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I have for some years specialized in Open Book Problem Type examinations in Econometrics and enclose some recent material of mine which spans our current Graduate Econometrics Sequence here at Yale. It is organized as follows:

(1) Graduate Introduction to Econometrics: Final Examination 1980

(2) Graduate Econometrics: Final Examination 1980

(3) Graduate Econometrics: Mid-term Examination 1981

(4) Graduate Advanced Econometrics Problem Sequence 1980

All of this material involves new problems, ranging from introductory to advanced work. As such, it should reflect well the current style and content of our Econometrics Sequence at Yale.

One final point occurs to me. I may well wish to use some of this material in later years myself in my <u>Exercises in Econometrics</u> series (published by Philip Allan and Ballinger).

Peter C. B. Phillips
March 19, 1981

YALE UNIVERSITY

GRADUATE INTRODUCTION TO ECONOMETRICS

FINAL EXAMINATION

Spring 1980

Author: P.C.B. Phillips

Time allowed: *THREE Hours.*

Instructions: *Answer THREE questions.*

This is an OPEN BOOK examination. ANY reference material allowed.

1. In the model

$$y_t = \beta_1 x_{1t} + \beta_2 x_{2t} + \beta_3 x_{3t} + u_t$$

the x_{it} are non-random and u_t is a serially independent random disturbance with $E(u_t) = 0$ and $E(u_t^2) = \sigma^2$ for all t. The coefficients in the model are known to be related to a more basic economic parameter α according to the equations

$$\beta_1 + \beta_2 = \alpha$$

$$\beta_1 + \beta_3 = -\alpha \quad .$$

Given the following sample second moment matrix (obtained from a sample of 100 observations)

	y	x_1	x_2	x_3
y	20	8	-1	1
x_1	8	6	$-\frac{7}{2}$	$\frac{3}{2}$
x_2	-1	$-\frac{7}{2}$	4	$-\frac{1}{2}$
x_3	1	$\frac{3}{2}$	$-\frac{1}{2}$	5

(a) Find the best linear unbiased estimate of α .

(b) Without carrying out the calculations, show how to estimate the variance of α .

2. In the model

$$y_1 = \beta_1 + u_1$$

$$y_2 = u_2$$

$$y_3 = \beta_2 + u_3$$

$$y_4 = u_4$$

the y_t $(t = 1, \ldots ,4)$ are observable economic variables, β_1 and β_2 are unknown parameters and the u_t $(t = 1, \ldots ,4)$ are random disturbances with zero means and covariance matrix

$$\frac{\sigma^2}{3} \begin{bmatrix} 4 & 2 & 1 & \frac{1}{2} \\ 2 & 4 & 2 & 1 \\ 1 & 2 & 4 & 2 \\ \frac{1}{2} & 1 & 2 & 4 \end{bmatrix}$$

where $\sigma^2 > 0$ is an unknown parameter.

(a) Suggest a model for the random disturbances that is consistent with the stated properties of u_1, u_2, u_3, u_4 .

(b) Using only observations y_1 and y_2 find the best linear unbiased estimate of the parameter β_1 .

(c) If it is known that $\beta_2 = 0$, and all observations y_1, y_2, y_3, y_4 are available, find the best linear unbiased estimate of β_1 .

(d) Assuming that all observations y_1, y_2, y_3, y_4 are again available but that β_2 is unknown, find the best linear unbiased estimates of β_1 and β_2.

(e) Comment on your results in (b), (c) and (d).

3. An investigator studying the demand for oranges specifies the relationship

$$Q_t = a_0 p_t^{a_1} q_t^{a_2} y_t^{a_3} e^{u_t}$$

where

Q_t = purchases of oranges in period t

p_t = price of oranges in period t

q_t = price index of other fruit in period t

y_t = nominal income in period t

u_t = random disturbance.

Using $T = 50$ time series observations, the following regression is estimated by ordinary least squares

$$\ell n Q_t = \text{const} + \underset{(0.32)}{0.15 \ell n p_t} + \underset{(0.39)}{0.21 \ell n q_t} + \underset{(0.25)}{0.62 \ell n y_t}$$

$$R^2 = 0.81 \qquad DW = 0.43$$

(Standard errors are given in parentheses.)

