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TIME SERIES REGRESSION WITH MIXTURES OF INTEGRATED PROCESSES

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The paper develops a statistical theory for regressions with integrated regressors of unknown order and unknown cointegrating dimension. In practice, we are often unsure whether unit roots or cointegration is present in time series data, and we are also uncertain about the order of integration in some cases. This paper addresses issues of estimation and inference in cases of such uncertainty. Phillips (1995, *Econometrica* 63, 1023–1078) developed a theory for time series regressions with an unknown mixture of I(0) and I(1) variables and established that the method of *fully modified ordinary least squares* (FM-OLS) is applicable to models (including vector autoregressions) with some unit roots and unknown cointegrating rank. This paper extends these results to models that contain some I(0), I(1), and I(2) regressors. The theory and methods here are applicable to cointegrating regressions that include unknown numbers of I(0), I(1), and I(2) variables and an unknown degree of cointegration. Such models require a somewhat different approach than that of Phillips (1995). The paper proposes a *residual-based fully modified ordinary least-squares* (RBFM-OLS) procedure, which employs residuals from a first-order autoregression of the first differences of the entire regressor set in the construction of the FM-OLS estimator. The asymptotic theory for the RBFM-OLS estimator is developed and is shown to be normal for all the stationary coefficients and mixed normal for all the nonstationary coefficients. Under Gaussian assumptions, estimation of the cointegration space by RBFM-OLS is optimal even though the dimension of the space is unknown.

1. INTRODUCTION

Many researchers have investigated time series regressions with integrated processes, and there is now a large literature. There are fully developed statistical theories available for regressions with I(1) and a mixture of I(0), I(1),

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and I(2) variables, but all the existing theories either presume knowledge of the order of cointegration and the number of unit roots in the model or require pretest procedures to determine these. In practice, we are often unsure whether unit roots or cointegration is present in data, and we are also uncertain about the order of integration in some cases. This paper addresses issues of estimation and inference in cases of such uncertainty and develops a statistical theory for regressions with integrated regressors of unknown order and unknown cointegrating dimension.

Recently, Phillips (1995) developed a theory for time series regressions with an unknown mixture of I(0) and I(1) variables and applied it to vector autoregressions (VAR's) with some unit roots and cointegration. His results do not require any prior knowledge of the number and location of unit roots in the systems and are applicable to models with any mixture of I(0) and I(1) processes. For instance, he established that the method of *fully modified ordinary least squares* (FM-OLS), developed in Phillips and Hansen (1990), is applicable to models, including VAR's, with some unit roots and unknown cointegrating rank.

The present paper extends the results in Phillips (1995) to models that contain some I(0), I(1), and I(2) regressors. Models that include I(2) variables are important because some major macroeconomic variables such as money stock and price levels have been characterized as I(2) processes in some recent empirical work (see, e.g., Johansen, 1992; King, Plosser, Stock, and Watson, 1991), and many VAR macroeconomic models include at least one such variable. The theory and methods that are developed in this paper are applicable to cointegrating regressions that include unknown numbers of I(0), I(1), and I(2) variables and an unknown degree of cointegration.

It turns out that models with unknown mixtures of I(0), I(1), and I(2) regressors require a somewhat different approach than that of Phillips (1995). This paper explains the difficulty and proposes a *residual-based fully modified ordinary least-squares* (RBFM-OLS) procedure, which employs residuals from a first-order autoregression of the first differences of the entire regressor set in the construction of the FM-OLS estimator. It is shown that the limit theory of the RBFM-OLS estimator in the stationary direction is normal and is asymptotically equivalent to that of the unrestricted OLS estimator, and that the RBFM-OLS estimator for the nonstationary coefficients has a nuisance parameter–free mixed normal limit distribution, which is optimal under Gaussian assumptions for cointegrated models with I(1) and I(2) processes (cf. Kitamura, 1995). One consequence of the normal and mixed normal limit theory for the RBFM-OLS estimators is that we can conduct hypothesis testing using Wald statistics, which have limit distributions that are linear combinations of independent chi-squared variates, along lines that are similar to those of Phillips (1995, pp. 19–20) in the I(0) and I(1) case.

The framework of this paper enables us to study the asymptotic behavior of the RBFM-OLS estimator in a general class of time series models that have various types of cointegration, covering CI(2,2) and CI(2,1) systems, in Granger's notation, as well as the usual CI(1,1) systems. The additional generality comes from introducing I(2) regressors into the model. The procedure proposed in the paper provides an approach to unrestricted regressions for fairly general time series that allows for integrated processes of orders up to 2 and the possibility of cointegration among the regressors.

The paper proceeds as follows. Section 2 gives the regression model and outlines the assumptions that we need for the theoretical development. Section 3 gives the limit theory for the OLS estimates in models with some I(0), I(1), and I(2) regressors. Section 4 considers models with cointegrated regressors when there is no prior knowledge about the degree of integration of the regressors and the directions of cointegration and develops the limit theory of the new RBFM-OLS estimator. Section 5 concludes the paper and summarizes its main results. A rather large number of useful subsidiary limit theories is needed for our development, and these are collected in Appendix A, with all proofs and technical details following in Appendix B.

The following terminology and notations are used in the paper. Following earlier work, we call $\Omega = \sum_{k=-\infty}^{\infty} E(u_k u_0')$ the long-run variance matrix of the stationary time series u_t and write $\operatorname{lrvar}(u_t) = \Omega$. Similarly, we denote long-run covariance matrices by $\operatorname{lrcov}(\cdot)$ and call matrices of one-sided sums of covariances, like $\Delta = \sum_{k=0}^{\infty} E(u_k u_0')$, one-sided long-run covariance matrices. We use $BM(\Omega)$ to denote a vector Brownian motion with covariance matrix Ω and write integrals with respect to the Lebesque measure, such as $\int_0^1 B(s)\,ds$, simply as $\int_0^1 B$. The notation $y_t = \mathrm{I}(d)$ signifies that the time series y_t is integrated of order d, so that $\Delta^d y_t = \mathrm{I}(0)$, and this requires that $\operatorname{lrvar}(\Delta^d y_t) > 0$. The inequality >0 denotes positive definite when applied to matrices. We use the symbols $\xrightarrow{\mathcal{D}}$, $\xrightarrow{\mathcal{P}}$, $\equiv$, and $:=$ to signify convergence in distribution, convergence in probability, equality in distribution, and notational definition, respectively. We also use $\|A\|$ to signify the matrix norm $(\operatorname{tr}(A'A))^{1/2}$, $|A|$ to denote the determinant of A, $\operatorname{vec}(A)$ to stack the rows of a matrix A into a column vector, and $[x]$ to denote the large integer $\leq x$. All the limits given in the paper are taken as the sample size $T \to \infty$.

2. THE MODEL AND ASSUMPTIONS

We consider the model given by

$$y_t = Ax_t + u_{0t}, \tag{1}$$

where A is an $(n \times m)$ coefficient matrix and x_t is an $m = (m_1 + m_2 + m_3)$-dimensional vector of *cointegrated* or possibly stationary regressors. We use an $(m \times m)$ orthogonal matrix $H = (H_1, H_2, H_3)$, which rotates the regressor space to separate out the I(0), I(1), and I(2) components of the regressors in model (1). We then specify the regressors according to the following equations:

$H_1' x_t = x_{1t} \sim (m_1 \times 1) - \mathrm{I}(0)$ process

$H_2' x_t = x_{2t} \sim (m_2 \times 1) - \mathrm{I}(1)$ process

$H_3' x_t = x_{3t} \sim (m_3 \times 1) - \mathrm{I}(2)$ process

with the data generating processes defined as

$$x_{1t} = u_{1t},$$

$$\Delta x_{2t} = u_{2t},$$

$$\Delta^2 x_{3t} = u_{3t},$$

where Δ is the usual difference operator. We now rewrite model (1) using the preceding specification of x_t as

$$y_t = A_1 x_{1t} + A_2 x_{2t} + A_3 x_{3t} + u_{0t}, \tag{2}$$

where $A_1 = AH_1$, $A_2 = AH_2$, and $A_3 = AH_3$. It is clear from (2) that the nonstationary regressors are cointegrated with the dependent variable y_t and that the I(1) and I(2) components that constitute the nonstationary part are possibly cointegrated themselves. Hence, model (2) covers cointegrations of the forms CI(2,1) and CI(2,2) as well as the most widely studied CI(1,1) in Granger's notation. The model of course also allows for full rank I(1) and/or full rank I(2) regressors, which are not associated with any form of cointegration. We construct data matrices from the variables in (2) and conventionally denote them by uppercase letters. Then, (2) is written as

$$Y' = A_1 X_1' + A_2 X_2' + A_3 X_3' + U_0',$$

with $X_1 = U_1$, $\Delta X_2 = U_2$, $\Delta^2 X_3 = U_3$ and where $Y' = (y_1, \ldots, y_T)$.

There is literature available on procedures to estimate and pretest for the direction of cointegration and the rank of the cointegrating space of the regressors, along with numerous tests for nonstationary characteristics of the data. However, it is well-known that there are cases where different procedures do not concur and lead to different inferences, even when they are based on the same data set. Thus, our goal is to develop a methodology that is not dependent on any prior specification of the data so that we can proceed without any information about the nonstationary characteristics of the data and the cointegration space itself, that is, with H unknown. With the approach developed in this paper, one can simply treat the model given in (1) as a time series regression without pretesting the regressors for unit roots, double unit roots, and cointegration, that is, without being specific about the I(2), I(1), and I(0) components of the data.

We let the error process $u_t = (u_{0t}', u_{1t}', u_{2t}', u_{3t}')'$ be an $(n+m)$-vector stationary process and define $\varphi_t = u_{0t} \otimes u_{1t}$. We make the following assumption.

Assumption 1. u_t is a linear process that satisfies

(a) $u_t = C(L)\epsilon_t = \sum_{j=0}^{\infty} C_j \epsilon_{t-j}$, with $\sum_{j=0}^{\infty} j^{\alpha} \|C_j\| < \infty$ for some $\alpha > 1$ and $|C(1)| \neq 0$;
(b) ϵ_t is i.i.d. with zero mean, variance $\Sigma_{\epsilon\epsilon} > 0$, and finite fourth-order cumulants; and
(c) $E(\varphi_{t,j}) = E(u_{0t+j} \otimes u_{1t}) = 0$ for all $j \geq 0$.

Assumption 1 ensures the validity of the functional central limit theory for both u_t and $u_t u_t'$ (cf. Phillips and Solo, 1992). In particular, we have

$$\frac{1}{\sqrt{T}} \sum_{t=1}^{T} \varphi_t \xrightarrow{\mathfrak{D}} \mathcal{N}(0, \Omega_{\varphi\varphi}) \tag{3}$$

with

$$\Omega_{\varphi\varphi} = \sum_{j=-\infty}^{\infty} E(u_{0t} u_{0t+j}' \otimes u_{1t} u_{1t+j}'). \tag{4}$$

We also have

$$\frac{1}{\sqrt{T}} \sum_{t=1}^{[T\cdot]} u_t \xrightarrow{\mathfrak{D}} B(\cdot) \equiv BM(\Omega), \tag{5}$$

where $B = (B_0', B_1', B_2', B_3')'$ is an $(n+m)$-vector Brownian motion with covariance matrix

$$\begin{aligned}
\Omega &= C(1)\Sigma_{\epsilon\epsilon}C(1)' \\
&= \begin{array}{c} \begin{bmatrix} \Omega_{00} & \Omega_{01} & \Omega_{02} & \Omega_{03} \\ \Omega_{10} & \Omega_{11} & \Omega_{12} & \Omega_{13} \\ \Omega_{20} & \Omega_{21} & \Omega_{22} & \Omega_{23} \\ \Omega_{30} & \Omega_{31} & \Omega_{32} & \Omega_{33} \end{bmatrix} \end{array} \begin{array}{l} n \\ m_1 \\ m_2 \\ m_3 \end{array} \\
&= \sum_{j=-\infty}^{\infty} E(u_j u_0') \\
&= \Sigma + \Lambda + \Lambda',
\end{aligned}$$

where

$$\Sigma = E(u_0 u_0'),$$

$$\Lambda = \sum_{j=1}^{\infty} E(u_j u_0').$$

We also define the one-sided long-run covariance matrix as

$$\Delta = \Sigma + \Lambda = \sum_{j=0}^{\infty} E(u_j u_0') = \sum_{j=0}^{\infty} \Gamma(j),$$

where $\Gamma(j)$'s are autocovariances. We partition the variance matrix, Σ, and the one-sided long-run covariance, Δ, of u_t into cell submatrices Σ_{ij} and Δ_{ij} (for $i,j = 0,1,2,3$) conformably with the partition of u_t. When u_{0t} and u_{1s} are independent for all t and s, $\Omega_{\varphi\varphi}$ in (4) reduces to $\sum_{j=-\infty}^{\infty} E(u_{0t}u_{0t+j}') \otimes E(u_{1t}u_{1t+j}')$. And when, in addition, u_{0t} is i.i.d. $(0,\Sigma_{00})$, $\Omega_{\varphi\varphi}$ becomes simply $\Sigma_{00} \otimes \Sigma_{11}$.

In later analysis, we use spectral estimates of Ω and Δ that are based on kernel smoothing of the component sample autocovariances. The general form of the kernel estimate can be found in Priestley (1981) or Hannan (1970). We use the following general form for kernel estimates of Ω and Δ, that is,

$$\hat{\Omega} = \Sigma_{j=-T+1}^{T-1} w(j/K)\hat{\Gamma}(j), \tag{6}$$

$$\hat{\Delta} = \Sigma_{j=0}^{T-1} w(j/K)\hat{\Gamma}(j), \tag{7}$$

where $w(\cdot)$ is a kernel function, K is a lag truncation or bandwidth parameter, and $\hat{\Gamma}(\cdot)$ is a sample autocovariance. Truncation in the sums given in (6) and (7) occurs when $w(j/K) = 0$ for $|j| \geq K$.

As in the analysis for the I(1) cointegrated model studied in Phillips (1995), kernel estimation of both Ω and Δ plays an important role in developing the limit theory for our I(2) cointegrated system. The conditions employed in the aforementioned paper are generally sufficient for our analysis, as we will discuss later, and we will use the same class of admissible kernels as in that paper, namely, the following assumption.

Assumption 2. The kernel function $w(\cdot): \mathbb{R} \to [-1,1]$ is a twice continuously differentiable even function with

(a) $w(0) = 1$, $w'(0) = 0$, $w''(0) \neq 0$; and
(b) $w(x) = 0$, $|x| \geq 1$, with

$$\lim_{|x| \to 1} \frac{w(x)}{(1 - |x|)^2} = \text{constant}.$$

Assumption 2 allows for the commonly used Parzen and Tukey–Hanning kernels.

