F\,|\,q^2)$ which is the central F distribution with m and n d.f.