Noting that the own price elasticity has the wrong sign in this regression and that neither price coefficient is significantly different from zero, the investigator attempts to improve the results by a constrained regression which yields

$$\ell n Q_t = \text{const} - \underset{(0.09)}{0.24} \ell n \left(\frac{p_t}{q_t}\right) + \underset{(0.21)}{0.52} \ell n \left(\frac{y_t}{q_t}\right)$$

$R^2 = 0.73$ $\qquad$ $DW = 0.55$

The investigator is happy with the outcome of his regression because it embodies a constraint that has a meaningful economic interpretation and the estimates obtained are of the right sign and also seem to be statistically significant. However, a colleague points out to him that the Durbin Watson statistic is rather low and suggests that he run a new regression including $\ell n Q_{t-1}$ as a regressor. The results are

$$\ell n Q_t = \text{const} - \underset{(0.05)}{0.17} \ell n \left(\frac{p_t}{q_t}\right) + \underset{(0.28)}{0.34} \ell n \left(\frac{y_t}{q_t}\right) + \underset{(0.135)}{0.65} \ell n Q_{t-1}$$

$R^2 = 0.89$ $\qquad$ $DW = 1.8$

Both investigators are satisfied with the outcome of this final regression.

Critically appraise the approach to model specification, estimation and diagnostic checking taken in this empirical investigation, indicating areas of strength and weakness in the exercise. Try to justify your criticisms and your support in the light of econometric theory and the empirical evidence reported.

4. In the model

$$y_{1t} = \alpha x_{1t} + u_{1t}$$

$$y_{2t} = \beta x_{2t} + u_{2t}$$

The x_{it} are non-random exogenous variables and the u_{it} are serially independent random disturbances that are normally distributed with zero means and second moments

$$E(u_{1t}^2) = 1 \quad , \quad E(u_{2t}^2) = 2 \quad , \quad E(u_{1t}\, u_{2t}) = 1$$

for all values of t . The sample second moment matrix below was calculated from 20 sample observations:

	y_1	y_2	x_1	x_2
y_1	10	-5	1	-1
y_2	-5	15	5	1
x_1	1	5	11	1
x_2	-1	1	1	1

(a) Find the best linear unbiased estimates of the parameters α and β in the above model.

(b) Test the null hypothesis

$$H_0: \ \alpha = \beta$$

against the alternative

$$H_1: \ \alpha \neq \beta \ .$$

5. An investigator has specified the following two models and proposes to use them in some empirical work with macroeconomic time series data:

$$c_t = \alpha_1 y_t + \alpha_2\, m_{t-1} + u_{1t}$$
$$i_t = \beta_1 y_t + \beta_2\, r_t + u_{2t} \qquad \text{(Model 1)}$$
$$y_t = c_t + i_t$$

jointly dependent variables: c_t, i_t, y_t

predetermined variables: r_t, m_{t-1}

$$m_t = \gamma_1\, r_t + \gamma_2\, m_{t-1} + v_{1t}$$
$$r_t = \delta_1\, m_t + \delta_2\, m_{t-1} + \delta_3\, y_t + v_{2t} \qquad \text{(Model 2)}$$

jointly dependent variables: m_t, r_t

predetermined variables: m_{t-1}, y_t .

(a) Assess the identifiability of the parameters that appear as coefficients in the above two models (treating the two models separately).

(b) Obtain the reduced form equation for y_t in model (1) and the reduced form equation for r_t in model (2).

(c) Assess the identifiability of the two-equation model comprising the reduced form equation for y_t in model (1) (an IS curve) and the reduced form equation for r_t in model (2) (an LM curve).

6. The two equations of a competitive model of the meat market are:

Demand: $y_{1t} = a_1\, y_{2t} + a_2\, x_{1t} + u_{1t}$

Supply: $y_{1t} = b_1 y_{2t} + b_2 x_{2t} + b_3 x_{3t} + u_{2t}$

where

y_1 = meat consumption per capita (dollars p.a.) and

y_2 = meat price (1935 = 100) are jointly dependent variables;

x_1 = disposable income per capita (dollars p.a.),

x_2 = unit cost of meat processing (1935 = 100) and

x_3 = cost of agricultural production (1935 = 100) are exogenous variables;

u_{1t}, u_{2t} are stochastic disturbances.