Assumption 2(b′).

$$w(x) = O(x^{-2}), \quad \text{as } |x| \to 1.$$

If, instead of part (b) of Assumption 2, Assumption 2(b′) is assumed, then the Bartlett–Priestley or quadratic spectral kernel becomes admissible (see, e.g., Priestley, 1981, p. 463). We mention that Assumption 2 gives sufficient but not necessary conditions on the kernels for our later analyses.

We also have to be explicit about the bandwidth expansion rate of K as $T \to \infty$. We use the expansion rate order symbol O_e defined in Phillips (1995) to conveniently characterize rates of expansion of $K = K(T)$ as $T \to \infty$, that is, the following definition.

DEFINITION 1. *The expansion rate order symbol O_e is defined as*

$$K = O_e(T^k)$$

if

$$K \sim c_T T^k \quad \text{as } T \to \infty,$$

where c_T is slowly varying *at infinity, that is,* $\lim_{T\to\infty} c_{Tx}/c_T \to 1$ *for any constant $x > 0$.*

Using the definition $K = O_e(T^k)$, we now impose the following condition on how the bandwidth parameter K grows as $T \to \infty$ in the next assumption.

Assumption 3. The bandwidth parameter K in the kernel estimates in (6) and (7) has an expansion rate of the form $K = O_e(T^k)$ for some $k \in (0,1)$. We specify the following explicit rates:

BW(a): $K = O_e(T^k)$ for some $k \in (\frac{1}{4},\frac{1}{2})$;
BW(b): $K = O_e(T^k)$ for some $k \in (\frac{1}{4},1)$;
BW(c): $K = O_e(T^k)$ for some $k \in (0,\frac{1}{2})$;
BW(d): $K = O_e(T^k)$ for some $k \in (0,1)$.

Under Assumption 3, $K \sim c_T T^k$ for some slowly varying function c_T. Thus, for any $k \in (0,1)$, we have

$$\frac{K}{T} = \frac{c_T T^k}{T} = c_T T^{k-1} \to 0 \quad \text{as } T \to \infty,$$

and for any $k > \frac{1}{4}$, say $k = \frac{1}{4} + \delta$ with $\delta > 0$, we have

$$\frac{K^4}{T} = \frac{(c_T T^{1/4+\delta})^4}{T} = c_T^4 T^{4\delta} \to \infty.$$

Similarly, for $k = \frac{1}{2} - \delta$, we have

$$\frac{K^2}{T} = \frac{(c_T T^{1/2-\delta})^2}{T} = c_T^2 T^{-2\delta} \to 0.$$

Hence, BW(a) implies that, as $T \to \infty$, $\sqrt{T}/K^2 + K^2/T \to 0$ holds for $k \in (\frac{1}{4},\frac{1}{2})$. We note that conditions BW(a) and BW(b) do not include the optimal growth rate $K \sim cT^{1/5}$ (cf. Andrews, 1991), with c a constant, that applies when minimizing the asymptotic mean squared error of kernel estimates such as $\hat{\Omega}$ with kernels that satisfy the conditions in Assumption 2. But condition BW(c) does include the optimal growth rate. We mention here that we do not attempt to achieve an optimal expansion rate for K in the cases that require BW(a) or BW(b), because our goal is to achieve efficient estimation of model (1), not optimal kernel estimation of Ω.

3. OLS ESTIMATION

We first study the limit theory of the OLS estimator for the coefficient matrix A in model (1) to help motivate our new estimator. Define a subscript coupling notation b by

$b =$ "2,3,"

and use this to rewrite model (2) as

$$y_t = A_1 x_{1t} + A_b x_{bt} + u_{0t}, \tag{2'}$$

where $x_{1t} = u_{1t}$, $A_b = (A_2, A_3) = AH_b$, and $x_{bt} = H_b' x_t = (x_{2t}', x_{3t}')'$. With model (2′), we consider jointly the I(1) and I(2) regressors, the nonstationary part of x_t, separately from the stationary component. It is convenient to formulate the asymptotic theory in terms of the component submatrices $A_1 = AH_1$ and $A_b = AH_b$ in model (2′) that correspond to the stationary and nonstationary elements of the regressors. Before we proceed, we introduce an additional notational device to simplify some expressions in our later development. We use X_h to denote the H-transformed X, namely,

$$X_h = XH = (XH_1, XH_b) = (X_1, X_b)$$

and X_b^D to mean that the I(1) and I(2) regressors in X_b are normalized by D_T defined here as

$$X_b^D = X_b D_T^{-1} = (T^{-1}X_2, T^{-2}X_3) \quad \text{with } D_T = \begin{pmatrix} TI_{m_2} & 0 \\ 0 & T^2 I_{m_3} \end{pmatrix}.$$

The OLS estimator of A in (1) is defined as $\hat{A} = Y'X(X'X)^{-1}$, and the limit theory for OLS follows directly from Lemma 2.1 of Park and Phillips (1989). We state it here for convenience.

PROPOSITION 1. *Under Assumption* 1, *we have*

(a) $\sqrt{T}(\hat{A} - A)H_1 \xrightarrow{\mathfrak{D}} \mathcal{N}(0, (I \otimes \Sigma_{11}^{-1})\Omega_{\varphi\varphi}(I \otimes \Sigma_{11}^{-1}))$,
(b) $(\hat{A} - A)H_b D_T \xrightarrow{\mathfrak{D}} (\int_0^1 dB_0 \bar{B}_b' + \Delta_{0b})(\int_0^1 \bar{B}_b \bar{B}_b')^{-1}$,

where $\bar{B}_b' = (B_2', \bar{B}_3')$ *with* $\bar{B}_3(r) = \int_0^r B_3(s)\,ds$, *and* $\Delta_{0b} = (\Delta_{02}, 0)$.

Remark. We decompose the Brownian motion $B_0(r)$, using Lemma 3.1 of Phillips (1991b), as

$$B_0(r) = B_{0\cdot b}(r) + \Omega_{0b}\Omega_{bb}^{-1}B_b(r),$$

where $B_{0\cdot b} = BM(\Omega_{00\cdot b})$ and $\Omega_{00\cdot b} = \Omega_{00} - \Omega_{0b}\Omega_{bb}^{-1}\Omega_{b0}$. Then, the limit distribution of the OLS estimator given in Proposition 1(b) can be alternatively written as

$$\left(\int_0^1 d(B_{0\cdot b} + \Omega_{0b}\Omega_{bb}^{-1}B_b)\bar{B}_b' + \Delta_{0b}\right)\left(\int_0^1 \bar{B}_b\bar{B}_b'\right)^{-1}$$
$$= \int_0^1 dB_{0\cdot b}\bar{B}_b'\left(\int_0^1 \bar{B}_b\bar{B}_b'\right)^{-1} + \Omega_{0b}\Omega_{bb}^{-1}\int_0^1 dB_b\bar{B}_b'\left(\int_0^1 \bar{B}_b\bar{B}_b'\right)^{-1}$$
$$+ \Delta_{0b}\left(\int_0^1 \bar{B}_b\bar{B}_b'\right)^{-1}. \quad (8)$$

We note that the first term in the preceding expression is mixed normal with the conditional covariance matrix

$$\Omega_{00\cdot b} \otimes \left(\int_0^1 \bar{B}_b\bar{B}_b'\right)^{-1},$$

which is the variance matrix of the optimal estimator under Gaussian assumptions for cointegrated models with I(1) and I(2) processes (see Kitamura, 1995; Phillips, 1991a). The second term involves the long-run covariance Ω_{0b} between the equation error u_{0t} and u_{bt} that drives the nonstationary regressors x_{bt} and a mixture of unit root and double unit root limit processes. Finally, the third term involves serial correlation between the equation error and the I(1) regressors x_{2t}. As shown in Park and Phillips (1989, Lemma 2.1), there is no serial correlation effect in the limit distribution between the equation error and x_{3t} because the signal from the I(2) regressor is so strong relative to the effects of serial correlation between the equation error and the past history of the shocks $\{u_{3s}; s \leq t\}$ that generate the I(2) regressor. The second and third terms together measure the extent of the second-order simultaneous equation bias that results from the endogeneity in x_{bt} associated with the cointegration linkage between y_t and x_{bt}, which is apparent from (2′). The second order simultaneous equation bias then brings a miscentering, an asymmetry, and nonscale nuisance parameter dependency to the limit distribution of $\hat{A}_b$.

With Assumption 1, we allow the errors to be correlated both contemporaneously and over time. These correlations produce the endogeneity and serial correlation effects that are manifested in the limit theory given in Proposition 1. The fully modified estimator in the next section is a regression estimator that is designed to remove these effects of endogeneity and serial correlation in the limit theory.

If we had prior knowledge about the nonstationary characteristics of the data (i.e., if H were known a priori), we would be able to remove the endogeneity and serial correlation problems in the limit distribution of the OLS estimator by using endogeneity-corrected dependent variables and by modifying the estimator to eliminate any serial correlation effects in the limit. This idea underlies the construction of the original fully modified estimator in Phillips and Hansen (1990). Of course, there is no need for such corrections as far as the stationary components of the system are concerned, because the OLS estimator of these components already has good asymptotic properties under our error conditions, as shown in Proposition 1. For the nonstationary component submatrices, however, we do need to undertake such adjustments to the data and the estimator if we are to obtain a limit theory with a mixed normal distribution and if we are to have any hope of attaining a regression equivalent of an optimal estimator under Gaussian assumptions. The FM correction terms are designed to remove the endogeneity in the nonstationary regressors x_{bt}, which results from the cointegration linkages with the dependent variable y_t, and the serial correlation effects between the error u_{0t} and the innovations that drive the I(1) regressors x_{2t}.

When H is known, we have model (2′), where the regressors are explicitly specified as I(0), I(1), and I(2). We can then apply the necessary adjustments only where they are needed to obtain a new estimator with improved asymptotic properties. The necessary correction terms here can easily be constructed by using the estimated sample autocovariance $\hat{\Gamma}(\cdot)$ in (6) and (7) defined as

$$\hat{\Gamma}(j) = T^{-1}\sum{}' \hat{u}_t \hat{u}_{t-j}',$$

where $\sum'$ signifies summation over $1 \leq t,\ t-j \leq T$, and $\hat{u}_t = (\hat{u}_{0t}', u_{1t}', u_{2t}', u_{3t}')'$ with $\hat{u}_{0t} = y_t - \hat{A}x_t$, the first-stage OLS residuals, $u_{1t} = H_1'x_t$, the regressors in the stationary direction, $u_{2t} = H_2'\Delta x_t$, the first difference of the regressors in I(1) direction, and $u_{3t} = H_3'\Delta^2 x_t$, the second difference of the regressors in I(2) direction. More precisely, an efficient asymptotically mixed normal estimator of A_b in (2′) can be defined as

$$\hat{A}^{++} = (\hat{A}_1^{++}, \hat{A}_b^{++}) = (Y'X_1, Y^{++}X_b - T\hat{O}^{++})(X'X)^{-1}$$

where

$$Y^{++\prime} = Y' - \hat{\Omega}_{u_0u_b}\hat{\Omega}_{u_bu_b}^{-1}u_b'$$

$$\hat{\Delta}^{++} = (\hat{\Delta}_{\hat{u}_0u_2}^{+}, 0)$$

with

$$\hat{\Delta}_{\hat{u}_0u_2}^{+} = \hat{\Delta}_{\hat{u}_0u_2} - \hat{\Omega}_{\hat{u}_0u_b}\hat{\Omega}_{u_bu_b}^{-1}\hat{\Delta}_{u_bu_2}.$$

It can be shown straightforwardly that the limiting distribution of $\hat{A}_b^{++}$ is identical to that of the first term in (8). The use of $\hat{u}_{0t}$ in the place of u_{0t} will not affect our results as $\hat{A} \overset{\mathcal{P}}{\to} A$ under Assumption 1.

In general, however, we do not have precise information about the configuration of the regressors; that is, the components x_{1t}, x_{2t}, and x_{3t} are not known a priori, thereby eliminating the possibility of constructing the necessary correction terms. The next section develops a procedure that makes all the adjustments that are necessary to obtain an asymptotically optimal estimator under Gaussian assumptions without any prior knowledge about the nonstationary characteristics of the data.

4. FM-OLS AND RBFM-OLS ESTIMATION

We develop a version of the FM procedure that is robust to the precise specification of the "integratedness" of the regressors x_t in model (1). Phillips (1995) provided an extensive account of this problem for models with cointegrated I(1) variables. That reference shows how the usual FM procedure works with cointegrated regressors. The limit theory of the FM-OLS estimator for the stationary component remains the same as the OLS estimator, that is, it is still Gaussian, but the FM-OLS estimator of the coefficients of the I(1) regressors is asymptotically mixed normal. To achieve the asymptotic equivalence of FM-OLS and OLS for the stationary component, the bandwidth parameter expansion rate k, which is used for the construction of kernel estimates of long-run covariance matrices, is required to be in the interval $(\frac{1}{4},1)$. The limit theory for the nonstationary component holds when the expansion rate $k \in (0,\frac{2}{3})$. Hence, the FM-OLS estimator as a whole achieves the optimal (conditional) Gaussian limit distribution for $k \in (\frac{1}{4},\frac{2}{3})$ in Phillips' mixed I(0) and I(1) model.

In this section, we extend the analysis of Phillips's paper by adding I(2) processes to the model, as specified in (2) earlier. Our procedure follows the basic idea of the FM methodology in the sense that we transform the data to correct for potential endogeneities in the regressors and modify the estimator to correct further for any remaining serial correlation effects. However, the usual correction terms employed in Phillips (1995) and Phillips and Hansen (1990) that are based on the first difference of the regressors do not work completely in I(2) regressions, as the first differences of the I(2) regressors are still I(1), and the FM-OLS estimator based on such correction terms has a nonstandard limit theory (for an illustration and discussion of this result, see Phillips and Chang, 1994). It is also not sufficient to base correction terms on a vector of first and second differences of the data. Thus, the problem of assigning corrections is substantially more complex in models with I(0), I(1), and I(2) components.

We show that the new FM correction terms defined later lead to optimal estimators under Gaussian assumptions. In particular, we show that the results in Phillips (1995) continue to apply in models with an unknown mixture of I(0), I(1), and I(2) regressors under exactly the same conditions except for the ranges of allowable bandwidth parameter expansion rates. We show that the limit theory for our FM-OLS estimator, given in (18) later, in the stationary direction is still normal, but it requires a bandwidth parameter expansion rate $K = O_e(T^k)$ for $k \in (\frac{1}{4},\frac{1}{2})$ for the limit theory to hold. We also show that the joint FM-OLS estimator for the I(1) and I(2) coefficients attains the optimal mixed normal distribution under a bandwidth parameter expansion rate $k \in (0,\frac{1}{2})$.

