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## UNCERTAINTY IN ECONOMICS

*See* ESSU vol. 2

## UNIT-ROOT TESTS

Many observed time series* display nonstationary characteristics. Some grow in a secular way over long periods of time; others appear to wander around in a random way as if they have no fixed population mean. These characteristics are especially evident in time series that represent aggregate economic behavior (such as gross domestic product), financial time series (such as indexes of stock prices), and political opinion poll series (such as presidential popularity data). Any attempt to explain or forecast series of this type requires that a mechanism be introduced to capture the nonstationary elements in the series, or that the series be transformed in some way to achieve stationarity. The problem is particularly delicate in the multivariate case, where several time series may have nonstationary characteristics and the interrelationships of these variables are the main object of study. Figure 1 graphs the monthly leading economic indicators time series for the U.S. economy over the period 1948:1–1994:1. Also shown in the figure is the regression line of a linear trend. The time series shows evidence of growth over time as well as a tendency to wander randomly away from the linear trend line. A successful statistical model of the time series needs to deal with both these features of the data. *See* TIME SERIES, NONSTATIONARY.

One way of modeling nonstationarity is to use deterministic trending functions such as time polynomials to represent secular characteristics such as growth over time. In this approach, a time series $y_t$ is broken down into two components, one to capture trend* and another to capture stationary fluctuations. A general model of this form is

$$y_t = h_t + y_t^s, \quad h_t = \gamma' x_t \qquad (t = 1, \ldots, n), \qquad (1)$$

where $y_t^s$ is a *stationary time series**, $x_t$ is an $m$-vector of deterministic trends, and $\gamma$ is a vector of $m$ parameters. In this case, $y_t$ is known as a *trend-stationary* time series. The simplest example is a linear trend. Then $\gamma' x_t = \gamma_0 + \gamma_1 t$, and the time series $y_t$ is stationary about this deterministic linear trend. A more general example where the trends are piecewise higher-order polynomials is given in (10) below.

An unsatisfactory feature of trend-stationary models (like the linear trend line in Fig. 1) is that no random elements appear in the trending mechanism and only the stationary component is subject to stochastic shocks. Models with autoregressive *unit roots* are a simple attempt to deal with this shortcoming. In such models



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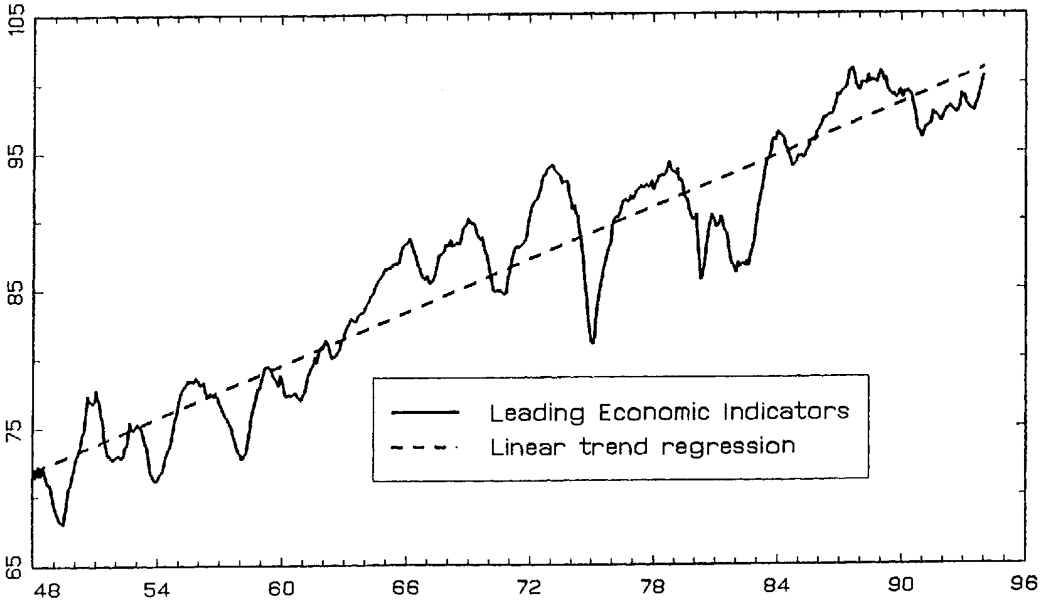


**Figure 1** Monthly U.S. economic time series, 1948:1–1994:1.

the trend is permitted to have both deterministic and stochastic elements. For example, in (1) the deterministic trend $h_t$ can be retained, and the process $y_t^s$ can be modeled as the nonstationary autoregression

$$y_t^s = \alpha y_{t-1}^s + u_t \quad (t = 1, \ldots, n) \quad \text{with} \quad \alpha = 1. \quad (2)$$

In this model there is an autoregressive root of unity (corresponding to the solution of the characteristic equation $1 - \alpha L = 0$), and the shock $u_t$ is stationary. *Unit-root tests* usually seek to determine whether data support this model or a trend-stationary alternative. In a unit-root test the null hypothesis is that the autoregressive* parameter $\alpha = 1$ in (2). The process $y_t^s$ is then *difference-stationary* in the sense that the first differences $\Delta y_t^s = u_t$ are stationary. Unit-root tests are typically one-sided tests against the alternative hypothesis that $|\alpha| < 1$. Under the alternative hypothesis, the process $y_t^s$ is stationary, and then, $y_t$ in (1) is trend-stationary. Unit-root tests can therefore be interpreted as tests of difference stationarity versus trend stationarity.