(a) Estimate the parameters a_1 and a_2 by two stage least squares, and test the hypothesis

$$H_0 : a_2 = 0 \text{ against } H_1 : a_2 \neq 0$$

given the following sample second moment matrix (based on 36 observations):

	y_1	y_2	x_1	x_2	x_3
y_1	10	0	1	0	-1
y_2	0	10	-1	-1	0
x_1	1	-1	1	0	0
x_2	0	-1	0	1	0
x_3	-1	0	0	0	1

(b) Repeat part (a) of the question using instrumental variables estimates of a_1 and a_2 obtained with x_{2t} as an instrument for y_{2t} and with x_{1t} as its own instrument.

YALE UNIVERSITY

GRADUATE ECONOMETRICS

FINAL EXAMINATION

Spring 1980

Author: P.C.B. Phillips

Time allowed: *THREE Hours.*

Instructions: *Answer THREE questions.*

This is an OPEN BOOK examination. ANY reference material allowed.

1. Two economic variables η_t and ξ_t are assumed to be related by the equation

$$\eta_t = \alpha\xi_t \qquad (t = 1, \ldots ,T)$$

where α is an unknown parameter. Observable variables y_t and x_t are known to be related to η_t and ξ_t according to the equations

$$y_t = \eta_t + u_t$$

$$x_t = \xi_t + v_t$$

where u_t and v_t are serially independent random disturbances which are distributed independently of t and of η_s and ξ_s $(s = 1, \ldots ,T)$ with zero means and higher moments given by

$$E(u_t^2) = \sigma_u^2 \text{ , } E(u_t^2) = \sigma_u^2 \text{ , } E(u_t^3) = 0 \text{ , } E(u_t^4) = 3\sigma_u^4 \text{ ,}$$

$$E(u_t v_t) = 0$$

for all t .

Given that $\xi_t = m(1+r^t)$, where $m > 0$ and $0 < r < 1$ are constants:

(a) Find the limit in probability as $T \to \infty$ of α^*, the estimator of α obtained from an ordinary least squares regression of y_t on x_t;

(b) for the special case in which $\sigma_u^2 = 0$ so that $y_t = \eta_t$ with probability one, prove that $\sqrt{T}(\alpha^* - \text{plim}\ \alpha^*)$ has a limiting normal distribution and find the variance of this distribution.

2. In the model

$$y_1 = 2\beta_1 + \beta_2 + \beta_3 + u_1$$

$$y_2 = \beta_1 + \beta_2 + u_2$$

$$y_3 = \beta_1 + \beta_3 + u_3$$

The $y_t (t = 1, 2, 3)$ are observable economic variables, $\beta_1, \beta_2, \beta_3$ are unknown parameters and the $u_t (t = 1, 2, 3)$ are random disturbances distributed with zero means, constant variance $E(u_t^2) = \sigma^2$ for all t and $E(u_t u_s) = 0$ for $t \neq s$.

(a) Show that $\beta_1 - \beta_2 + 2\beta_3$ is an estimable function.

(b) Find the best linear unbiased estimate of $\beta_1 - \beta_2 + 2\beta_3$.

(c) Find an unbiased estimate of the variance σ^2.

[Hint: If M is an $n \times n$ positive semi definite matrix of rank $n-1$ and d is a non-zero vector for which $Md = 0$ then a generalized inverse of M is given by the matrix $[M + dd']^{-1}$.]

3. An investigator studying the demand for money specifies the relationship

$$m_t = \alpha_0 + \alpha_1 y_t + \alpha_2 r_t + u_t$$

where

m_t = volume of money in period t

y_t = nominal income in period t

r_t = rate of interest in period t

u_t = random disturbance.

The relationship is estimated by ordinary least squares (OLS) from a time series of $T = 50$ observations, for which the correlations between m_t and t and y_t and t are 0.995 and 0.980 respectively. The result is

$$m_t = \text{const} + \underset{(0.05)}{2.11}\, y_t - \underset{(30.11)}{4.01}\, r_t$$

$$R^2 = 0.99 \qquad DW = 0.51$$

where standard errors are in parentheses. Since the coefficient of r_t is insignificant he drops this variable but because of the low DW (Durbin-Watson) statistic he decides to include m_{t-1} as a regressor. Re-estimation of the equation by OLS yields

$$m_t = \text{const} + \underset{(0.04)}{0.9}\, y_t + \underset{(0.14)}{0.7}\, m_{t-1}$$

$$R^2 = 0.996 \qquad DW = 2.3$$

which the investigator accepts as a satisfactory relationship.