We proceed by considering the following first-order vector autoregression of the first differences of the entire regressor set on their first lags, namely,

$$\Delta x_t = \hat{J}\Delta x_{t-1} + \hat{v}_t, \tag{9}$$

which can be respecified as

$$H'\Delta x_t = H'\hat{J}HH'\Delta x_{t-1} + H'\hat{v}_t,$$

that is,

$$\Delta x_{1t} = \hat{J}_{11}\Delta x_{1t-1} + \hat{J}_{12}\Delta x_{2t-1} + \hat{J}_{13}\Delta x_{3t-1} + \hat{v}_{1t}, \tag{10}$$

$$\Delta x_{2t} = \hat{J}_{21}\Delta x_{1t-1} + \hat{J}_{22}\Delta x_{2t-1} + \hat{J}_{23}\Delta x_{3t-1} + \hat{v}_{2t}, \tag{11}$$

$$\Delta x_{3t} = \hat{J}_{31}\Delta x_{1t-1} + \hat{J}_{32}\Delta x_{2t-1} + \hat{J}_{33}\Delta x_{3t-1} + \hat{v}_{3t}. \tag{12}$$

We note that plim $\hat{J}_{13}$ and plim $\hat{J}_{23}$, the probability limits of the coefficient matrices on the I(1) regressor Δx_{3t-1} in regressions (10) and (11), are 0 (otherwise, the regression would be spurious, which it is not). Also, note that regression (12) is a full rank I(1) regression and, hence, plim $\hat{J}_{33}$ is the identity matrix. Then, equations (10)–(12) can be reformulated as regression equations for the model

$$\begin{bmatrix} \Delta x_{1t} \\ \Delta x_{2t} \\ \Delta^2 x_{3t} \end{bmatrix} = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \\ J_{31} & J_{32} \end{bmatrix} \begin{pmatrix} \Delta x_{1t-1} \\ \Delta x_{2t-1} \end{pmatrix} + \begin{bmatrix} v_{1t} \\ v_{2t} \\ v_{3t} \end{bmatrix},$$

where $(v_{1t}', v_{2t}', v_{3t}')' =: v_{ht}$ is a stationary process. The residuals in the preceding equation are defined as

$$v_{ht} = u_{ht}^* + G_h u_{ht-1}, \tag{13}$$

where

$$u_{ht}^* = \begin{bmatrix} \Delta u_{1t} \\ u_{2t} \\ u_{3t} \end{bmatrix}, \qquad u_{ht} = \begin{pmatrix} \Delta u_{1t} \\ u_{2t} \end{pmatrix} \tag{14}$$

and

$$G_h = \begin{bmatrix} -J_{11} & -J_{12} \\ -J_{21} & -J_{22} \\ -J_{31} & -J_{32} \end{bmatrix} := \begin{bmatrix} L_1 & L_2 \\ M_1 & M_2 \\ N_1 & N_2 \end{bmatrix}. \tag{15}$$

Note that the residuals v_{ht} contain the linear combination of u_{bt} that is necessary for correcting the endogeneity in I(1) and I(2) components jointly.

Because v_{ht} is not observed, we use the residuals from regression (9) already given, that is,

$$\hat{v}_t = \Delta x_t - \hat{J}\Delta x_{t-1},$$

which can be similarly respecified as $\hat{v}_{ht} := H'\hat{v}_t = H'\Delta x_t - H'\hat{J}HH'\Delta x_{t-1}$, namely,

$$\hat{v}_{ht} = \begin{bmatrix} \Delta x_{1t} \\ \Delta x_{2t} \\ \Delta x_{3t} \end{bmatrix} - \begin{bmatrix} \hat{J}_{11} & \hat{J}_{12} & \hat{J}_{13} \\ \hat{J}_{21} & \hat{J}_{22} & \hat{J}_{23} \\ \hat{J}_{31} & \hat{J}_{32} & \hat{J}_{33} \end{bmatrix} \begin{bmatrix} \Delta x_{1t-1} \\ \Delta x_{2t-1} \\ \Delta x_{3t-1} \end{bmatrix}.$$

Now, $\operatorname{plim} J_{13} = \operatorname{plim} J_{23} = 0$, $\operatorname{plim} J_{33} = I$, and we may conveniently write $\hat{J}_{13} = \hat{J}_{23} = O_p(T^{-1})$ and $\hat{J}_{33} = I + O_p(T^{-1})$, because these OLS estimators of the coefficients of the I(1) variables are T-consistent. Then, the right-hand side of the preceding equation becomes

$$\begin{bmatrix} \Delta x_{1t} \\ \Delta x_{2t} \\ \Delta^2 x_{3t} \end{bmatrix} - \begin{bmatrix} \hat{J}_{11} & \hat{J}_{12} \\ \hat{J}_{21} & \hat{J}_{22} \\ \hat{J}_{31} & \hat{J}_{32} \end{bmatrix} \begin{pmatrix} \Delta x_{1t-1} \\ \Delta x_{2t-1} \end{pmatrix} + O_p(T^{-1}) \begin{bmatrix} \Delta x_{3t-1} \\ \Delta x_{3t-1} \\ \Delta x_{3t-1} \end{bmatrix},$$

giving the following expression for $\hat{v}_{ht}$

$$\hat{v}_{ht} = u_{ht}^* + \hat{G}_h u_{ht-1} + O_p(T^{-1/2}), \tag{16}$$

where $\hat{G}_h$ is defined as

$$\hat{G}_h' := \begin{pmatrix} \hat{L}_1' & \hat{M}_1' & \hat{N}_1' \\ \hat{L}_2' & \hat{M}_2' & \hat{N}_2' \end{pmatrix}. \tag{17}$$

Next, we define our FM-OLS of the coefficient matrix A in the I(2) cointegrated system given in (1) as

$$\hat{A}^+ = (Y^{+\prime}X - T\hat{\Delta}^+)(X'X)^{-1}, \tag{18}$$

where

$$y_t^+ = y_t - \hat{\Omega}_{\hat{u}_0\hat{v}}\hat{\Omega}_{\hat{v}\hat{v}}^{-1}\hat{v}_t, \tag{19}$$

and

$$\hat{\Delta}^+ = \hat{\Delta}_{\hat{u}_0\Delta x} - \hat{\Omega}_{\hat{u}_0\hat{v}}\hat{\Omega}_{\hat{v}\hat{v}}^{-1}\hat{\Delta}_{\hat{v}\Delta x}. \tag{20}$$

Because the correction terms provided in (19) and (20) are constructed using the residual $\hat{v}_t$, we call the estimator defined in (18) a *residual-based fully modified ordinary least-squares* (RBFM-OLS) estimator. Notice that we do not need any information on H for the construction of the estimator $\hat{A}^+$.

We denote the long-run and one-sided long-run covariance matrices between the errors u_{0t} and v_{ht}, the H-transformed v_t, as Ω_{0v_h} and Δ_{0v_h} and define their kernel estimates as in (6) and (7) in terms of the sample autocovariances. We similarly denote the long-run variance and one-sided long-run variance matrices of v_{ht} as $\Omega_{v_h v_h}$ and $\Delta_{v_h v_h}$ and define their kernel estimates in the same way as (6) and (7). We use the following notation to represent kernel estimates of these long-run covariances that are defined by the sample autocovariances associated with the lagged variable $u_{ht-1} = (\Delta u'_{1t-1}, u'_{2t-1})'$:

$$ {}^{+}\hat{\Omega} = \sum_{j=-T+1}^{T-1} w(j/K)\hat{\Gamma}(j+1), \tag{21} $$

$$ {}^{-}\hat{\Omega} = \sum_{j=-T+1}^{T-1} w(j/K)\hat{\Gamma}(j-1), \tag{22} $$

and

$$ {}^{-}\hat{\Delta} = \sum_{j=0}^{T-1} w(j/K)\hat{\Gamma}(j-1), \tag{23} $$

where the sample autocovariances are constructed from the first-stage OLS residuals $\hat{u}_{0t}$ and the residuals $\hat{v}_{ht}$ from (16). Define $\hat{\zeta}_t = (\hat{u}'_{0t}, \hat{v}'_{ht})'$. Then, the sample autocovariances in the preceding equations are given by

$$ \hat{\Gamma}(j+1) = T^{-1} \sum{}' \hat{\zeta}_t \hat{\zeta}'_{t-(j+1)}, $$
$$ \hat{\Gamma}(j-1) = T^{-1} \sum{}' \hat{\zeta}_t \hat{\zeta}'_{t-(j-1)}, $$

where $\sum'$ signifies summation over $1 \le t$, $t - j \mp 1 \le T$. As can be seen from the definition of v_{ht} given in (13), these estimates involve components such as $\hat{\Omega}_{u_0 u_h^*}$, ${}^{+}\hat{\Omega}_{u_0 u_h}$, $\hat{\Omega}_{u_h^* u_h^*}$, ${}^{-}\hat{\Omega}_{u_h u_h^*}$, and ${}^{+}\hat{\Omega}_{u_0 u_h^*}$, as well as the components $\hat{\Omega}_{u_0 u_h}$ and $\hat{\Omega}_{u_h u_h}$ that appear in the analysis of cointegrated I(1) systems in Phillips (1995). The asymptotics for these kernel estimates can be constructed as in Lemma 8.1 of Phillips (1995) with some modifications regarding ${}^{+}\hat{\Omega}$, ${}^{-}\hat{\Omega}$, and ${}^{-}\hat{\Delta}$, and it turns out that Assumption KL in the aforementioned paper is sufficient to deal with the degeneracies in the submatrices of the component long-run covariances constituting $\Omega_{v_h v_h}$ corresponding to the I(-1) difference Δu_{1t}. As already outlined in Assumptions 1 and 2, the error conditions and the kernel conditions that we need for the asymptotics here are exactly the same as those given in Phillips (1995). All the asymptotic results regarding the estimation of these long-run covariance matrices that we need for our subsequent theory are collected together in Appendices A and B. They provide a set of important subsidiary results for out main development here.

We now present the limit theory for the RBFM-OLS estimator $\hat{A}^+$ defined in (18) when $H = (H_1, H_2, H_3)$, that is, when our model allows for integrated processes of orders up to 2 and the possibility of cointegration among the regressors.

THEOREM 2. *When* $H = (H_1, H_2, H_3)$, *the following limit theory for* $\hat{A}^+$ *holds under Assumptions* 1 *and* 2:

(a) $\sqrt{T}(\hat{A}^+ - A)H_1 \xrightarrow{\mathfrak{D}} \mathcal{N}(0, (I \otimes \Sigma_{11}^{-1})\Omega_{\varphi\varphi}(I \otimes \Sigma_{11}^{-1}))$,
(b) $(\hat{A}^+ - A)H_b D_T \xrightarrow{\mathfrak{D}} \int_0^1 dB_{0\cdot b}\bar{B}_b'(\int_0^1 \bar{B}_b \bar{B}_b')^{-1} \equiv \mathcal{MN}(0, \Omega_{00\cdot b} \otimes (\int_0^1 \bar{B}_b \bar{B}_b')^{-1})$.

Part (a) *holds for the bandwidth parameter expansion rate* BW(a); *that is,* $K = O_e(T^k)$ *for* $k \in (\frac{1}{4}, \frac{1}{2})$. *Part* (b) *holds under* BW(c); *that is, for* $k \in (0, \frac{1}{2})$. *Thus, parts* (a) *and* (b) *both hold under* BW(a).

Remark. Note that, by construction, the RBFM correction terms have no effect on the limit distribution of the stationary component coefficient submatrix for bandwidth expansion rates $k \in (\frac{1}{4}, \frac{1}{2})$. Consequently, the limit theory of the RBFM-OLS $\hat{A}^+$ in the stationary direction is normal and asymptotically equivalent to that of the OLS estimator for $k \in (\frac{1}{4}, \frac{1}{2})$. The limit distribution of the RBFM-OLS estimator for the nonstationary regressors is mixed normal for $k \in (0, \frac{1}{2})$ and, thus, is free from the endogeneity and serial correlation problems, appearing in the limit distribution of the OLS estimator given in Propositions 1(b) and (8). One important consequence of this property is that we can conduct hypothesis testing based on asymptotic chi-squared and mixed chi-squared tests using classical procedures such as the Wald test, as shown in Remark 4.4(f)–(i) and Theorem 4.5 in Phillips (1995). Hence, Theorem 2 shows that the limit theory of an I(1) cointegrated system given in Phillips (1995) continues to hold for our I(2) cointegrated system, but now a much wider range of cointegration type is permitted.

The following corollaries give results for some specific versions of model (1). The model configurations can conveniently be specified in terms of the rotation matrix H, which prescribes the stationary/nonstationary characteristics of the regressors and the type of cointegration involved. For example, when $H = (H_1, H_2)$, the model is a cointegrated I(1) system covering cointegration of the form CI(1,1). The limit theory for this case is given in the next corollary.

COROLLARY 3. *When* $H = (H_1, H_2)$, *the following limit theory for* $\hat{A}^+$ *holds under Assumptions* 1 *and* 2:

(a) $\sqrt{T}(\hat{A}^+ - A)H_1 \xrightarrow{\mathfrak{D}} \mathcal{N}(0, (I \otimes \Sigma_{11}^{-1})\Omega_{\varphi\varphi}(I \otimes \Sigma_{11}^{-1}))$,
(b) $T(\hat{A}^+ - A)H_2 \xrightarrow{\mathfrak{D}} \int_0^1 dB_{0\cdot 2}B_2'(\int_0^1 B_2 B_2')^{-1} \equiv \mathcal{MN}(0, \Omega_{00\cdot 2} \otimes (\int_0^1 B_2 B_2')^{-1})$,

where $B_{0\cdot 2} = B_0 - \Omega_{02}\Omega_{22}^{-1}B_2 \equiv BM(\Omega_{00\cdot 2})$ *and* $\Omega_{00\cdot 2} = \Omega_{00} - \Omega_{02}\Omega_{22}^{-1}\Omega_{20}$. *Part* (a) *holds with the bandwidth parameter expansion rate* $K = O_e(T^k)$ *for* $k \in (\frac{1}{4}, \frac{1}{2})$. *Part* (b) *holds for* $k \in (0, \frac{2}{3})$. *Both parts* (a) *and* (b) *hold when* $k \in (\frac{1}{4}, \frac{1}{2})$.