If the initial condition in (2) is set at $t = 0$, the output of the model can be written in terms of accumulated shocks as $y_t^s = \sum_{j=1}^{t} u_j + y_0^s$. In view of this representation, $y_t^s$ is often called an *integrated process* of order one [written as $I(1)$]. The term *stochastic trend* is also in common use, and is explained by the fact that $y_t^s$ is of stochastic order $O_p(t^{1/2})$ under very general conditions, i.e., the variance of $y_t^s$ is of order $O(t)$ and the standardized quantity $t^{-1/2}y_t^s$ satisfies a central limit theorem* as $t \to \infty$. The simplest example of a stochastic trend is a random walk*. In this case, the shocks $u_t$ are independently and identically distributed (i.i.d.) with zero mean and constant variance $\sigma^2$. A more general case occurs when the stationary shocks $u_t$ in (2) are generated by the linear process $u_t = C(L)\epsilon_t$, whose innovations $\epsilon_t$ are iid$(0, \sigma^2)$, and where $C(L)$ is a polynomial in the lag operator $L$ for which $Ly_t = y_{t-1}$. More specifically, if

$$C(L) = \sum_{j=0}^{\infty} c_j L^j,$$

$$\sum_{j=0}^{\infty} c_j^2 < \infty, \quad C(1) \neq 0, \quad (3)$$

then the process $u_t$ is *covariance-stationary* and has positive spectral density at the origin, given by the expression $(\sigma^2/2\pi)C(1)^2$. The latter property ensures that the unit root in $y_t^s$

does not cancel (as it would if the process $u_t$ had a *moving-average unit root*, in which case the spectral density would be zero at the origin). If the summability condition in (3) is strengthened to $\sum_{j=0}^{\infty} j^{1/2}|c_j| < \infty$, then $y_t^s$ satisfies an invariance principle* or functional central limit theorem* (see Phillips and Solo [31] for a demonstration), and this is an important element in the development of the asymptotic theory of all unit-root tests. Thus, $n^{-1/2}y_{[nr]}^s \Rightarrow B(r)$, a Brownian motion* with variance $\omega^2 = \sigma^2 C(1)^2$, where $[nr]$ signifies the integer part of $nr$, $\Rightarrow$ signifies *weak convergence* and $r \in [0, 1]$ is some fraction of the sample data. The parameter $\omega^2$ is called the *long-run variance* of $u_t$.

The literature on unit-root tests is vast. Most of the research has appeared since 1980, but an important early contribution came in 1958 from White [39], who first recognized the vital role played by invariance principles in the asymptotic theory of time series with a unit root. The first explicit research on unit-root tests dealt with Gaussian random walks and was done by Dickey and Fuller [4, 5]. Solo [37], Phillips [24], and Chan and Wei [3] developed more general limit theories using invariance principles. Subsequently, an immense variety of tests have been developed, inspired in large part by the need to allow for more general processes than random walks in empirical applications. This entry covers the main principles of testing, the commonly used tests in practical work, and recent developments.

Under certain conditions, (1) and (2) can be combined to give the regression model

$$y_t = \beta' x_t + \alpha y_{t-1} + u_t, \qquad (4)$$

where $\beta$ is an $m$-vector of deterministic trend coefficients. This formulation usually involves raising the degree of the deterministic trends to ensure that the maximum trend degrees in (4) and (1) are the same, which results in some inefficiency in the regression because there are surplus trend variables in (4). There is an alternative approach that avoids this problem of redundant variables and it will be discussed below. Asymptotic theory assumes that there exists a matrix $D_n$ and a piecewise continuous function $X(r)$ such that $D_n^{-1}x_{[nr]} \to X(r)$ as $n \to \infty$ uniformly in $r \in [0, 1]$. $X(r)$ is then the limiting trend function.

The stationary process $u_t$ in (4) may be treated in a parametric or a nonparametric way, leading to two classes of unit-root tests. One relies on casting the stationary part of the process in terms of a parametric model (commonly an autoregression). The other is parametric only in its treatment of the regression coefficient $\alpha$, being nonparametric with regard to the general stationary part of the process. The approach is therefore said to be semiparametric.