Comment on the approach to model specification, estimation and diagnostic checking taken in this empirical investigation. Support your comments by evaluating the empirical results reported and in the light of your knowledge of econometric theory.

4. In the two equation regression model

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 & 0 \\ 0 & x_2 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

y_i $(i = 1, 2)$ is a vector of T observations on the i'th dependent variable, x_i is a matrix of observations on m_i non-random independent variables, the β_i are vectors of unknown coefficients and the u_i are vectors of disturbances for which

$$E(u_1) = E(u_2) = 0$$
$$E(u_1 u_1') = \sigma_{11} I_T \ , \ E(u_2 u_2') = \sigma_{22} I_T \ , \ E(u_1 u_2') = \sigma_{12} I_T \ .$$

It is assumed that the matrix

$$\sum = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix}$$

is positive definite and known.

(a) If $\hat{\beta}_1$ and $\hat{\beta}_2$ denote the Zellner seemingly unrelated regression estimator of β_1 and β_2 respectively, show that the covariance matrices of these estimators are given by

$$\sigma_{11}(1-\rho^2) \left[x_1' x_1 - \rho^2 x_1' x_2 (x_2' x_2)^{-1} x_2' x_1 \right]^{-1}$$

and

$$\sigma_{22}(1-\rho^2) \left[x_2' x_2 - \rho^2 x_2' x_1 (x_1' x_1)^{-1} x_1' x_2 \right]^{-1} \ ,$$

respectively, where $\rho = \sigma_{12}/(\sigma_{11}\sigma_{22})^{\frac{1}{2}}$.

(b) Hence, prove the result that the single equation least squares estimators of β_1 and β_2 are as efficient as $\hat{\beta}_1$ and $\hat{\beta}_2$ <u>if and only if</u> either $\rho = 0$ or the range spaces of x_1 and x_2 are equivalent.

5. In the simultaneous equations model

$$By_t + Cx_t = u_t$$

y_t is a vector of observations in period t on n endogenous variables, x_t is a vector of observations in period t on m exogenous variables, u_t is a vector of serially independent random disturbances distributed with zero means and non-singular covariance matrix and $B(n\times n)$ and $C(n\times m)$ are matrices of coefficients. B is non-singular.

Let b' and c' denote the first rows of the matrices B and C respectively. It is known that the elements of b and c satisfy the following restrictions (which include the normalization rule)

$$\Phi_1 b + \Phi_2 c = \phi$$

where $\Phi = \left[\Phi_1 \vdots \Phi_2\right]$ is a known $r\times(n+m)$ matrix and ϕ is a known vector.

(a) Show that if the first equation is identified then

(i) There exists a vector $x \neq 0$ for which

$$\left(\Phi_1 - \Phi_2 A'\right) x = \phi$$

(ii) rank $\left(\Phi_1 - \Phi_2 A'\right) = n$

where $A = B^{-1} C$ is the reduced form coefficient matrix.

(b) For the special case where $n = 2$ and the first equation is

$$b_{11}y_{1t} + b_{12}y_{2t} + c_{11}x_{1t} + c_{12}x_{2t} + c_{13}x_{3t}$$
$$+ c_{14}x_{4t} = u_{1t}$$

find the restrictions on the reduced form matrix A implied by the restrictions

$$b_{12} = c_{11} \qquad c_{12} = c_{13} \qquad c_{11} + c_{13} = 1$$

with the normalization $b_{11} = 1$.

6. In the model

$$y_{1t} = \alpha_1 y_{2t} + \alpha_2 x_{1t} + u_{1t}$$
$$y_{2t} = \beta_1 y_{1t} + \beta_2 x_{2t} + u_{2t}$$

the y_{it} are endogenous variables and the x_{it} are exogenous variables which we assume to be non-random and whose moment matrix we assume to have a finite positive definite limit as the sample size tends to infinity. The random disturbances u_{it} are serially independent and are distributed with zero means and second moments

$$E(u_{1t}^2) = 1 \ , \ E(u_{1t}u_{2t}) = 1 \ , \ E(u_{2t}^2) = 2$$

for all t .