Remark. Corollary 3 covers the case that is considered in Theorem 4.1 of Phillips (1995). Our theory gives exactly the same results except that the range of the allowable bandwidth expansion rate for part (a) is a little narrower. In Phillips (1995), the Gaussian limit theory for the stationary coefficient is valid for $k \in (\frac{1}{4},1)$, and the required bandwidth rate for the I(1) coefficient to be mixed normal is $k \in (0,\frac{2}{3})$. Hence, roughly speaking, we may say that we are giving up the range of rates $k \in (\frac{1}{2},\frac{2}{3})$ by introducing the additional generality of I(2) regressors into the model.

COROLLARY 4. *When* $H = (H_1, H_3)$, *the following limit theory for* $\hat{A}^+$ *holds under Assumptions* 1 *and* 2:

(a) $\sqrt{T}(\hat{A}^+ - A)H_1 \xrightarrow{\mathfrak{D}} \mathcal{N}(0,(I \otimes \Sigma_{11}^{-1})\Omega_{\varphi\varphi}(I \otimes \Sigma_{11}^{-1}))$,

(b) $T^2(\hat{A}^+ - A)H_3 \xrightarrow{\mathfrak{D}} \int_0^1 dB_{0\cdot 3}\bar{B}_3'(\int_0^1 \bar{B}_3\bar{B}_3')^{-1} \equiv \mathcal{MN}(0,\Omega_{00\cdot 3} \otimes (\int_0^1 \bar{B}_3\bar{B}_3')^{-1})$,

where $B_{0\cdot 3} = B_0 - \Omega_{03}\Omega_{33}^{-1}B_3 \equiv BM(\Omega_{00\cdot 3})$ *and* $\Omega_{00\cdot 3} = \Omega_{00} - \Omega_{03}\Omega_{33}^{-1}\Omega_{30}$. *Part* (a) *holds with the bandwidth parameter expansion rate* $K = O_e(T^k)$ *for* $k \in (\frac{1}{4},\frac{1}{2})$. *Part* (b) *holds for* $k \in (0,\frac{1}{2})$. *Both parts* (a) *and* (b) *hold for* $k \in (\frac{1}{4},\frac{1}{2})$.

Remark. Corollary 4 covers cases where all of the nonstationary variables are I(2) and some linear combinations of the I(2) variables are cointegrated and become stationary, that is, a CI(2,2) case. There may well be I(1) variables, but they are all cointegrated and absorbed into the stationary component. Chang and Phillips (1994) showed that the FM correction terms that are based on the second difference of the regressor set $\Delta^2 x_t$ do work completely for models with an unknown mixture of I(0) and I(2) regressors. The limit theories developed in the aforementioned paper are equivalent to those given earlier except for the allowable bandwidth expansion rates—the required ranges of k for parts (a) and (b) to hold are $k \in (\frac{1}{8},1)$ and $k \in (0,1)$, respectively.

COROLLARY 5 (Stationary Case). *When* $H = H_1$, *under Assumptions* 1 *and* 2, *we have*

$$\sqrt{T}(\hat{A}^+ - A) \xrightarrow{\mathfrak{D}} \mathcal{N}(0,(I \otimes \Sigma_{11}^{-1})\Omega_{\varphi\varphi}(I \otimes \Sigma_{11}^{-1})),$$

with the bandwidth expansion rate $K = O_e(T^k)$ *for* $k \in (\frac{1}{4},1)$.

The following corollaries cover cases where the possibility of cointegration among regressors is excluded. The regressor x_t may be full rank I(1) or full rank I(2) processes. Of course, it may well be a mixture of full rank I(1) and I(2) processes.

COROLLARY 6 (Full Rank I(1) Case). *When* $H = H_2$, *and under Assumptions* 1 *and* 2 *and* BW(d), *that is,* $K = O_e(T^k)$ *for* $k \in (0,1)$, *the following holds*:

$$T(\hat{A}^{+} - A) \xrightarrow{\mathfrak{D}} \int_0^1 dB_{0\cdot 2} B_2' \left(\int_0^1 B_2 B_2' \right)^{-1}$$

$$\equiv \mathcal{MN}\left(0, \Omega_{00\cdot 2} \otimes \left(\int_0^1 B_2 B_2' \right)^{-1} \right).$$

Remark. The same result was given in Phillips and Hansen (1990), where the fully modified regression method originated. Many other researchers have investigated this case, and several methods are now available in the literature, such as the spectral cointegrating regression method of Phillips (1991b), the canonical cointegrating regression by Park (1992), the modified regression method by Saikkonen (1992), the dynamic OLS and GLS by Stock and Watson (1993), the full system maximum likelihood estimation by Phillips (1991a), and the nonlinear least-squares method of Phillips and Loretan (1991). These estimators are asymptotically equivalent to the full-system maximum likelihood estimators and optimal in the sense that they are median unbiased in the limit and that their variance matrices are equivalent to that of the optimal estimator under Gaussian assumptions (for further discussion on this point, see Phillips, 1991a).

COROLLARY 7 (Full Rank I(2) Case). *When* $H = H_3$, *and under Assumptions* 1 *and* 2 *and* BW(c), *that is*, $K = O_e(T^k)$ *for* $k \in (0, \frac{1}{2})$, *the following holds*:

$$T^2(\hat{A}^{+} - A) \xrightarrow{\mathfrak{D}} \int_0^1 dB_{0\cdot 3} \bar{B}_3' \left(\int_0^1 \bar{B}_3 \bar{B}_3' \right)^{-1}$$

$$\equiv \mathcal{MN}\left(0, \Omega_{00\cdot 3} \otimes \left(\int_0^1 \bar{B}_3 \bar{B}_3' \right)^{-1} \right).$$

Remark. Chang (1993) considered the FM estimation in full rank I(2) regressions and provided the same limit theory as given earlier. That paper also addresses the issue of possible misspecification in using the original FM-OLS regressors when, in fact, the regressors are I(2) and shows that the limit theory for such a misspecified FM-OLS estimator is nonstandard and depends on nonscale nuisance parameters. The simulation study in that paper compares the finite sample properties of the OLS, misspecified FM-OLS, and the correctly specified FM-I(2) estimators and shows that, overall, the estimated distribution of the correctly specified FM-I(2) is better centered and suffers much less from asymmetry than those of the OLS and misspecified FM-OLS. This simulation confirms the asymptotic theory in Chang (1993) that the FM estimation procedure is beneficial only when it is conducted with a correctly specified model; otherwise, it just adds more bias, dispersion, and asymmetry to the distribution.

COROLLARY 8 (Full Rank I(1)/I(2) Case). *When* $H = (H_2, H_3) = H_b$, *and under Assumptions* 1 *and* 2 *and* BW(c), *that is*, $K = O_e(T^k)$ *for* $k \in (0, \frac{1}{2})$, *the following holds*:

$$(\hat{A}^+ + A)D_T \xrightarrow{\mathfrak{D}} \int_0^1 dB_{0\cdot b}\bar{B}_b'\left(\int_0^1 \bar{B}_b\bar{B}_b'\right)^{-1}$$

$$\equiv \mathcal{MN}\left(0, \Omega_{00\cdot b} \otimes \left(\int_0^1 \bar{B}_b\bar{B}_b'\right)^{-1}\right).$$

Remark. With prior information about the configuration of the cointegration space and the order of integration of the regressors, Chang (1993) derived the limit theory for the FM-OLS estimator for this case. Kitamura (1995) obtained the same limit theory by employing the full-system maximum likelihood estimator procedure suggested in Phillips (1991a), but that procedure also presumes knowledge of the order of cointegration and the number of unit roots and double unit roots in the model.

5. CONCLUSION

Motivated by earlier work in Phillips (1995) that applies the FM-OLS principle to possibly cointegrated I(1) models, this paper proposes an RBFM-OLS procedure. The framework of the paper enables us to study the asymptotic behavior of the RBFM-OLS estimator in a general class of time series models, which allows a wider than usual range of cointegration and covers the CI(1,1), CI(2,2), and CI(2,1) systems in Granger's notation, thereby including I(2) regressors in the model. The theory shows that the RBFM-OLS estimator is consistent and optimal for the nonstationary coefficients under Gaussian assumptions and is robust to the specification of the nonstationary characteristics of the regressors and the precise configurations of cointegration space. In particular, it is shown that the limit theory of the RBFM-OLS estimator in the stationary direction is normal and is asymptotically equivalent to that of the unrestricted OLS estimator and that the RBFM-OLS estimator for the nonstationary coefficients has a nuisance parameter-free mixed normal limit distribution. Moreover, optimal estimation of the cointegration space is attained under Gaussian assumptions without prior information about the presence or number of unit roots and/or double unit roots and cointegrating relations. As discussed in Phillips (1995, pp. 19–20), this property in turn simplifies inference for the RBFM regression estimates. In particular, we can conduct hypothesis testing using Wald statistics that have limit distributions that are linear combinations of independent chi-squared variates.

The next step in this approach is to apply the RBFM-OLS regression in the context of VAR's that include unknown numbers of I(0), I(1), and I(2) variables and an unknown degree of cointegration. As in Phillips's analysis of I(1)

VAR's, this extension should lead to an RBFM-VAR procedure that will offer many of the advantages of unrestricted levels VAR while allowing for varying degrees of nonstationarity and cointegration. These extensions are presently under investigation and will be reported in later work.

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APPENDIX A: USEFUL LEMMAS AND PROOFS

For convenience, we assume that the kernel functions $w(\cdot)$, used in forming long-run covariance matrix estimates, satisfy the explicit kernel conditions given in Assumptions 2(a) and 2(b). Note that the kernel function $w(x)$ is truncated for $|x| > 1$ in Assumption 2(b), and this leads to the truncation of the sums in the kernel estimates given in (6) and (7), namely,

$$\hat{\Omega} = \sum_{j=-K+1}^{K-1} w(j/K)\hat{\Gamma}(j), \qquad \hat{\Delta} = \sum_{j=0}^{K-1} w(j/K)\hat{\Gamma}(j).$$

The same truncation is applied to ${}^{+}\hat{\Omega}$, ${}^{-}\hat{\Omega}$, and ${}^{-}\hat{\Delta}$ defined in (21) and (22). Therefore, the proofs given in this section hold for the Parzen and Tukey–Hanning kernels, as these kernels satisfy the truncation conditions. Phillips (1995) showed that his limit theory holds for untruncated kernels that satisfy Assumptions 2(a) and 2(b′) and illustrated the modifications to the proofs that are needed to deal with the untruncated sums in kernel estimates (6) and (7). The same modifications apply here to achieve extension of our results to untruncated kernels and, to save space, will not be detailed.

LEMMA 1. *Under Assumptions* 1 *and* 2 *and* BW(d) *of Assumption* 3, *the following hold*:

$$\text{(a)} \sum_{j=K_{L_\Omega}}^{K^U} \Delta w(l(j)/K)\hat{\Gamma}(j) = K^{-2}w''(0)\sum_{j=-\infty}^{\infty}(l(j)-1/2)\Gamma(j) + O_p(1/\sqrt{TK}),$$

$$\text{(b)} \sum_{j=K_{L_\Omega}}^{K^U} \Delta^2 w(l(j)/K)\hat{\Gamma}(j) = K^{-2}w''(0)\Omega + O_p(1/\sqrt{TK^3}),$$

$$\text{(c)} \sum_{j=K_{L_\Delta}}^{K^U} \Delta w(l(j)/K)\hat{\Gamma}(j) = K^{-2}w''(0)\sum_{j=K_{L_\Delta}}^{\infty}(l(j)-1/2)\Gamma(j) + O_p(1/\sqrt{TK}),$$

$$\text{(d)} \sum_{j=K_{L_\Delta}}^{K^U} \Delta^2 w(l(j)/K)\hat{\Gamma}(j) = K^{-2}w''(0)\sum_{j=K_{L_\Delta}}^{\infty}\Gamma(j) + O_p(1/\sqrt{TK^3}),$$

where $K^U \in \{K-1, K-2, K-3\}$, $K_{L\Omega} \in \{-K+1, -K+2, -K+3\}$, $K_{L_\Delta} \in \{-1,0,1\}$, *and* $l(j) = j$, $j \pm 1$ *or* $j \pm 2$. *The error terms of* $O_p(1/\sqrt{TK})$ *that appear in parts* (a) *and* (c) *are sharp. The same applies to the terms of* $O_p(1/\sqrt{TK^3})$ *that appear in parts* (b) *and* (d).