## THE DICKEY–FULLER TESTS AND SEMIPARAMETRIC EXTENSIONS

Let $\hat{\alpha}$ be the ordinary least-squares (OLS) estimator of $\alpha$ in (4). The Dickey–Fuller [4, 5] unit-root tests are based on the coefficient estimator $\hat{\alpha}$ and its regression $t$-ratio $t_{\hat{\alpha}}$. The basic idea of the tests is to access whether the observed $\hat{\alpha}$ is close enough to unity to support the hypothesis of the presence of a unit root in the true data-generating mechanism. Classical test procedures require a distribution theory to deliver critical values for the test statistics $\hat{\alpha}$ and $t_{\hat{\alpha}}$ under the null hypothesis that $\alpha = 1$. The finite sample distributions of these test statistics are complex and depend on unknown nuisance parameters* associated with the stationary process $u_t$. It is therefore customary to rely on asymptotic theory, where the results are simpler and the parameter dependences are clearly understood.

The large-sample theory for $\hat{\alpha}$ and $t_{\hat{\alpha}}$ is most simply obtained using invariance principles and involves functionals of Brownian motion. In the special case where there is no deterministic component in (4) and the shocks $u_t$ are iid$(0, \sigma^2)$, the limit theory for the test statistics is as follows: $n(\hat{\alpha} - 1) \Rightarrow (\int_0^1 W\,dW)(\int_0^1 W^2)^{-1}$, and $t_{\hat{\alpha}} \Rightarrow (\int_0^1 W\,dW)(\int_0^1 W^2)^{-1/2}$, where $W$ is standard Brownian motion. These limit distributions are commonly known as the *Dickey–Fuller distributions*, although the Brownian-motion forms were not used in refs. [4, 5] and were given later in refs. [3, 24, 37].

The limit distribution of $\hat{\alpha}$ is asymmetric and has a long left tail, as shown in Fig. 2. It was computed directly in [9]. In the general case where $u_t$ is stationary, the limit has an additional bias term that depends on the autocovariance* in $u_t$ through the nuisance parameter $\lambda = \sum_{j=1}^{\infty} E(u_0 u_j)$. This parameter and the related nuisance parameter $\omega^2$ may be consistently estimated by *kernel** techniques, using residuals from an OLS regression on (4). If $\hat{\omega}^2$ and $\hat{\lambda}$ are such estimates, then the following statistics provide general semiparametric tests of the unit-root hypothesis (Phillips [24]), which correct for possible autocorrelation in $u_t$:

$$Z_\alpha = n(\hat{\alpha} - 1) - \hat{\lambda}\left(n^{-2}\sum_{t=2}^{n} y_{X,t-1}^2\right)^{-1} \Rightarrow \left(\int_0^1 W_X\, dW\right)\left(\int_0^1 W_X^2\right)^{-1}, \quad (5)$$

$$Z_t = \hat{\sigma}_u \hat{\omega}^{-1} t_{\hat{\alpha}} - \hat{\lambda}\left[\hat{\omega}\left(n^{-2}\sum_{t=2}^{n} y_{X,t-1}^2\right)^{1/2}\right]^{-1} \Rightarrow \left(\int_0^1 W_X\, dW\right)\left(\int_0^1 W_X^2\right)^{-1/2}. \quad (6)$$

In these formulas, $y_{X,t}$ is the residual from a regression of $y_t$ on $x_t$, $\hat{\sigma}_u^2$ is the OLS estimator of $\sigma_u^2 = \mathrm{var}(u_t)$, and $W_X$ is the $L_2[0,1]$ Hilbert space projection of $W$ onto the space orthogonal to $X$, viz. $W_X(r) = W(r) - (\int_0^1 WX')(\int_0^1 XX')^{-1}X(r)$.

The limit variates that appear on the right side of (5) and (6) are free from the nuisance parameters $\beta$, $\omega^2$, and $\lambda$, and are used to construct critical values for the tests. This is typically done by large-scale simulations, since the limit distributions are nonstandard. Figure 2 shows how these distributions change by stretching out the left tail as we move from a regression with no trend to a regression with a linear trend. Computerized tabulations of the critical values are given in Ouliaris and Phillips [21] for the case of polynomial trends. In the case of the $Z_\alpha$-test, for instance, we reject the null hypothesis of a unit root at the 5% level if $Z_\alpha < \mathrm{cv}(Z_\alpha; 5\%)$, the 5% critical value of the test. Both the $Z_\alpha$ and $Z_t$ tests are one-sided. They measure the support in the data for a unit root against the alternative that the data are stationary about the deterministic trend $x_t$. When no deterministic trend appears in the model, the alternative hypothesis is just stationarity. In this case, the limit variates involve only the standard Brownian motion $W$. The $Z_\alpha$ and $Z_t$ tests were developed in Phillips [24] and extend the original unit-root tests of Dickey and Fuller based on the statistics $n(\hat{\alpha} - 1)$ and $t_\alpha$. Extensions of these semiparametric tests were obtained in refs. [20,