Find consistent estimates of the parameters α_1, α_2, β_1 and β_2 given the following sample second moment matrix (based on $T = 100$ observations) and test the null hypothesis

YALE UNIVERSITY

GRADUATE ECONOMETRICS

MID TERM EXAMINATION

Winter 1981

Author: P.C.B. Phillips

Time allowed: *ONE Hour and a HALF.*

Instructions: *Answer ONE question.*

This is an OPEN BOOK examination. ANY reference material allowed.

1. An investigator assumes that two observable economic variables are related by the equation

$$y_t = \alpha x_t + u_t \qquad (t = 1, \dots, T) \qquad (1)$$

where α is an unknown parameter and the u_t $(t = 1, \dots, T)$ are assumed to be independent and identically distributed (i.i.d.) $N(o, \sigma_u^2)$. In fact, the investigator's equation (1) is misspecified and the true relationship is given by

$$y_t = \alpha x_t + \beta z_t + \varepsilon_t \qquad (t = 1, \dots, T) \qquad (2)$$

which involves a third observable economic variable z_t , an additional unknown coefficient β and the ε_t are i.i.d. $N(o, \sigma_\varepsilon^2)$. In the true relationship (2) the x_t are i.i.d. $N(m_x, \sigma_x^2)$, the variable z_t is generated by the equation

$$z_t = \gamma w_t + \eta_t \qquad (t = 1, \dots, T) \qquad (3)$$

in which γ is an unknown parameter, the w_t are i.i.d. $N(m_w, \sigma_w^2)$ and the η_t are i.i.d. $N(o, \sigma_\eta^2)$. The random variables $\varepsilon_t, \eta_t, x_t$ and

w_t in equations (2) and (3) are statistically independent.

(a) Find the limit in probability as $T \to \infty$ of α^*, the least squares estimator of α in the investigator's equation (1).

(b) Find the limiting distribution as $T \to \infty$ of $\sqrt{T}\,(\alpha^* - \text{plim}\,\alpha^*)$ and derive explicitly a formula for the variance of this distribution.

2. The observable nx1 random vectors $y_1, \ldots, y_T$ and the non-random mx1 vectors $x_1, \ldots, x_T$ satisfy the system

$$y_t = Ax_t + u_t \qquad (t = 1, \ldots, T)$$

where A is an nxm matrix of unknown coefficients and the u_t $(t = 1, \ldots, T)$ are serially independent random vectors which are identically distributed as $N(o, \sigma^2\Sigma)$ where Σ is a known positive definite nxn matrix and σ^2 is an unknown scalar. It is assumed that $T > m$ and that the data matrix $X' = [x_1, \ldots, x_T]$ has full rank.

(a) Show that the best linear unbiased estimator of the coefficient matrix A is given by the ordinary least squares estimator $A^* = Y'X(X'X)^{-1}$ where $Y' = [y_1, \ldots, y_T]$.

(b) Construct a statistical test of the hypothesis that the rows of the matrix A are identical (i.e. that the coefficients of the explanatory variables are the same for all equations in the model). Find the exact sampling distribution of your test statistic and hence justify your testing procedure.

YALE UNIVERSITY

ADVANCED ECONOMETRICS

PROBLEM SEQUENCE IN LINEAR AND NON-LINEAR ESTIMATION

Fall 1980

Author: P.C.B. Phillips

1. In the model

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ -y_1 \end{bmatrix} + \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix} \quad \text{or} \quad x = y + \varepsilon$$

the scalar y_1 is unknown and the error vector ε is known to have zero mean and covariance matrix

$$Q = \begin{bmatrix} \sigma^2 & \sigma^2 \\ \sigma^2 & 2\sigma^2 \end{bmatrix} .$$

(a) Draw a diagram to illustrate

(1) the concentration ellipsoid of ε ;

(2) the linear subspace L in which the vector y is known to lie;

(3) the linear subspace K which is the principal conjugate of L with respect to the concentration ellipsoid of ε ;

(4) the projection estimator of y in L based on the observation x , whose concentration ellipsoid is contained in that of every other unbiased linear estimator of y .