LEMMA 2. *Under Assumptions* 1 *and* 2 *and* BW(d) *of Assumption* 3, *the following hold*:

(a) $\hat{\Omega}_{\Delta u_1 \Delta u_1} = -K^{-2}w''(0)\Omega_{11} + O_p(1/\sqrt{TK^3}) + o_p(K^{-2})$;
(b) ${}^{+}\hat{\Omega}_{\Delta u_1 \Delta u_1} = -K^{-2}w''(0)\Omega_{11} + O_p(1/\sqrt{TK^3}) + o_p(K^{-2})$;
(c) ${}^{-}\hat{\Omega}_{\Delta u_1 \Delta u_1} = -K^{-2}w''(0)\Omega_{11} + O_p(1/\sqrt{TK^3}) + o_p(K^{-2})$;
(d) $\hat{\Omega}_{u_0 \Delta u_1} = K^{-2}w''(0)\Phi_{01} + O_p(1/\sqrt{TK}) + o_p(K^{-2})$,
${}^{+}\hat{\Omega}_{u_0 \Delta u_1} = K^{-2}w''(0)\Phi_{01}^{+} + O_p(1/\sqrt{TK}) + o_p(K^{-2})$;
(e) $\hat{\Omega}_{u_b \Delta u_1} = K^{-2}w''(0)\Phi_{b1} + O_p(1/\sqrt{TK}) + o_p(K^{-2})$,
$\hat{\Omega}_{\Delta u_1 u_b} = -K^{-2}w''(0)\Phi_{1b} + O_p(1/\sqrt{TK}) + o_p(K^{-2})$;
(f) ${}^{+}\hat{\Omega}_{u_b \Delta u_1} = K^{-2}w''(0)\Phi_{b1}^{+} + O_p(1/\sqrt{TK}) + o_p(K^{-2})$;
(g) ${}^{-}\hat{\Omega}_{\Delta u_1 u_b} = -K^{-2}w''(0)\Phi_{1b}^{-} + O_p(1/\sqrt{TK}) + o_p(K^{-2})$;
(h) ${}^{+}\hat{\Omega}_{\Delta u_1 u_2} = -K^{-2}w''(0)\Phi_{12}^{+} + O_p(1/\sqrt{TK}) + o_p(K^{-2})$;
(i) ${}^{-}\hat{\Omega}_{u_2 \Delta u_1} = -K^{-2}w''(0)\Phi_{21}^{-} + O_p(1/\sqrt{TK}) + o_p(K^{-2})$,
${}^{-}\hat{\Omega}_{u_2 \Delta u_b} = -K^{-2}w''(0)\Phi_{2b}^{-} + O_p(1/\sqrt{TK}) + o_p(K^{-2})$;
(j) ${}^{+}\hat{\Omega}_{u_0 u_2} := \hat{\Omega}_{u_0 u_2} - \hat{\Omega}_{u_0 \Delta u_2} = \hat{\Omega}_{u_0 u_2} - K^{-2}w''(0)\Phi_{02} + O_p(1/\sqrt{TK})$;
(k) ${}^{+}\hat{\Omega}_{u_b u_2} := \hat{\Omega}_{u_b u_2} - \hat{\Omega}_{u_b \Delta u_2} = \hat{\Omega}_{u_b u_2} - K^{-2}w''(0)\Phi_{b2} + O_p(1/\sqrt{TK})$;
(l) ${}^{-}\hat{\Omega}_{u_2 u_b} := \hat{\Omega}_{u_2 u_b} - {}^{-}\hat{\Omega}_{u_2 \Delta u_b} = \hat{\Omega}_{u_2 u_b} + K^{-2}w''(0)\Phi_{2b}^{-} + O_p(1/\sqrt{TK})$;

where $\Phi_{i1} = \sum_{j=-\infty}^{\infty}(j-\frac{1}{2})\Gamma_{u_i u_1}(j)$, $\Phi_{i1}^{+} = \sum_{j=-\infty}^{\infty}(j-\frac{3}{2})\Gamma_{u_i u_1}(j)$, $\Phi_{1b} = \sum_{j=-\infty}^{\infty}(j+\frac{1}{2})\Gamma_{u_1 u_b}(j)$, $\Phi_{1b}^{-} = \sum_{j=-\infty}^{\infty}(j+\frac{3}{2})\Gamma_{u_1 u_b}(j)$, $\Phi_{12}^{+} = \sum_{j=-\infty}^{\infty}(j-\frac{1}{2})\Gamma_{u_1 u_2}(j)$, $\Phi_{2i}^{-} = \sum_{j=-\infty}^{\infty}(j+\frac{1}{2})\Gamma_{u_2 u_i}(j)$, *and* $\Phi_{i2} = \sum_{j=-\infty}^{\infty}(j+\frac{1}{2})\Gamma_{u_i u_2}(j)$ *for* $i = 0, b$. *The error terms of* $O_p(1/\sqrt{TK^3})$ *that appear in parts* (a)–(c) *are sharp. The same applies to the terms of* $O_p(1/\sqrt{TK})$ *that appear in parts* (d)–(l).

LEMMA 3. *Under Assumptions* 1 *and* 2 *and* BW(d) *of Assumption* 3, *the following hold*:

(a) $\hat{\Omega}_{u_0 v_h} = \hat{\Omega}_{u_0 u_b}G' + O_p(K^{-2}) + O_p(1/\sqrt{TK})$,
(b) $\hat{\Omega}_{v_h v_h} = G\hat{\Omega}_{u_b u_b}G' + O_p(K^{-2}) + O_p(1/\sqrt{TK})$,
(c) $\hat{\Omega}_{u_0 v_h}\hat{\Omega}_{v_h v_h}^{-1} = (\Omega_{0b}\Omega_{bb}^{-1}(G'G)^{-1/2} + o_p(1), O_p(1) + O_p(K^{3/2}T^{-1/2}))C'$,

(d) $\hat{\Omega}_{\hat{v}_h\hat{v}_h} = \hat{\Omega}_{v_h v_h} + O_p(K^{-2}) + O_p(TK^{-1/2})$,
(e) $\hat{\Omega}_{\hat{u}_0\hat{v}_h} = \hat{\Omega}_{u_0 v_h} + O_p(K^{-2}) + O_p(TK^{-1/2})$,

where

$$G' = \begin{pmatrix} L_2' & (I+M_2)' & N_2' \\ 0 & 0 & I \end{pmatrix}, \qquad C = (\underline{G}, \underline{G}_\perp) \in O(m) \quad \text{with } \underline{G} = G(G'G)^{-1/2}.$$

The error terms of $O_p(1/\sqrt{TK})$ in parts (a) *and* (b) *and of $O_p(K^{3/2}T^{-1/2})$ in part* (c) *are sharp. The same applies to the terms of $O_p(KT^{-1/2})$ that appear in parts* (d) *and* (e).

LEMMA 4. *Under Assumptions* 1 *and* 2 *and* BW(d) *of Assumption* 3, *the following hold*:

(a) $\hat{\Delta}_{\hat{u}_0\Delta u_1} = O_p(1/\sqrt{TK})$,
(b) $\hat{\Delta}_{\hat{u}_0 u_2} = \Delta_{02} + O_p(K/T)^{1/2}$,
(c) $\hat{\Delta}_{\hat{u}_0\Delta x_3} = O_p(K)$,
(d) $\hat{\Delta}_{u_b\Delta x_3} = O_p(K)$,
(e) $T^{-1}\Delta U_1' U_1 - \hat{\Delta}_{\Delta u_1\Delta u_1} = K^{-2}w''(0)\Psi_{11} + O_p(1/\sqrt{TK^3}) + o_p(K^{-2})$,
(f) $T^{-1}U_b' U_1 - \hat{\Delta}_{u_b\Delta u_1} = -K^{-2}w''(0)\Psi_{b1} + O_p(1/\sqrt{TK}) + o_p(K^{-2})$,
(g) $T^{-1}\Delta U_1' X_2 - \hat{\Delta}_{\Delta u_1 u_2} = K^{-2}w''(0)\Psi_{12} + O_p(T^{-1/2}) + o_p(K^{-2})$,
(h) $T^{-1}U_b' X_2 - \hat{\Delta}_{u_b u_2} := N_{b2T} \xrightarrow{\mathfrak{D}} \int_0^1 dB_b B_2'$,
(i) $T^{-2}\Delta U_1' X_3 - T^{-1}\hat{\Delta}_{\Delta u_1\Delta x_3} = O_p(T^{-1/2})$,
(j) $T^{-2}U_b' X_3 - T^{-1}\hat{\Delta}_{u_b\Delta x_3} = \bar{N}_{b3T} + O_p(K/T) \xrightarrow{\mathfrak{D}} \int_0^1 dB_b \bar{B}_3$,

where $\Psi_{11} = \{\Delta_{11} - (\frac{1}{2})\Sigma_{11}\}$, $\Psi_{b1} = \sum_{j=1}^{\infty}(j - \frac{1}{2})\Gamma_{u_b u_1}(j)$, $\Psi_{12} = \sum_{j=0}^{\infty}(j + \frac{1}{2}) \times \Gamma_{u_1 u_2}(j)$, *and* $\bar{N}_{b3T} \xrightarrow{\mathfrak{D}} \int_0^1 B_b \bar{B}_3'$. *The error terms of $O_p(1/\sqrt{TK})$ and $O_p(1/\sqrt{TK^3})$ that appear in parts* (a), (f)–(g), *and* (e) *are sharp. The same applies to the terms of $O_p(K/T)$ that appear in parts* (b) *and* (j).

LEMMA 5. *Under Assumptions* 1 *and* 2 *and* BW(c) *of Assumption* 3 *the following hold*:

(a) $T^{-1}\Delta U_{1-1}' U_1 - {}^{-}\hat{\Delta}_{\Delta u_1\Delta u_1} = K^{-2}w''(0)\Psi_{11}^{-} + O_p(T^{-1}) + O_p(1/\sqrt{TK^3})$,
(b) $T^{-1}U_{2-1}' U_1 - {}^{-}\hat{\Delta}_{u_2\Delta u_1} = -K^{-2}w''(0)\Psi_{21}^{-} + O_p(1/\sqrt{TK}) + o_p(K^{-2})$,
(c) $T^{-1}\Delta U_{1-1}' X_2 - {}^{-}\hat{\Delta}_{\Delta u_1 u_2} = K^{-2}w''(0)\Psi_{12}^{-} + O_p(1/\sqrt{TK}) + o_p(K^{-2})$,
(d) $T^{-1}U_{2-1}' X_2 - {}^{-}\hat{\Delta}_{u_2 u_2} = N_{22T} + O_p(K^{-2}) + O_p(T^{-1/2})$,
(e) $T^{-2}\Delta U_{1-1}' X_3 - T^{-1}{}^{-}\hat{\Delta}_{\Delta u_1\Delta x_3} = O_p(T^{-1/2})$,
(f) $T^{-2}U_{2-1}' X_3 - T^{-1}{}^{-}\hat{\Delta}_{u_2\Delta x_3} = \bar{N}_{23T} + O_p(K/T) + O_p(T^{-1/2})$,

where $\Psi_{11}^{-} = \{\Delta_{11} + (\frac{1}{2})\Gamma_{u_1 u_1}(-1)\}$, $\Psi_{21}^{-} = \sum_{j=1}^{\infty}(j + \frac{1}{2})\Gamma_{u_2 u_1}(j)$, $\Psi_{12}^{-} = \sum_{j=1}^{\infty}(j + \frac{3}{2}) \times \Gamma_{u_1 u_2}(j)$, *and* $\bar{N}_{23T} \xrightarrow{\mathfrak{D}} \int_0^1 dB_2 \bar{B}_3'$.

LEMMA 6. *Under Assumptions* 1 *and* 2 *and* BW(d) *of Assumption* 3, *the following hold*:

(a) $T^{-1}\hat{V}_h' U_1 - \hat{\Delta}_{\hat{v}_h\Delta u_1} = O_p(K^{-2}) + O_p(1/\sqrt{TK})$,
(b) $(\hat{V}_h' X_b - T\hat{\Delta}_{\hat{v}_h\Delta x_b})D_T^{-1} = G\bar{N}_{bbT} + O_p(K/T) + O_p(K^{-2}) + O_p(T^{-1/2})$,

(c) $T^{-1}\hat{\Omega}_{u_0 v_h}\hat{\Omega}_{v_h v_h}^{-1}(\hat{V}_h' U_1 - T\hat{\Delta}_{\hat{v}_h \Delta u_1}) = O_p(T^{1/2}K^{-2}) + O_p(KT^{-1/2}) + O_p(K^{-1/2})$,

(d) $\hat{\Omega}_{u_0 v_h}\hat{\Omega}_{v_h v_h}^{-1}(\hat{V}_h' X_b - T\hat{\Delta}_{\hat{v}_h \Delta x_b})D_T^{-1} = \Omega_{0b}\Omega_{bb}^{-1}\bar{N}_{bbT} + O_p(K^{5/2}T^{-3/2}) + O_p(K^{3/2}/T) + o_p(1)$,

where $\bar{N}_{bbT} \xrightarrow{\mathfrak{D}} \int_0^1 dB_b \bar{B}_b'$. *The error terms of* $O_p(T^{1/2}K^{-2})$, $O_p(K^{5/2}T^{-3/2})$, *and* $O_p(K^{3/2}/T)$ *that appear in parts* (c) *and* (d) *are sharp.*

The proofs given here follow closely the proofs for Lemma 8.1 in Phillips (1995). Before we proceed, we recall some of the facts used to establish the results in that paper. Under the summability condition given in Assumption 1(a), we have

$$\sum_{j=0}^{\infty} j^{\alpha}\|\Gamma(j)\| < \infty, \tag{A.1}$$

which implies

$$\Gamma(K) = E(u_t u_{t-K}') = o(K^{-\alpha}), \tag{A.2}$$

as $K \to \infty$. As shown, for example, in Hannan (1970, p. 212), we also have

$$\operatorname{var}(\hat{\Gamma}(K)) = O(T^{-1}), \tag{A.3}$$

which together with the result in (A.2) gives the following order of magnitude for the sample autocovariances:

$$\hat{\Gamma}(K) = O_p(T^{-1/2}) + o(K^{-\alpha}). \tag{A.4}$$

Similarly, the same result applies to $\hat{\Gamma}(K-1)$, $\hat{\Gamma}(-K)$. At several points in later proofs, we also encounter terms such as $w((K-1)/K)\hat{\Gamma}(K)$, which appear as remainder terms in Taylor expansions of kernel estimates defined with sample autocovariances involving I(-1) processes like Δu_{1t}. For these expressions, we use Assumption 2(b) and the result in (A.4) to write

$$\begin{aligned} w((K-1)/K)\hat{\Gamma}(K) &= w(1-(1/K))\hat{\Gamma}(K) \\ &= O(K^{-2})\{O_p(T^{-1/2}) + o(K^{-\alpha})\} \\ &= o_p(K^{-2}), \end{aligned} \tag{A.5}$$

because $K = O_e(T^k)$ with $k \in (0,1)$.

There are a few other related terms that appear in our Taylor expansions when we express weighted sums of differences of sample autocovariances in terms of weighted sums of sample autocovariances with weights that involve differences of the kernel function $w(\cdot)$. As will become clear, especially in the proofs for Lemmas 2, 4, and 5, weighted sums involving the first differences of the kernel $\Delta w(\cdot)$ are used to approximate the kernel estimates that involve one I(-1) process, and weighted sums involving the second differences $\Delta^2 w(\cdot)$ are used to approximate kernel estimates defined with two I(-1) processes.

Now we proceed to prove the results in Lemma 1, where we provide approximation statements for such kernel estimates defined with differences of the kernel functions.

Proof of Lemma 1.