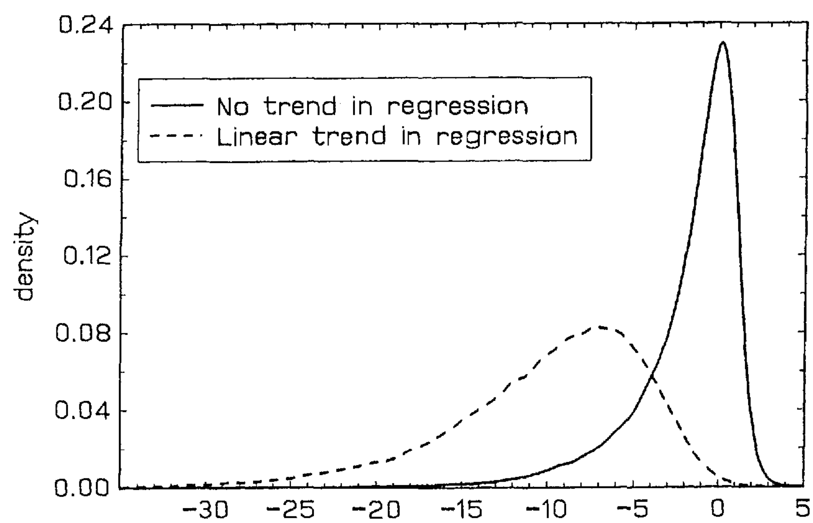


**Figure 2** Unit-root limit densities.

22, 23, 28] and are covered by the above formulas.

To illustrate, the model (4) was estimated with a linear deterministic trend for the data shown in Fig. 1. The calculated values of the coefficient-based test statistics are as follows: $n(\hat{\alpha} - 1) = -7.38$; $Z_\alpha = -13.25$. The asymptotic 5% critical value of the limit distribution of the $Z_\alpha$-statistic is $-21.21$ (cf. the density given by the broken line in Fig. 2). These tests do not reject the null of a unit root in the time series, while allowing for the presence of a linear trend. The $t$-ratio test statistics are $t_{\hat{\alpha}} = -1.92$, $Z_t = -2.56$. The asymptotic 5% critical value of the $Z_t$-statistic is $-3.43$. Again, the tests do not reject the null hypothesis of a unit root in the series. Note that the calculated values of the Dickey–Fuller statistics $n(\hat{\alpha} - 1)$ and $t_{\hat{\alpha}}$ are further from the critical values than are the semiparametric statistics $Z_\alpha$ and $Z_t$. The semiparametric corrections in the $Z$-tests for autocorrelation in the residual process $u_t$ are nonnegligible, but in this case they do not make a difference in the outcome of the unit-root tests.

## THE VON NEUMANN-RATIO LAGRANGE MULTIPLIER TEST

The von Neumann (VN) ratio is the ratio of the sample variances of the differences and the levels of a time series. For Gaussian data this ratio leads to well-known tests of *serial correlation** that have good finite-sample properties. Sargan and Bhargava [34] suggested the use of this statistic for testing the Gaussian-random-walk hypothesis. Using nonparametric estimates of the nuisance parameter $\omega^2$, it is a simple matter to rescale the VN ratio to give a unit-root test for the model (1) and (2). Using a different approach and working with polynomial trends, Schmidt and Phillips [35] showed that for a Gaussian likelihood the Lagrange multiplier (LM) principle leads to a VN test, and can be generalized by using a nonparametric estimate of $\omega^2$.

If $y_t^s$ were observable, the VN ratio would take the form $\mathrm{VN} = \sum_{t=2}^n (\Delta y_t^s)^2 / \sum_{t=1}^n (y_t^s)^2$. The process $y_t^s$ is, in fact, unobserved, but may be estimated from (1). Note that, under the null hypothesis and after differences are taken, this equation is trend-stationary, so that by the Grenander–Rosenblatt theorem [10, Chap. 7] the trend function can be efficiently estimated by an OLS regression. Let $\Delta\hat{y}_t^s = \Delta y_t - \Delta\hat{h}_t$ be the residuals from this *detrending regression*, and let $\hat{y}_t^s = \sum_{j=2}^t \Delta\hat{y}_j^s$ be the associated estimate of $y_t^s$. Also, let $\tilde{y}_t^s = \hat{y}_t^s - \tilde{\beta}' x_t$ be the residuals from an OLS regression of $\hat{y}_t^s$ on $x_t$. Then, rescaling the VN ratio leads to the following two test statistics:

$$R_{VN} = \frac{\hat{\omega}^2}{\hat{\sigma}_u^2} \frac{n^{-1}\sum_{t=2}^n (\Delta\hat{y}_t^s)^2}{n^{-2}\sum_{t=1}^n (\hat{y}_t^s)^2} \Rightarrow \left[\int_0^1 \hat{V}_X^2\right]^{-1},$$

$$\tilde{R}_{VN} = \frac{\hat{\omega}^2}{\hat{\sigma}_u^2} \frac{n^{-1}\sum_{t=2}^n (\Delta\tilde{y}_t^s)^2}{n^{-2}\sum_{t=1}^n (\tilde{y}_t^s)^2} \Rightarrow \left[\int_0^1 \tilde{V}_X^2\right]^{-1}. \tag{7}$$

The limit process $V_X(r)$ in (7) is a *generalized Brownian bridge** and $\tilde{V}_X(r)$ is a detrended generalized Brownian bridge. For example, in the case of a linear trend, $V_X(r) = W(r) - rW(1)$ is a standard *Brownian bridge* and $\tilde{V}_X(r) = V(r) - \int_0^1 V$ is a demeaned version of a standard Brownian bridge.