(b) If $x_1 = 10$ and $x_2 = 1$, calculate the best linear unbiased estimator of y .

2. In the linear model

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \end{bmatrix} \quad \text{or} \quad x = y + \varepsilon$$

the mean of ε is zero and its covariance matrix is given by

$$Q = \begin{bmatrix} \sigma^2 & 0 & 0 \\ 0 & 4\sigma^2 & 4\sigma^2 \\ 0 & 4\sigma^2 & 4\sigma^2 \end{bmatrix}$$

(a) Find the linear transformation of R^3 into itself which takes the support of ε into the leading 2-dimensional subspace of R^3, the concentration ellipsoid of ε into the unit circle in this subspace and the linear subspace L which contains the vector y into the first coordinate axis.

(b) If H is the matrix of the transformation in (a), find the best linear unbiased estimator of Hy. Use this estimator to deduce the corresponding estimator of y_1.

(c) Show that the estimator of y_1 obtained in (b) is identical to the estimator obtained from the projection on L along the principal conjugate subspace of L with respect to the concentration ellipsoid of ε in the original (untransformed) space.

3. In the linear model

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ -y_1 \end{bmatrix} + \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix} \quad \text{or} \quad x + y + \varepsilon$$

the mean of ε is zero and its covariance matrix is given by

$$Q = \begin{bmatrix} \sigma^2 & 2\sigma^2 \\ 2\sigma^2 & 4\sigma^2 \end{bmatrix} .$$

(a) Find the concentration ellipsoid of ε .

(b) Find the best linear unbiased estimator of y_1 . Does your estimator have any other interesting properties?

(c) Compare the variance of your estimator in (b) with that of the least squares estimator of y_1 .

4. In the model

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_1 + y_2 \\ y_2 \end{bmatrix} + \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \end{bmatrix} \quad \text{or} \quad x = y + \varepsilon$$

the scalars y_1 and y_2 are unknown and the error vector ε is known to have zero mean and covariance matrix

$$Q = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(a) Draw a figure to illustrate:

(1) the concentration ellipsoid of ε ;

(2) the linear subspace L in which the mean vector y is known to lie;

(3) the projection estimator of y in L based on the observation x, whose concentration ellipsoid is contained in that of every other unbiased linear estimator of y.

(b) If the observation vector x takes the value

$$x = \begin{bmatrix} 1 \\ 5 \\ 10 \end{bmatrix}$$

calculate the best linear unbiased estimator of y.

5. In the model

$$x_{1t} = a_{11}z_{1t} + a_{12}z_{2t} + u_{1t}$$

$$x_{2t} = a_{21}z_{1t} + a_{22}z_{2t} + u_{2t}$$

the z_{it} are non-random and the u_{it} are serially independent random disturbances with zero means and covariance matrix

$$\Sigma = \begin{bmatrix} \sigma^2 & -\sigma^2 \\ -\sigma^2 & \sigma^2 \end{bmatrix}$$

for all values of t.

(a) Find the best linear unbiased estimates of a_{11}, a_{12}, a_{21} and a_{22} in the above model, given the following matrix of sample second moments of the data.

	x_1	x_2	z_1	z_2
z_1	3	2	9	1
z_2	3	4	1	9

(b) If it is known that $a_{12} = 0$, find the best linear unbiased estimates of a_{11} , a_{21} and a_{22} .

(c) Discuss the role of the singularity of the error covariance matrix Σ in your answers to (a) and (b). Do we need a generalized inverse of Σ to find the best linear unbiased estimates in part (b)?

6. In the model

$$x_{1t} = \alpha z_{1t} + \beta z_{2t} + u_{1t}$$
$$x_{2t} = \alpha z_{2t} + u_{2t}$$
$$x_{3t} = \beta z_{1t} + u_{3t}$$

the z_{it} are non-random and the u_{it} are random disturbances with zero means and covariance matrix

$$\begin{bmatrix} 2\sigma^2 & \sigma^2 & \sigma^2 \\ \sigma^2 & \sigma^2 & 0 \\ \sigma^2 & 0 & \sigma^2 \end{bmatrix}$$

for each value of t . The u_{it} are serially independent. Find the best linear unbiased estimates of α and β in the above model given the following matrix of second moments based on a sample of size T :

	x_1	x_2	x_3	z_1	z_2
z_1	1	2	3	9	-3
z_2	1	-1	-2	-3	9

7. The random n-vector x is assumed to have an unknown mean vector y, which is known to lie in a certain subspace L of dimension p in n dimensional Euclidean space, and a known covariance matrix Q of rank $r < n$. L is assumed to lie in the range space (or support) of Q.