(a) We consider the case with $l(j) = j + 1$ for convenience. We start by writing

$$\sum_{j=K_{L_\Omega}}^{K^U} \Delta w((j+1)/K)\hat{\Gamma}(j) = \{\Sigma_{\mathcal{B}_*} + \Sigma_{\mathcal{B}^*}\}\Delta w((j+1)/K)\hat{\Gamma}(j), \quad \textbf{(A.6)}$$

where $\mathcal{B}_* = \{j : |j| \le K^*\}$ and $\mathcal{B}^* = \{j : |j| > K^*, K_{L_\Omega} \le j \le K^U\}$ for some $K^* = K^\lambda$ with $\lambda \in (0,1)$. Under Assumption 2, we can use the following approximation by the Taylor expansion for $\Delta w((j+1)/K)$ when $|j| \le K^*$ and $K \to \infty$ as

$$\begin{aligned}
\Delta w((j+1)/K) &= w((j+1)/K) - w(j/K) \\
&= w'(j/K)(1/K) + (\tfrac{1}{2})w''(j/K)(1/K^2) + o(1/K^2) \\
&= \{w'(0) + w''(0)(j/K) + o(1/K)\}(1/K) \\
&\quad + (\tfrac{1}{2})\{w''(0) + o(1)\}(1/K^2) + o(1/K^2) \\
&= w''(0)(j/K^2) + (\tfrac{1}{2})w''(0)(1/K^2) + o(1/K^2) \\
&= K^{-2}w''(0)(j + \tfrac{1}{2})(1 + o(1)).
\end{aligned}$$

We then use the preceding results to rewrite the first sum in (A.6) as

$$\sum_{|j| \le K^*} \Delta w((j+1)/K)\hat{\Gamma}(j) = K^{-2}w''(0) \underbrace{\sum_{|j| \le K^*} (j + \tfrac{1}{2})\hat{\Gamma}(j)}(1 + o(1)).$$

The mean of the underbraced term is

$$\sum_{|j| \le K^*} (j + \tfrac{1}{2})E(\hat{\Gamma}(j)) = \sum_{|j| \le K^*} (j + \tfrac{1}{2})(1 - |j|/T)\Gamma(j),$$

which converges, as $K \to \infty$, to

$$\sum_{j=-\infty}^{\infty} (j + \tfrac{1}{2})\Gamma(j).$$

Hence, the mean of the first sum in (A.6) scaled by K^2 converges to

$$w''(0) \sum_{j=-\infty}^{\infty} (j + \tfrac{1}{2})\Gamma(j), \quad \textbf{(A.7)}$$

as $K \to \infty$. Now recall that the Taylor expansion used earlier can take the following alternate form:

$$\Delta w((j+1)/K) = K^{-1}w'(\theta_j),$$

for $j/K < \theta_j < (j+1)/K$. Then, we can write the second sum in (A.6) as
$\Sigma_{\mathcal{B}^*} \Delta w((j+1)/K)\hat{\Gamma}(j) = K^{-1} \sum_{|j|>K^*} w'(\theta_j)\hat{\Gamma}(j)$,

whose mean is given by

$$K^{-1} \sum_{|j|>K^*} w'(\theta_j)(1 - |j|/T)\Gamma(j).$$

Recall that $w'(\cdot)$ is uniformly bounded under Assumption 2. That is, $|w'(x)| \leq M_{w'}$, $\forall x \in \mathcal{R}$, for some constant $M_{w'}$. Then, the modulus of the preceding mean is dominated by

$$K^{-1}\left\{ \sup_{|j|<K} |w'(\theta_j)| \right\} \sum_{|j|>K^*} \|\Gamma(j)\| \leq K^{-1} M_{w'} \underbrace{\sum_{|j|>K^*} \sum_{s=0}^{\infty} \|C_s\| \, \|C_{s-j}\|}$$
$$= O(K^{-1-\alpha\lambda}).$$

For the last line, we use the result that the preceding underbraced term is $O(K^{-\alpha\lambda})$, which is given in Phillips (1995, p. 49). The preceding result then implies that the mean of the second sum in (A.6) is $o(K^{-2})$ as $K \to \infty$ for $\lambda \in (1/\alpha, 1)$ because $\alpha > 1$ from Assumption 1(a). Hence, so long as we choose $K^* = K^\lambda$ with such $\lambda \in (1/\alpha, 1)$, the mean of (A.6) times K^2 is dominated by that of the first sum, whose limit is given in (A.7).

Now we consider the variance matrix of (A.6). First, we expand $\Delta w((j+1)/K)$ as

$$\Delta w((j+1)/K) = K^{-1} w'(j/K)(1 + O(1/K))$$

because $w''(\cdot)$ is uniformly bounded under Assumption 2. Then we rewrite (A.6) as

$$K^{-1} \sum_{K_{L_\Omega}}^{K^U} w'(j/K)\hat{\Gamma}(j)(1 + O(1/K)). \qquad \textbf{(A.8)}$$

We can apply Theorem 9 of Hannan (1970, p. 280) to find the asymptotic variance matrix of (A.8) because its leading term has the same form as a spectral estimate at the origin and $w'(\cdot)$ is continuous and uniformly bounded under Assumption 2. As in Phillips (1995, p. 50), we can analyze the asymptotic variance of the dominant term of (A.8) in this way, as follows:

$$\lim_{T\to\infty} KT \operatorname{var}\left\{ \operatorname{vec}\left\{ K^{-1} \sum_{j=K_{L_\Omega}}^{K^U} w'(j/K)\hat{\Gamma}(j) \right\}\right\}$$
$$= \lim_{T\to\infty} KTK^{-2} \operatorname{var}\left\{ \operatorname{vec}\left\{ \sum_{j=K_{L_\Omega}}^{K^U} w'(j/K)\hat{\Gamma}(j) \right\}\right\}$$
$$= \lim_{T\to\infty} \frac{T}{K} \operatorname{var}\left\{ \operatorname{vec}\left\{ \sum_{j=K_{L_\Omega}}^{K^U} w'(j/K)\hat{\Gamma}(j) \right\}\right\}$$
$$= \text{constant}.$$

This expression implies that the variance of the dominant term in (A.8) is $O(1/(TK))$. We deduce from (A.6)-(A.8) and the preceding result that

$$\sum_{K_{L_\Omega}}^{K^U} \Delta w((j+1)/K)\hat{\Gamma}(j) = K^{-2}w''(0)\sum_{j=-\infty}^{\infty}(j+\tfrac{1}{2})\Gamma(j) + O_p(1/\sqrt{TK}).$$

The same analysis can be carried out for $l(j) = j, j-1$ and $j \pm 2$, by using

$$\Delta w(l(j)/K) = K^{-2}w''(0)((l(j)-1)+\tfrac{1}{2})(1+o(1))$$

and

$$\Delta w(l(j)/K) = K^{-1}w'(\theta_{l(j)-1})(1+O(1/K)),$$

for $(l(j)-1)/K < \theta_{l(j)-1} < l(j)/K$. These formulae then lead to the reduction

$$\sum_{j=K_{L_\Omega}}^{K^U} \Delta w(l(j)/K)\hat{\Gamma}(j) = \sum_{j=-\infty}^{\infty}(l(j)-\tfrac{1}{2})\Gamma(j) + O_p(1/\sqrt{TK}),$$

as required.

(b) Again we consider the case with $l(j) = j+1$ for convenience. We start by doing a second-order Taylor expansion of $\Delta^2 w((j+1)/K)$, as follows:

$$\begin{aligned}\Delta^2 w((j+1)/K) &= \Delta\{w((j+1)/K) - w(j/K)\}\\ &= w((j+1)/K) - 2w(j/K) + w((j-1)/K)\\ &= \{w((j+1)/K) - w(j/K)\} - \{w((j-1)/K) - w(j/K)\}\\ &= \{w'(j/K)(1/K) + (\tfrac{1}{2})w''(j/K)(1/K)^2 + o(1/K^2)\}\\ &\quad + \{w'(j/K)(-1/K) + (\tfrac{1}{2})w''(j/K)(-1/K)^2 - o(1/K^2)\}\\ &= K^{-2}w''(j/K) + o(K^{-2}).\end{aligned}$$

Note that $w''(\cdot)$ is also continuous and uniformly bounded under Assumption 2. Hence, we can show just as in part (a) that the variance matrix of

$$\sum_{j=K_{L_\Omega}}^{K^U} \Delta^2 w((j+1)/K)\hat{\Gamma}(j) \tag{A.9}$$

is

$$O(1/TK^3). \tag{A.10}$$

Next, we consider the mean of (A.9). For this, we decompose the sum in (A.9) as we did earlier in (A.6) as

$$\sum_{j=K_{L_\Omega}}^{K^U} \Delta^2 w((j+1)/K)\hat{\Gamma}(j) = \{\Sigma_{\mathcal{B}_*} + \Sigma_{\mathcal{B}^*}\}\Delta^2 w((j+1)/K)\hat{\Gamma}(j). \tag{A.11}$$

Under Assumption 2, we can use the following one-step Taylor expansion for $\Delta^2 w((j+1)/K)$ at 0 when $|j| \le K^*$ and $K \to \infty$ as

$$\Delta^2 w((j+1)/K) = K^{-2}w''(0)(1+o(1)).$$

We then use the preceding result to rewrite the first sum in (A.11) as

$$K^{-2}w''(0)\sum_{|j|\le K^*}\hat{\Gamma}(j)(1+o(1)). \tag{A.12}$$

Now it is clear from (A.12) that the mean of the first sum in (A.11) scaled by K^2 is

$$w''(0) \sum_{|j| \le K^*} E(\hat{\Gamma}(j)) \to w''(0)\Omega, \tag{A.13}$$

as $K \to \infty$. For the mean of the second sum in (A.11), we apply the same analysis as that following (A.7) in part (a). First, write

$$\Delta^2 w((j+1)/K) = K^{-2} w''(\theta_j)$$

for $j/K < \theta_j < (j+1)/K$. Then, the second sum in (A.11) is

$$\Sigma_{\mathcal{B}^*} \Delta^2 w((j+1)/K)\hat{\Gamma}(j) = K^{-2} \sum_{|j|>K^*} w''(\theta_j)\hat{\Gamma}(j),$$

whose mean is given by

$$K^{-2} \sum_{|j|>K^*} w''(\theta_j)(1 - |j|/T)\Gamma(j).$$

Under Assumption 2, $|w''(x)| \le M_{w''}$, $\forall x \in \Re$, for some constant $M_{w''}$. Then, the modulus of the preceding mean is dominated by

$$K^{-2}\left\{\sup_{|j|<K} |w''(\theta_j)|\right\} \sum_{|j|>K^*} \|\Gamma(j)\| \le K^{-2} M_{w''} \underbrace{\sum_{|j|>K^*} \sum_{s=0}^{\infty} \|C_s\| \, \|C_{s-j}\|}$$
$$= O(K^{-2-\alpha\lambda}).$$

As discussed in the proof of part (a), the preceding underbraced term is of order $O(K^{-\alpha\lambda})$. Then, the preceding result implies that the mean of the second sum in (A.11) is $o(K^{-2})$ for all $\lambda \in (0,1)$ because $\alpha > 1$ from Assumption 1(a). Finally, we deduce from (A.9)–(A.13) and the preceding result that

$$\sum_{j=K_{L_\Omega}}^{K^U} \Delta^2 w((j+1)/K)\hat{\Gamma}(j) = K^{-2} w''(0)\Omega + O_p(1/\sqrt{TK^3}).$$

Note that the preceding result does not depend on $l(j)$, thereby establishing the required result.

(c)–(d) We can follow exactly the same line of proofs as in parts (a) and (b) to establish the results in parts (c) and (d), with the replacement of the lower limit of summations in those analyses by $K_{L_\Delta} \in \{-1,0,1\}$. ■

Proof of Lemma 2.

(a) The sample autocovariance function of Δu_{1t} used in the construction of $\hat{\Omega}_{\Delta u_1 \Delta u_1}$ can be decomposed as

$$\begin{aligned}
\hat{\Gamma}_{\Delta u_1 \Delta u_1}(j) &= T^{-1} \textstyle\sum' \Delta u_{1t} \Delta u'_{1t-j} \\
&= T^{-1} \textstyle\sum' \{u_{1t} - u_{1t-1}\}\{u_{1t-j} - u_{1t-j-1}\}' \\
&= T^{-1} \textstyle\sum' \{u_{1t}u'_{1t-j} - u_{1t-1}u'_{1t-j} - u_{1t}u'_{1t-j-1} + u_{1t-1}u'_{1t-j-1}\} \\
&= \hat{\Gamma}_{u_1u_1}(j) - \hat{\Gamma}_{u_1u_1}(j-1) - \hat{\Gamma}_{u_1u_1}(j+1) + \hat{\Gamma}_{u_1u_1}(j) \\
&= -\hat{\Gamma}_{u_1u_1}(j-1) + 2\hat{\Gamma}_{u_1u_1}(j) - \hat{\Gamma}_{u_1u_1}(j+1) \\
&= -\Delta^2 \hat{\Gamma}_{u_1u_1}(j+1).
\end{aligned}$$

Then, it follows that

$$\hat{\Omega}_{\Delta u_1 \Delta u_1} = -\sum_{j=-K+1}^{K-1} w(j/K)\{\hat{\Gamma}_{u_1u_1}(j-1) - 2\hat{\Gamma}_{u_1u_1}(j) + \hat{\Gamma}_{u_1u_1}(j+1)\}$$
$$= -\sum_{j=-K+2}^{K-2} \{w((j+1)/K) - 2w(j/K) + w((j-1)/K)\}\hat{\Gamma}_{u_1u_1}(j) + \xi, \quad \textbf{(A.14)}$$

where

$$\xi = -w((K-1)/K)\{-2\hat{\Gamma}_{u_1u_1}(K-1) + \hat{\Gamma}_{u_1u_1}(K)\}$$
$$- w((K-2)/K)\hat{\Gamma}_{u_1u_1}(K-1) - w((-K+2)/K)\hat{\Gamma}_{u_1u_1}(-K+1)$$
$$- w((-K+1)/K)\{\hat{\Gamma}_{u_1u_1}(-K) - 2\hat{\Gamma}_{u_1u_1}(-K+1)\}.$$

According to (A.5), each of the terms constituting ξ is of $o_p(K^{-2})$. Correspondingly, we have $\xi = o_p(K^{-2})$. We also note that

$$w((j+1)/K) - 2w(j/K) + w((j-1)/K) = \Delta w((j+1)/K) - \Delta w(j/K)$$
$$= \Delta^2 w((j+1)/K),$$

which, together with (A.14), yields

$$\hat{\Omega}_{\Delta u_1 \Delta u_1} = -\sum_{j=-K+2}^{K-2} \Delta^2 w((j+1)/K)\hat{\Gamma}_{u_1u_1}(j) + o_p(K^{-2}). \quad \textbf{(A.15)}$$

Now we apply the result in Lemma 1(b) to approximate the sum in (A.15) as

$$-K^{-2}w''(0)\Omega_{11} + O_p(1/\sqrt{TK^3}),$$

which establishes the stated result.