Critical values of the limit variate shown in (7) are obtained by simulation. The statistics are positive almost surely, and the tests are one-sided. MacNeill [18] and Schmidt and Phillips [35] provide tabulations in the case where $h_t$ is a linear trend. The presence of a unit root is rejected at the 5% level if $R_{\mathrm{VN}} > \mathrm{cv}(R_{\mathrm{VN}}, 5\%)$, or if $\tilde{R}_{\mathrm{VN}} > \mathrm{cv}(\tilde{R}_{\mathrm{VN}}, 5\%)$.

## THE PARAMETRIC ADF TEST

The most common parametric unit-root test is based on the following autoregressive approximation to (4):

$$\Delta y_t = a y_{t-1} + \sum_{j=1}^k \varphi_j \Delta y_{t-j} + \beta' x_t + \epsilon_t. \tag{8}$$

As $k \to \infty$ we can expect the autoregressive approximation to give an increasingly accurate representation of the true process. The unit-root hypothesis in (4) corresponds to the hypothesis

$a = 0$ in (8). The hypothesis is tested by means of the regression $t$-ratio statistic on the coefficient $a$. This statistic has the same limit distribution (and critical values) as the $Z_t$-test given in (6) above, provided $k \to \infty$ at an appropriate rate as $n \to \infty$ [32]. The test is known as the *augmented Dickey–Fuller* (ADF) *test.*

## EFFICIENT DETRENDING BY QUASI-DIFFERENCING

As discussed above, the VN-ratio LM test $R_{\text{VN}}$ is constructed using an efficient detrending regression under the null hypothesis, in contrast to the regression (4), where there are redundant trending regressors. One way to improve the power of unit-root tests is to perform the detrending regression in a way that is efficient under the alternative hypothesis as well, an idea that was suggested in ref. [7] in the context of the removal of means and linear trends. Alternatives that are close to unity can often be well modeled using the local alternative [25]

$$\alpha = \exp(n^{-1}c) \approx 1 + n^{-1}c \qquad (9)$$

for some fixed $c = \bar{c}$, say, given the sample size $n$. Quasi-differencing rather than differencing can now be used in the detrending regression. Such a regression leads to estimates of the trend coefficients that are asymptotically more efficient than an OLS regression in levels [16], and this result justifies the modified test procedure that follows.

To be specific, define the *quasi-difference* $\Delta_{\bar{c}} y_t = (1 - L - n^{-1}\bar{c}L)y_t = \Delta y_t - n^{-1}\bar{c}y_{t-1}$, and run the detrending OLS regression $\Delta_{\bar{c}} y_t = \tilde{\gamma}' \Delta_{\bar{c}} x_t + \Delta_{\bar{c}} \tilde{y}_t^s$. Using the fitted coefficients $\tilde{\gamma}$, the levels data are detrended according to $\tilde{y}_t = y_t - \tilde{\gamma}' x_t$, and $\tilde{y}_t$ can be used in the construction of all of the above unit-root tests. For example, the modified semiparametric $Z_\alpha$-test has the form $\tilde{Z}_\alpha = n(\tilde{\alpha} - 1) - \tilde{\lambda}(n^{-2}\sum_{t=2}^n \tilde{y}_{t-1}^2)^{-1}$, where $\tilde{\lambda}$ is a consistent estimator of $\lambda$, and $\tilde{\alpha}$ is the coefficient in the regression of $\tilde{y}_t$ on $\tilde{y}_{t-1}$. New critical values are needed for the $\tilde{Z}_\alpha$-test, and the limit theory depends not only on the trend functions, as it does in (5), but also on the localizing parameter $\bar{c}$ that is used in the quasi-differencing. A good default choice of $\bar{c}$ seems to be the value for which local asymptotic power is 50% [7, 14].

## A POINT OPTIMAL TEST

When the model for $y_t$ is a Gaussian AR(1) with unit error variance [*see* AUTOREGRESSIVE MOVING-AVERAGE (ARMA) MODELS], the Neyman–Pearson lemma* can be used to construct the most powerful test of a unit root against a simple point alternative. This is a *point optimal test* (POT [14]) for a unit root at the alternative that is selected. Taking a specific local alternative with $c = \bar{c}$ in (9), using quasi-differencing to detrend, and using a consistent nonparametric estimate $\hat{\omega}^2$ of the nuisance parameter $\omega^2$, the POT test statistic for a unit root in (1) and (2) has the form $\tilde{P}_{\bar{c}} = \hat{\omega}^{-2}[\bar{c}^2 n^{-2}\sum_{t=2}^n(\tilde{y}_{t-1}^s)^2 - \bar{c}n^{-1}\tilde{y}_n^s]$, which was given by Elliot et al. [7] in the case where there is a linear trend in (1). The test is performed by comparing the observed value of the statistic with the critical value obtained by simulation. The presence of a unit root in the data is rejected at the 5% level if $\tilde{P}_{\bar{c}} < \text{cv}(\tilde{P}_{\bar{c}}, 5\%)$, i.e., if $\tilde{P}_{\bar{c}}$ is too small. Note that in the construction of $\tilde{P}_{\bar{c}}$, the estimate $\hat{\omega}^2$ is used and this is obtained in the same way as in the $Z_t$-test, i.e., using residuals from the regression (4).