   (a) If y^* is the Gauss Markov (best linear unbiased) estimator of y, prove that y^* is the vector of L which minimizes the quantity $(x-y)'Q^{-}(x-y)$ for all y in L and any choice of generalized inverse Q^{-} of Q.

   (b) Prove that y^* is identical to the ordinary least squares estimator of y if and only if K, the principal conjugate subspace of L with respect to the concentration ellipsoid of x is orthogonal to the subspace L.

   (c) Hence, prove that y^* is identical to the ordinary least squares estimator of y if and only if the image of L under the transformation Q is identical to L.

8. In the model

$$\begin{bmatrix} x_{1t} \\ x_{2t} \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix} \quad \text{or} \quad x_t = y + \varepsilon_t \quad (t = 1, \ldots, T)$$

it is known that $y_2 = y_1^2$, and the ε_t are serially independent and normally distributed with zero mean vector and covariance matrix

$$\Omega = \begin{bmatrix} 2\sigma^2 & \sigma^2 \\ \sigma^2 & \sigma^2 \end{bmatrix}$$

for each value of t .

(a) Given the following sample moments obtained from $T = 100$ observations

	x_1	x_2
x_1	9	-1
x_2	-1	9

$\bar{x}_1 = 1{\cdot}75 \qquad \bar{x}_2 = 1{\cdot}75$

find asymptotically efficient estimates of y_1 and y_2 .

(b) Illustrate your estimates on a diagram.

(c) How would you estimate σ^2 ?

9. In the model

$$x_t = a_1 z_{1t} + a_2 z_{2t} + \varepsilon_t \qquad t = 1, \ldots, T$$

the z_{it} are non-random and the ε_t are normally distributed with $E(\varepsilon_t) = 0$, $E(\varepsilon_t^2) = \sigma^2$ and $E(\varepsilon_t \varepsilon_s) = 0$, $s \neq t$. a_1 and a_2 satisfy the restriction $a_1^2 a_2 = 1$.

(a) Explain how to find asymptotically efficient estimators $\hat{a}_1$ and $\hat{a}_2$ of a_1 and a_2 and compute the asymptotic variance of $\sqrt{T}(a_1 - a_2)$ when the true values of the parameters are $\sigma^2 = \frac{1}{4}$, $a_1 = 1$, $a_2 = 1$ and

$$\lim_{T\to\infty} \frac{1}{T} \sum_t \begin{bmatrix} z_{1t} & z_{1t}z_{2t} \\ z_{2t}z_{1t} & z_{2t} \end{bmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} .$$

(b) Compare your result with the asymptotic variance of $\sqrt{T}(a_1^* - a_2^*)$ where a_1^* and a_2^* denote the ordinary least squares estimators of a_1 and a_2 .

10. In the model

$$x_t = ax_t z_t + bz_t + u_t$$

z_t is non-random the u_t $(t = 1, \ldots, T)$ are independent, identically distributed random disturbances with zero mean and variance σ^2 .

(a) Explain how you would obtain consistent estimators of the parameters a and b in the above model by the method of instrumental variables (with instruments z_t and a constant). Indicate assumptions over and above those already given which will ensure that your estimators are consistent.

(b) Find the covariance matrix of the limiting distribution of $\sqrt{T}(\hat{a} - a , \hat{b} - b)$ where $\hat{a}$ and $\hat{b}$ denote the instrumental variable estimates of part (a) and

$$a = 1 , b = 2 , \sigma^2 = 1$$

and

$$\lim_{T\to\infty} \frac{1}{T}\sum_{t=1}^{T} z_t = 1 , \lim_{T\to\infty} \frac{1}{T}\sum_{t=1}^{T} z_t^2 = 2 , \lim_{T\to\infty} \frac{1}{T}\sum_{t=1}^{T} \frac{z_t^i}{1-z_t} = \frac{2^{i+1}}{2^i-1} .$$