(b) We can similarly show that

$$\hat{\Gamma}_{\Delta u_1 \Delta u_1}(j+1) = -\{\hat{\Gamma}_{u_1u_1}(j) - 2\hat{\Gamma}_{u_1u_1}(j+1) + \hat{\Gamma}_{u_1u_1}(j+2)\}.$$

Then,

$$^{+}\hat{\Omega}_{\Delta u_1 \Delta u_1} = \sum_{j=-K+1}^{K-1} w(j/K)\hat{\Gamma}_{\Delta u_1 \Delta u_1}(j+1)$$
$$= -\sum_{j=-K+1}^{K-1} w(j/K)\{\hat{\Gamma}_{u_1u_1}(j) - 2\hat{\Gamma}_{u_1u_1}(j+1) + \hat{\Gamma}_{u_1u_1}(j+2)\}$$
$$= -\sum_{j=-K+3}^{K-1} \{w(j/K) - 2w((j-1)/K) + w((j-2)/K)\}$$
$$\times \hat{\Gamma}_{u_1u_1}(j) + {}^{+}\xi,$$

where

$$^{+}\xi = -w((K-1)/K)\{-2\hat{\Gamma}_{u_1u_1}(K) + \hat{\Gamma}_{u_1u_1}(K+1)\}$$
$$- w((K-2)/K)\hat{\Gamma}_{u_1u_1}(K) - w((-K+2)/K)\hat{\Gamma}_{u_1u_1}(-K+2)$$
$$- w((-K+1)/K)\{\hat{\Gamma}_{u_1u_1}(-K+1) - 2\hat{\Gamma}_{u_1u_1}(-K+2)\}.$$

Again, by the result in (A.5), ${}^{+}\xi = o_p(K^{-2})$. Then we have

$$ {}^{+}\hat{\Omega}_{\Delta u_1 \Delta u_1} = -\sum_{j=-K+3}^{K-1} \Delta^2 w(j/K)\hat{\Gamma}_{u_1 u_1}(j) + o_p(K^{-2}). $$

As in part (a), we apply the result in part (b) of Lemma 1 to the first term in the preceding expression. Doing this establishes the required result.

(c) Similarly, we have

$$ \begin{aligned} {}^{-}\hat{\Omega}_{\Delta u_1 \Delta u_1} &= \sum_{j=-K+1}^{K-1} w(j/K)\hat{\Gamma}_{\Delta u_1 \Delta u_1}(j-1) \\ &= -\sum_{j=-K+1}^{K-1} w(j/K)\Delta^2 \hat{\Gamma}_{u_1 u_1}(j) \\ &= -\sum_{j=-K+1}^{K-3} \Delta^2 w((j+2)/K)\hat{\Gamma}_{u_1 u_1}(j) + {}^{-}\xi, \end{aligned} $$

where

$$ \begin{aligned} {}^{-}\xi = & -w((K-1)/K)\{-2\hat{\Gamma}_{u_1 u_1}(K-2) + \hat{\Gamma}_{u_1 u_1}(K-1)\} \\ & - w((K-2)/K)\hat{\Gamma}_{u_1 u_1}(K-2) - w((-K+2)/K)\hat{\Gamma}_{u_1 u_1}(-K) \\ & - w((-K+1)/K)\{\hat{\Gamma}_{u_1 u_1}(-K-1) - 2\hat{\Gamma}_{u_1 u_1}(-K)\}. \end{aligned} $$

By the results in (A.5) and part (b) of Lemma 1, we have

$$ {}^{-}\hat{\Omega}_{\Delta u_1 \Delta u_1} = -K^{-2} w''(0)\Omega_{11} + O_p(1/\sqrt{TK^3}) + o_p(K^{-2}), $$

as required.

(d) The first part of part (d) follows directly from part (b) of Lemma 8.1 in Phillips (1995), and, by using (A.5) and the results in part (a) of Lemma 1, the second part is written as follows:

$$ \begin{aligned} {}^{+}\hat{\Omega}_{u_0 \Delta u_1} &= \sum_{j=-K+1}^{K-1} w(j/K)\hat{\Gamma}_{u_0 \Delta u_1}(j+1) \\ &= \sum_{j=-K+1}^{K-1} w(j/K)\{\hat{\Gamma}_{u_0 u_1}(j+1) - \hat{\Gamma}_{u_0 u_1}(j+2)\} \\ &= \sum_{j=-K+3}^{K} \{w((j-1)/K) - w((j-2)/K)\}\hat{\Gamma}_{u_0 u_1}(j) \\ &\quad - w((K-1)/K)\hat{\Gamma}_{u_0 u_1}(K+1) + w((-K+1)/K)\hat{\Gamma}_{u_0 u_1}(-K+2) \\ &= \sum_{j=-K+3}^{K} \Delta w((j-1)/K)\hat{\Gamma}_{u_0 u_1}(j) + o_p(K^{-2}) \\ &= K^{-2} w''(0)\Phi_{01}^{+} + O_p(1/\sqrt{TK}) + o_p(K^{-2}), \end{aligned} $$

where $\Phi_{01}^{+} = \sum_{j=-\infty}^{\infty}(j - \frac{3}{2})\Gamma_{u_0 u_1}(j)$. The result in (A.5) is used to approximate the residual terms by $o_p(K^{-2})$ and that in Lemma 1(a) establishes the last line with $l(j) = j - 1$. This then proves the second part of part (d).

(e)–(i) The proofs for parts (e)–(i) are similar to the proof for part (d) and hence omitted.

(j)

$$
\begin{aligned}
{}^{+}\hat{\Omega}_{u_0u_2} &= \sum_{j=-K+1}^{K-1} w(j/K)\hat{\Gamma}_{u_0u_2}(j+1) \\
&= \sum_{j=-K+1}^{K-1} w(j/K)T^{-1}\sum{}' u_{0t}u'_{2t-(j+1)} \\
&= \sum_{j=-K+1}^{K-1} w(j/K)T^{-1}\sum{}' u_{0t}(u_{2t-j} - u_{2t-j} + u_{2t-(j+1)})' \\
&= \sum_{j=-K+1}^{K-1} w(j/K)T^{-1}\sum{}' (u_{0t}u'_{2t-j} - u_{0t}\Delta u'_{2t-j}) \\
&= \sum_{j=-K+1}^{K-1} w(j/K)\{\hat{\Gamma}_{u_0u_2}(j) - \hat{\Gamma}_{u_0\Delta u_2}(j)\} \\
&= \hat{\Omega}_{u_0u_2} - \hat{\Omega}_{u_0\Delta u_2} \\
&= \hat{\Omega}_{u_0u_2} + O_p(K^{-2}) + O_p(1/\sqrt{TK}) + o_p(K^{-2}).
\end{aligned}
$$

The last line holds by the result in Lemma 8.1(b) in Phillips (1995), and this proves part (j).

(k)–(l) The proofs for parts (k) and (l) are essentially identical to the proof for part (j) and, thus, not presented here. ∎

Proof of Lemma 3.

(a) Using the definition of G_h given in (15), we let

$$
G_h := (G_{h1}, G_{h2}), \quad \text{where } G_{h1} = \begin{pmatrix} L_1 \\ M_1 \\ N_1 \end{pmatrix} \quad \text{and} \quad G_{h2} = \begin{pmatrix} L_2 \\ M_2 \\ N_2 \end{pmatrix}.
$$

Consider

$$
\begin{aligned}
\hat{\Omega}_{u_0v_h} &= \sum_{j=-K+1}^{K-1} w(j/K)\hat{\Gamma}_{u_0v_h}(j) \\
&= \sum_{j=-K+1}^{K-1} w(j/K)T^{-1}\sum{}' u_{0t}(u^*_{ht-j} + G_h u_{ht-1-j})' \\
&= \sum_{j=-K+1}^{K-1} w(j/K)T^{-1}\sum{}' \{u_{0t}u^{*\prime}_{ht-j} + u_{0t}u'_{ht-1-j}G'_h\} \\
&= \sum_{j=-K+1}^{K-1} w(j/K)\{\hat{\Gamma}_{u_0u^*_h}(j) + \hat{\Gamma}_{u_0u_h}(j+1)G'_h\} \\
&= \hat{\Omega}_{u_0u^*_h} + {}^{+}\hat{\Omega}_{u_0u_h}G'_h \\
&= (\hat{\Omega}_{u_0\Delta u_1}, \hat{\Omega}_{u_0u_h}) + ({}^{+}\hat{\Omega}_{u_0\Delta u_1}, {}^{+}\hat{\Omega}_{u_0u_2})G'_h.
\end{aligned}
$$

Define

$$
f_{KT} := K^{-2} + 1/\sqrt{TK}, \tag{A.16}
$$

and use the results in Lemmas 2(d) and 2(j) to write

$$\begin{aligned}\hat{\Omega}_{u_0 v_h} &= (O_p(f_{KT}), \hat{\Omega}_{u_0 u_b}) + (O_p(f_{KT}), \hat{\Omega}_{u_0 u_2} + O_p(f_{KT}))G_h' \\ &= (0, \hat{\Omega}_{u_0 u_b}) + (0, \hat{\Omega}_{u_0 u_2})G_h' + (O_p(f_{KT}), O_p(f_{KT})) \\ &= (0, \hat{\Omega}_{u_0 u_b}) + \hat{\Omega}_{u_0 u_2} G_{h2}' + (O_p(f_{KT}), O_p(f_{KT})) \\ &= (0, \hat{\Omega}_{u_0 u_b}) + \hat{\Omega}_{u_0 u_2}(L_2', M_2', N_2') + (O_p(f_{KT}), O_p(f_{KT})) \\ &= (\hat{\Omega}_{u_0 u_2} L_2', \hat{\Omega}_{u_0 u_2}(I + M_2)', \{\hat{\Omega}_{u_0 u_2} N_2' + \hat{\Omega}_{u_0 u_3}\}) + (O_p(f_{KT}), O_p(f_{KT})) \\ &= \hat{\Omega}_{u_0 u_b} \begin{pmatrix} L_2' & I + M_2' & N_2' \\ 0 & 0 & I \end{pmatrix} + (O_p(f_{KT}), O_p(f_{KT})),\end{aligned}$$

this establishes the required result.

(b) First, write the sample autocovariance of v_{ht} as

$$\begin{aligned}\hat{\Gamma}_{v_h v_h}(j) &= T^{-1} \sum{}' v_{ht} v_{ht-j}' \\ &= T^{-1} \sum{}' \{u_{ht}^* + G_h u_{ht-1}\}\{u_{ht-j}^* + G_h u_{ht-1-j}\}' \\ &= T^{-1} \sum{}' \{u_{ht}^* u_{ht-j}^{*\prime} + G_h u_{ht-1} u_{ht-j}^{*\prime} + u_{ht}^* u_{ht-1-j}' G_h' \\ &\qquad + G_h u_{ht-1} u_{ht-1-j}' G_h'\} \\ &= \hat{\Gamma}_{u_h^* u_h^*}(j) + G_h \hat{\Gamma}_{u_h u_h^*}(j-1) + \hat{\Gamma}_{u_h^* u_h}(j+1) G_h' + G_h \hat{\Gamma}_{u_h u_h}(j) G_h'.\end{aligned}$$

This expression decomposes the kernel estimate $\hat{\Omega}_{v_h v_h}$ as follows:

$$\begin{aligned}\hat{\Omega}_{v_h v_h} &= \sum_{j=-K+1}^{K-1} w(j/K)\hat{\Gamma}_{v_h v_h}(j) \\ &= \sum_{j=-K+1}^{K-1} w(j/K)\hat{\Gamma}_{u_h^* u_h^*}(j) + G_h\left(\sum_{j=-K+1}^{K-1} w(j/K)\hat{\Gamma}_{u_h u_h^*}(j-1)\right) \\ &\quad + \left(\sum_{j=-K+1}^{K-1} w(j/K)\hat{\Gamma}_{u_h^* u_h}(j+1)\right)G_h' \\ &\quad + G_h\left(\sum_{j=-K+1}^{K-1} w(j/K)\hat{\Gamma}_{u_h u_h}(j)\right)G_h' \\ &= \hat{\Omega}_{u_h^* u_h^*} + G_h {}^{-}\hat{\Omega}_{u_h u_h^*} + {}^{+}\hat{\Omega}_{u_h^* u_h} G_h' + G_h \hat{\Omega}_{u_h u_h} G_h'.\end{aligned}$$

The definitions for ${}^{+}\hat{\Omega}$ and ${}^{-}\hat{\Omega}$ given in (21) and (22) are used in the last line. By the results in parts (a) and (e) of Lemma 2, we can write the first component in the preceding expression as follows:

$$\begin{aligned}\hat{\Omega}_{u_h^* u_h^*} &= \begin{pmatrix} \hat{\Omega}_{\Delta u_1 \Delta u_1} & \hat{\Omega}_{\Delta u_1 u_b} \\ \hat{\Omega}_{u_b \Delta u_1} & \hat{\Omega}_{u_b u_b} \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ 0 & \hat{\Omega}_{u_b u_b} \end{pmatrix} + K^{-2} w''(0) \begin{pmatrix} -\Omega_{11} & -\Phi_{1b} \\ \Phi_{b1} & 0 \end{pmatrix} \\ &\quad + \begin{pmatrix} O_p(1/\sqrt{TK^3}) + o_p(K^{-2}) & O_p(1/\sqrt{TK}) + o_p(K^{-2}) \\ O_p(1/\sqrt{TK}) + o_p(K^{-2}) & 0 \end{pmatrix} \text{edit}.\end{aligned}$$