## ASYMPTOTIC PROPERTIES AND LOCAL POWER

The above test statistics are asymptotically similar* in the sense that their limit distributions are free of nuisance parameters. But the limit distributions do depend on whether the data have been prefiltered in any way by a preliminary regression. The tests are also consistent against stationary alternatives provided that any nonparametric estimator of $\omega^2$ that is used in the test converges in probability to a positive limit as $n \to \infty$. The latter condition is important, and it typically fails when estimates of $\omega^2$ are constructed using first differences or quasi-differences of the data rather than regression residuals [27].

Rates of divergence of the statistics under the alternative are also available. For instance, when $|\alpha| < 1$, we have $Z_\alpha$, $\tilde{Z}_\alpha$, $R_{\mathrm{VN}} = O_p(n)$ and $Z_t$, ADF $= O_p(n^{1/2})$ as $n \to \infty$ [27]. Thus, coefficient-based tests that rely on the estimated autoregressive coefficient and the VN-ratio LM tests diverge at a faster rate than tests that are based on the regression $t$-ratio. We may therefore expect such tests to have greater power than $t$-ratio tests, and this is generally borne out in simulations. Heuristically, the $t$-ratio tests suffer because there is no need to estimate a scale parameter when estimating the autoregressive coefficient $\alpha$.

Under the local alternative (9), the limit theory can be used to analyze local asymptotic power. When (2) and (9) hold, $y_t^s$ behaves asymptotically like a linear diffusion rather than Brownian motion, i.e., $n^{-1/2}y_{[nr]}^s \Rightarrow J_c(r) = \int_0^r e^{(r-s)c}\,dW(s)$. The limit distributions of the unit-root test statistics then involve functionals of $J_c(r)$ [25]. The local asymptotic theory can be used to construct asymptotic power envelopes for unit-root tests by taking the limit distribution of the POT statistic under the local alternative $c = \overline{c}$, and then varying the parameter $\overline{c}$.

## FINITE SAMPLE PROPERTIES OF UNIT-ROOT TESTS

Extensive simulations* have been conducted to explore the finite sample performance of unit-root tests. One general conclusion that emerges is that the discriminatory power in all of the tests between models with a root at unity and a root close to unity is low. For instance, the power is less than 30% for $\alpha \in [0.90, 1.0)$ and $n = 100$. The power is reduced further by detrending the data. Both these features mirror the asymptotic theory. One interesting finding from simulation studies is the extent of the finite sample size distortion of the tests in cases where the true model is close to a trend-stationary process. For example, if $u_t$ in (2) follows a moving-average* process $u_t = \epsilon_t + \theta\epsilon_{t-1}$ with $\theta$ large and negative, then the sample trajectories of $y_t^s$ more closely resemble those of a stationary process than a random walk. In such cases there is a tendency for all of the tests to overreject the null of a unit root. Tests that are based directly on autoregressive coefficient estimates, like $Z_\alpha$, tend to be more affected by size distortion than the other tests. This is because the bias in the first-order autoregressive estimator is large in this case, not only in finite samples but even in the asymptotic distribution (7), where the miscentering is measured by the bias parameter $\lambda = \theta\sigma_\epsilon^2$. Good estimates of the bias parameter are needed to control the size distortion. Since $\lambda$ is estimated in a nonparametric way by kernel methods, it is usually not estimated at a $\sqrt{n}$ rate. Recent attempts to improve the estimation of this parameter using data-determined bandwidth* choices [1], prefiltering [2], and data-based model selection and prefiltering [16] offer some promise, the latter reference showing that $\sqrt{n}$ rates of estimation are achievable in these estimates when consistent *model selection** techniques are used to determine the prefilter.

The parametric ADF procedure is less affected by size distortions when the true model is close to stationarity, but generally has much less power than the other tests. With this test, the power is further reduced by the inclusion of additional lagged dependent regressors in (4). Again, model selection methods like BIC [36] are useful in this respect and provide some increase in the finite-sample power of the ADF test.

Since detrending the data reduces power, surplus trend variables in regressions like (4) will do so also. Hence, efficient detrending procedures can be expected to benefit all the tests. Simulations confirm [38] that detrending by regression in quasidifferences seems to be the most successful method so far for increasing finite-sample (and asymptotic) power.

## TRENDS WITH STRUCTURAL BREAKS

Breaks in deterministic trend functions are often employed to capture changes in trend. This possibility is already included in the specification of $h_t$ in (1). For instance, the trend function

$$h_t = \sum_{j=0}^{p} f_j t^j + \sum_{j=0}^{p} f_{m,j} t_m^j,$$

$$\text{where} \quad t_m^j = \begin{cases} 0, & t \in \{1, \ldots, m\}, \\ (t - m)^j, & t \in \{m + 1, \ldots, n\} \end{cases} \tag{10}$$

has a time polynomial of degree $p$ (the first component) and a similar time polynomial with different coefficients (the second component) that initiates at the point $t = m + 1$. This trend function therefore allows for the presence of a structural change in the polynomial trend at the data point $t = m + 1$. Suppose $\mu = \lim_{n\to\infty}(m/n) > 0$ is the limit of the fraction of the sample where this structural change occurs. Then the limiting trend function $X(r)$ corresponding to (10) has a similar break at the point $\mu$. The unit-root tests given above, including those that make use of efficient detrending procedures, continue to apply for such broken trend functions. Indeed, (10) may be extended further to allow for multiple break points in the sample and in the limit process without affecting the theory.

In order to construct unit-root tests that allow for breaking trends like (10) it is necessary to specify the break point $m$. (Correspondingly, the limit theory depends on limit processes that depend on the break point $\mu$.) In effect, the break point is exogenously determined. Perron [23] considered linear trends with single break points in this way. An alternative perspective is that any break points are endogenous to the data and unit-root tests should take account of this fact. Alternative unit-root tests have been suggested [40] that endogenize the break point by choosing the value of $m$ that gives the least favorable view of the unit-root hypothesis. This has been done for the parametric ADF test and for linear trends with breaks. If ADF($m$) denotes the ADF statistic given by the $t$-ratio for $\alpha$ in the ADF regression (4) with a broken trend function like (10), then the trend-break ADF statistic is

$$\mathrm{ADF}(\hat{m}) = \min_{\underset{\sim}{m} \le m \le \tilde{m}} \mathrm{ADF}(m)$$

$$\text{where} \quad \underset{\sim}{m} = [n\underset{\sim}{\mu}], \quad \tilde{m} = [n\tilde{\mu}], \quad \text{and} \quad 0 < \underset{\sim}{\mu} < \tilde{\mu} < 1. \tag{11}$$

The limit theory for this trend-break ADF statistic is given by

$$\mathrm{ADF}(\hat{m}) \Rightarrow \inf_{\mu \in [\underset{\sim}{\mu}, \tilde{\mu}]} \left( \int_0^1 W_X \, dW \right) \left( \int_0^1 W_X^2 \right)^{-1/2}, \tag{12}$$

where the limit process $X(r)$ that appears in this functional on the right side is now dependent on the trend break point $\mu$ over which the functional is minimized. Critical values of the limiting test statistic (12) are further out in the tail than those of the exogenous trend-break statistic, so it is harder to reject the null hypothesis of a unit root when the break point is considered to be endogenous. Simulations indicate that the introduction of trend break functions leads to further reductions in the power of unit-root tests. Sample trajectories of a random walk are often similar to those of a process that is stationary about a broken trend for some particular break point (more so when several break points are permitted in the trend). So reductions in the power of unit-root tests against competing models of this type should not be unexpected.

## SEASONAL UNIT-ROOT TESTS

The parametric ADF test has been extended to the case of seasonal unit roots. In order to accommodate fourth-differencing the autoregressive model is written in the new form

$$\Delta_4 y_t = \alpha_1 y_{1t-1} + \alpha_2 y_{2t-1} + \alpha_3 y_{3t-2} + \alpha_4 y_{3t-1} + \sum_{j=1}^{p} \varphi_j \Delta_4 y_{t-j} + \epsilon_t, \tag{13}$$

where $\Delta_4 = 1 - L^4$, $y_{1t} = (1 + L)(1 + L^2)y_t$, $y_{2t} = -(1 - L)(1 + L^2)y_t$, $y_{3t} = -(1 - L^2)y_t$. The data $y_{1t}, y_{2t}, y_{3t}$ retain the unit root at the zero frequency (long run), the semiannual frequency (two cycles per year), and the annual frequency (one cycle per year), respectively. When $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$, there are unit roots at the zero and all seasonal frequencies. To test the hypothesis of a unit root ($L = 1$), a $t$-ratio test of $\alpha_1 = 0$ is used. Similarly, the test for a semiannual root ($L = -1$) is based on a $t$-ratio test of $\alpha_2 = 0$, and the test for an annual root on the $t$-ratios for $\alpha_3 = 0$ or $\alpha_4 = 0$. Details of the implementation of this procedure are given in Hylleberg et al.

[12]. The limit theory is developed in Chan and Wei [3].

## BAYESIAN TESTS

While most practical work on unit-root testing has utilized classical procedures of the type discussed above, Bayesian methods offer certain advantages that are useful in empirical research. Foremost among these is the potential that these methods offer for embedding the unit-root hypothesis in the wider context of model specification. Whether or not a model such as (4) has a unit root can be viewed as part of the bigger issue of model determination. Model comparison techniques like *posterior odds* and *predictive odds* make it easy to assess the evidence in the data in support of the hypothesis $\alpha = 1$ at the same time as decisions are made concerning other features of the model, such as the lag order in the autoregression (4), the degree of the deterministic trend component, and the presence of trend breaks. Phillips and Ploberger [29, 30] explore this approach to unit-root testing and give an extension of the Schwarz criterion* [36] that can be used for this purpose in models with nonstationary data.

A second advantage of Bayesian methods in models with unit roots is that the asymptotic form of the posterior density is normal [13, 30], a result that facilitates large-sample Bayesian inference* and contrasts with the nonstandard asymptotic distribution theory of classical estimators and tests. Thus, a large-sample Bayesian confidence* set for the autoregressive parameter $\alpha$ in (4) can be constructed in the conventional way without having to appeal to any nonstandard limit theory. In this respect, Bayesian theory (which leads to a symmetric confidence set for $\alpha$) differs from classical statistical analysis, where the construction of valid confidence regions is awkward because of the discontinuity of the limit theory at $\alpha = 1$ (but may be accomplished using local asymptotics). This divergence can lead to quite different inferences being made from the two approaches with the same data. This is so even when the influence of the prior is negligible, as it is in very large samples. In small samples, the role of the prior is important, and time-series models raise special concerns about the construction of uninformative priors, primarily because a great deal is known about the properties of simple time-series models like autoregressions and their characteristic features in advance of data analysis. How this knowledge should be used or whether it should be ignored is a matter on which there is ongoing debate (see Phillips [26] and two recent themed issues of the *Journal of Applied Econometrics*, 1991, and *Econometric Theory*, 1994).

Third, Bayesian methods offer flexibility and convenience in analyzing models with possible unit roots and endogenous trend breaks. In such cases a prior distribution of break points is postulated (such as a uniform prior across potential break points), the posterior mass function is calculated, and the Bayes estimate of the break point is taken as the one with highest posterior mass [41]. This approach makes the analysis of multiple break points straightforward, a problem where classical asymptotic theory is much more complex.

## TESTING STATIONARITY

Adding a stationary component $v_t$ to (1) and (2) gives the model

$$y_t = h_t + y_t^s + v_t, \qquad y_t^s = y_{t-1}^s + u_t, \tag{14}$$

which decomposes the time series $y_t$ into a deterministic trend, a stochastic trend, and a stationary residual. The stochastic trend in (14) is annihilated when $\sigma_u^2 = 0$, which therefore corresponds to a null hypothesis of trend stationarity. Under Gaussian assumptions and i.i.d. error conditions, the hypothesis can be tested in a simple way using the LM principle, and the procedure is easily extended to more general cases where there is serial dependence, by using parametric [17] or semiparametric [15] methods. Defining $w_t = y_t^s + v_t$ and writing its differences as $\Delta w_t = (1 - \theta L)\eta_t$ where $\eta_t$ is stationary, it is clear that $\sigma_u^2 = 0$ in (14) corresponds to the null hypothesis of a moving-average unit root $\theta = 1$. Thus, there is a correspondence between testing for stationarity and testing for a moving-average unit root [33].

## APPLICATIONS, EMPIRICAL EVIDENCE AND FUTURE PROSPECTS

Most empirical applications of unit-root tests have been in the field of economics. Martingales* play a key role in the mathematical theory of efficient financial markets [6] and in the macroeconomic theory of the aggregate consumption behavior of rational economic agents [11]. In consequence, economists have been intrigued by the prospect of testing these theories. In the first modern attempt to do so using unit-root tests, Nelson and Plosser [19] tested fourteen historical macroeconomic time series for the United States by the ADF test and found empirical evidence to support a unit root for thirteen of these series (the exception being unemployment). Since then, these series have been retested with other methods, and hundreds of other time series have been examined in the literature. While it is recognized that the discriminatory power of unit-root tests is often low, there is a mounting body of evidence that many economic and financial time series are well characterized by models with roots at or near unity, as in the case of the leading economic indicators data graphed in Fig. 1.

In empirical applications to multiple time series*, the ADF and semiparametric $Z$ tests have been extensively used to test for the presence of cointegration* (or co-movement among variables with unit roots). The tests are used in the same way as unit-root tests and have the same null hypothesis, but the data are the residuals from an OLS regression among the variables, and the alternative hypothesis (of cointegration) is now the main hypothesis of interest [8, 27]. The model is analogous to (1), but both variables $y_t$ and $x_t$ have unit roots and $y_t^s$ is stationary.

Unit-root models, testing procedures, and unit-root asymptotics now occupy a central position in the econometric analysis of time series. This is partly because of the growing empirical evidence of stochastic trends in economic data, and partly because of the importance of the notion of shock persistence in economic theory. The scope for the use of these methods in empirical research in other fields like political science and communications seems substantial. Advances in computer technology will continue to facilitate the use of simulation methods in dealing with the nonstandard distributions that unit-root methods entail. The explosion of research over the last decade in the field of nonstationary time series and unit-root methods shows no sign of abating. The field is full of potential for future developments in statistical theory, in modeling, and in empirical applications.

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