Similarly, by the results in parts (b)-(d), (f), and (h)-(l) of Lemma 2, we have

$$
\begin{aligned}
G_h{}^{-}\hat{\Omega}_{u_h u_h^*} &= G_h\begin{pmatrix} {}^{-}\hat{\Omega}_{\Delta u_1 \Delta u_1} & {}^{-}\hat{\Omega}_{\Delta u_1 u_b} \\ {}^{-}\hat{\Omega}_{u_2 \Delta u_1} & {}^{-}\hat{\Omega}_{u_2 u_b} \end{pmatrix} \\
&= G_h\begin{pmatrix} 0 & 0 \\ 0 & \hat{\Omega}_{u_2 u_b} \end{pmatrix} + K^{-2} w''(0) G_h \begin{pmatrix} -\Omega_{11} & -\Phi_{1b}^{-} \\ -\Phi_{21}^{-} & \Phi_{2b}^{-} \end{pmatrix} \\
&\quad + \begin{pmatrix} O_p(1/\sqrt{TK}) + o_p(K^{-2}) & O_p(1/\sqrt{TK}) + o_p(K^{-2}) \\ O_p(1/\sqrt{TK}) + o_p(K^{-2}) & O_p(1/\sqrt{TK}) + o_p(K^{-2}) \end{pmatrix},
\end{aligned}
$$

$$
\begin{aligned}
{}^{+}\hat{\Omega}_{u_h^* u_h} G_h' &= \begin{pmatrix} {}^{+}\hat{\Omega}_{\Delta u_1 \Delta u_1} & {}^{+}\hat{\Omega}_{\Delta u_1 u_2} \\ {}^{+}\hat{\Omega}_{u_b \Delta u_1} & {}^{+}\hat{\Omega}_{u_b u_2} \end{pmatrix} G_h' \\
&= \begin{pmatrix} 0 & 0 \\ 0 & \hat{\Omega}_{u_b u_2} \end{pmatrix} G_h' + K^{-2} w''(0) \begin{pmatrix} -\Omega_{11} & -\Phi_{12}^{+} \\ \Phi_{b1}^{+} & -\Phi_{b2} \end{pmatrix} G_h' \\
&\quad + \begin{pmatrix} O_p(1/\sqrt{TK}) + o_p(K^{-2}) & O_p(1/\sqrt{TK}) + o_p(K^{-2}) \\ O_p(1/\sqrt{TK}) + o_p(K^{-2}) & O_p(1/\sqrt{TK}) + o_p(K^{-2}) \end{pmatrix},
\end{aligned}
$$

and

$$
\begin{aligned}
G_h \hat{\Omega}_{u_h u_h} G_h' &= G_h \begin{pmatrix} \hat{\Omega}_{\Delta u_1 \Delta u_1} & \hat{\Omega}_{\Delta u_1 u_2} \\ \hat{\Omega}_{u_2 \Delta u_1} & \hat{\Omega}_{u_2 u_2} \end{pmatrix} G_h' \\
&= G_h \begin{pmatrix} 0 & 0 \\ 0 & \hat{\Omega}_{u_2 u_2} \end{pmatrix} G_h' + K^{-2} w''(0) G_h \begin{pmatrix} -\Omega_{11} & \Phi_{12} \\ \Phi_{21} & 0 \end{pmatrix} G_h' \\
&\quad + \begin{pmatrix} O_p(1/\sqrt{TK}) + o_p(K^{-2}) & O_p(1/\sqrt{TK}) + o_p(K^{-2}) \\ O_p(1/\sqrt{TK}) + o_p(K^{-2}) & 0 \end{pmatrix}.
\end{aligned}
$$

Combining all these results, we now have

$$
\begin{aligned}
\hat{\Omega}_{v_h v_h} = {} & \overbrace{\begin{pmatrix} 0 & 0 \\ 0 & \hat{\Omega}_{u_b u_b} \end{pmatrix} + \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, G_{h2} \hat{\Omega}_{u_2 u_b} \right) + \begin{pmatrix} 0 & 0 \\ 0 & \hat{\Omega}_{u_b u_2} G_{h2}' \end{pmatrix} + G_{h2} \hat{\Omega}_{u_2 u_2} G_{h2}'}^{\hat{\Omega}^*} \\
& \underbrace{+ K^{-2} w''(0) \hat{\Omega}_R^* + O_p(1/\sqrt{TK}) + o_p(K^{-2})}_{O_p(f_{KT}^*)}, \qquad \textbf{(A.17)}
\end{aligned}
$$

where we denote the leading term and the remainder term as $\hat{\Omega}^*$ and $O_p(f_{KT}^*)$, respectively. The term $\hat{\Omega}_R^*$ in the remainder term is more explicitly expressed as

$$
\begin{aligned}
& \begin{pmatrix} -\Omega_{11} & \Phi_{1b} \\ \Phi_{b1} & 0 \end{pmatrix} + G_h \begin{pmatrix} -\Omega_{11} & -\Phi_{1b}^{-} \\ -\Phi_{21}^{-} & \Phi_{2b}^{-} \end{pmatrix} \\
& \quad + \begin{pmatrix} -\Omega_{11} & -\Phi_{12}^{+} \\ \Phi_{b1}^{+} & -\Phi_{b2} \end{pmatrix} G_h' + G_h \begin{pmatrix} -\Omega_{11} & \Phi_{12} \\ \Phi_{21} & 0 \end{pmatrix} G_h'. \qquad \textbf{(A.18)}
\end{aligned}
$$

We now decompose the leading term $\hat{\Omega}^*$ further as follows:

$$\begin{pmatrix} 0 & 0 \\ 0 & \hat{\Omega}_{u_b u_b} \end{pmatrix} + \left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{bmatrix} L_2 \\ M_2 \\ N_2 \end{bmatrix} \hat{\Omega}_{u_2 u_b} \right]$$

$$+ \begin{pmatrix} 0 & 0 \\ \hat{\Omega}_{u_b u_2}(L_2', M_2', N_2') \end{pmatrix} + \begin{bmatrix} L_2 \\ M_2 \\ N_2 \end{bmatrix} \hat{\Omega}_{u_2 u_2}(L_2', M_2', N_2')$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & \hat{\Omega}_{u_b u_b} \end{pmatrix} + \begin{bmatrix} 0 & L_2 \hat{\Omega}_{u_2 u_b} \\ 0 & \begin{pmatrix} M_2 \\ N_2 \end{pmatrix} \hat{\Omega}_{u_2 u_b} \end{bmatrix}$$

$$+ \begin{pmatrix} 0 & 0 \\ \hat{\Omega}_{u_b u_2} L_2' & \hat{\Omega}_{u_b u_2}(M_2', N_2') \end{pmatrix} + \begin{bmatrix} L_2 \\ M_2 \\ N_2 \end{bmatrix} \hat{\Omega}_{u_2 u_2}(L_2', M_2', N_2')$$

$$= \begin{bmatrix} L_2 \hat{\Omega}_{u_2 u_2} L_2' & L_2 \{ \hat{\Omega}_{u_2 u_2}(I + M_2),\ \hat{\Omega}_{u_2 u_2} N_2' + \hat{\Omega}_{u_3 u_2} \} \\ \left\{ \begin{pmatrix} \hat{\Omega}_{u_2 u_2} \\ \hat{\Omega}_{u_3 u_2} \end{pmatrix} + \begin{pmatrix} M_2 \\ N_2 \end{pmatrix} \hat{\Omega}_{u_2 u_2} \right\} L_2' & \left\{ \begin{array}{l} \hat{\Omega}_{u_b u_b} + \begin{pmatrix} M_2 \\ N_2 \end{pmatrix} \hat{\Omega}_{u_2 u_b} \\ \quad + \hat{\Omega}_{u_b u_2}(M_2', N_2') \\ \quad + \begin{pmatrix} M_2 \\ N_2 \end{pmatrix} \hat{\Omega}_{u_2 u_2}(M_2', N_2') \end{array} \right\} \end{bmatrix}$$

$$= \begin{bmatrix} L_2 \hat{\Omega}_{u_2 u_2} L_2' & L_2 \hat{\Omega}_{u_2 u_b} \begin{pmatrix} I + M_2' & N_2' \\ 0 & I \end{pmatrix} \\ \begin{pmatrix} I + M_2 & 0 \\ N_2 & I \end{pmatrix} \hat{\Omega}_{u_b u_2} L_2' & \begin{pmatrix} I + M_2 & 0 \\ N_2 & I \end{pmatrix} \hat{\Omega}_{u_b u_b} \begin{pmatrix} I + M_2' & N_2' \\ 0 & I \end{pmatrix} \end{bmatrix}$$

$$= \left[\begin{bmatrix} L_2 & 0 \\ (I + M_2) & 0 \\ N_2 & I \end{bmatrix} \hat{\Omega}_{u_b u_b} \begin{pmatrix} L_2' & (I + M_2)' & N_2' \\ 0 & 0 & I \end{pmatrix} \right].$$

Next, define

$$G' := \begin{pmatrix} L_2' & (I + M_2)' & N_2' \\ 0 & 0 & I \end{pmatrix}.$$

Then,

$$\hat{\Omega}^* = G \hat{\Omega}_{u_b u_b} G'.$$

This together with the preceding results finally gives

$$\hat{\Omega}_{v_h v_h} = G \hat{\Omega}_{u_b u_b} G' + O_p(K^{-2}) + O_p(1/\sqrt{TK}),$$

which is required.

(c) Let f_{KT} be defined as in (A.16), and let

$$\underset{\sim}{G} = G(G'G)^{-1/2}. \qquad \textbf{(A.19)}$$

We define

$$\hat{\underline{\Omega}}_{u_0u_b} := \hat{\Omega}_{u_0u_b}(G'G)^{1/2},$$

$$\hat{\underline{\Omega}}_{u_bu_b} := (G'G)^{1/2}\hat{\Omega}_{u_bu_b}(G'G)^{1/2},$$

and let $C := (\underline{G}, \underline{G}_\perp)$ be orthogonal. Then it follows from (A.17) and (A.18) that

$$\begin{aligned}
\hat{\Omega}_{u_0v_h}\hat{\Omega}_{v_hv_h}^{-1} &= (\hat{\underline{\Omega}}_{u_0u_b}\underline{G}' + O_p(f_{KT}))(\underline{G}\hat{\underline{\Omega}}_{u_bu_b}\underline{G}' + O_p(f_{KT}^*))^{-1} \\
&= (\hat{\underline{\Omega}}_{u_0u_b}\underline{G}'C + O_p(f_{KT}))(C'\underline{G}\hat{\underline{\Omega}}_{u_bu_b}\underline{G}'C + O_p(f_{KT}^*))^{-1}C' \\
&= (\hat{\underline{\Omega}}_{u_0u_b}(I,0) + O_p(f_{KT}))\left(\begin{pmatrix} I \\ 0 \end{pmatrix}\hat{\underline{\Omega}}_{u_bu_b}(I,0) + O_p(f_{KT}^*)\right)^{-1}C' \\
&= ((\hat{\underline{\Omega}}_{u_0u_b},0) + O_p(f_{KT}))\left(\begin{pmatrix} \hat{\underline{\Omega}}_{u_bu_b} & 0 \\ 0 & 0 \end{pmatrix} + O_p(f_{KT}^*)\right)^{-1}C' \\
&= (\hat{\underline{\Omega}}_{0b} + O_p(f_{KT}), O_p(f_{KT})) \\
&\quad \times \begin{pmatrix} \hat{\underline{\Omega}}_{bb} + (O_p(f_{KT}^*))_{11} & (O_p(f_{KT}^*))_{12} \\ (O_p(f_{KT}^*))_{21} & (O_p(f_{KT}^*))_{22} \end{pmatrix}^{-1} C' \\
&= (\underline{\Omega}_{0b} + o_p(1), O_p(f_{KT}))\begin{pmatrix} \underline{\Omega}_{bb} + o_p(1) & (O_p(f_{KT}^*))_{12} \\ (O_p(f_{KT}^*))_{21} & (O_p(f_{KT}^*))_{22} \end{pmatrix}^{-1} C'.
\end{aligned}$$

It is now convenient to denote the matrix to be inverted in the preceding equation by

$$\Theta = \begin{pmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{pmatrix} := \begin{pmatrix} \underline{\Omega}_{bb} + o_p(1) & (O_p(f_{KT}^*))_{12} \\ (O_p(f_{KT}^*))_{21} & (O_p(f_{KT}^*))_{22} \end{pmatrix}$$

and its inverse by

$$\Theta^{-1} = \begin{pmatrix} (\Theta^{-1})_{11} & (\Theta^{-1})_{12} \\ (\Theta^{-1})_{21} & (\Theta^{-1})_{22} \end{pmatrix}.$$

From (A.17) and (A.18), we have

$$\Theta_{22} = (O_p(f_{KT}^*))_{22} = K^{-2}w''(0)\Theta_{22}^* + O_p(1/\sqrt{TK}) + o_p(K^{-2}),$$

where $\Theta_{22}^* = (\hat{\Omega}_R^*)_{22}$ is further expressed as

$$\begin{aligned}
&\left[\begin{bmatrix} L_1 & L_2 \\ M_1 & M_2 \\ N_1 & N_2 \end{bmatrix}\begin{pmatrix} -\Omega_{11} & -\Phi_{1b}^- \\ -\Phi_{21}^- & \Phi_{2b}^- \end{pmatrix} + \begin{pmatrix} -\Omega_{11} & -\Phi_{12}^+ \\ \Phi_{b1}^+ & -\Phi_{b2} \end{pmatrix}\begin{pmatrix} L_1' & M_1' & N_1' \\ L_2' & M_2' & N_2' \end{pmatrix}\right]_{22} \\
&\quad = \begin{pmatrix} M_1 & M_2 \\ N_1 & N_2 \end{pmatrix}\begin{pmatrix} -\Phi_{1b}^- \\ \Phi_{2b}^- \end{pmatrix} + (\Phi_{b1}^+, -\Phi_{b2})\begin{pmatrix} M_1' & N_1' \\ M_2' & N_2' \end{pmatrix} \\
&\quad = -\begin{pmatrix} M_1 \\ N_1 \end{pmatrix}\sum_{j=-\infty}^{\infty}(j+\tfrac{3}{2})\Gamma_{1b}(j) + \begin{pmatrix} M_2 \\ N_2 \end{pmatrix}\sum_{j=-\infty}^{\infty}(j+\tfrac{1}{2})\Gamma_{2b}(j) \\
&\qquad + \sum_{j=-\infty}^{\infty}(j-\tfrac{3}{2})\Gamma_{b1}(j)\begin{pmatrix} M_1' \\ N_1' \end{pmatrix} - \sum_{j=-\infty}^{\infty}(j+\tfrac{1}{2})\Gamma_{b2}(j)\begin{pmatrix} M_2' \\ N_2' \end{pmatrix},
\end{aligned}$$

by using the definitions of $\hat{\Omega}_R^*$ and its component submatrices given in (A.18) and parts (g), (h), (k), and (l) of Lemma 2. We assume that the preceding limit matrix Θ_{22}^* is nonsingular. Then, for $K = O_e(T^k)$ with $k < \frac{1}{3}$, we have

$$K^{-2}\Theta_{22}^{-1} \xrightarrow{\mathcal{P}} -(1/w''(0))\Theta_{22}^{*-1}.$$

Applying the partitioned matrix inversion formulae to the submatrices of Θ^{-1} gives

$$(\Theta^{-1})_{11} = \Omega_{bb}^{-1} + o_p(1),$$
$$(\Theta^{-1})_{12} = -\Omega_{bb}^{-1}O_p(1) + o_p(1),$$
$$(\Theta^{-1})_{22} = O_p(K^2).$$

Therefore,

$$\begin{aligned}
\hat{\Omega}_{u_0 v_h}\hat{\Omega}_{v_h v_h}^{-1} &= (\Omega_{0b} + o_p(1), O_p(f_{KT}))\begin{pmatrix} \Omega_{bb}^{-1} + o_p(1) & O_p(1) \\ O_p(1) & O_p(K^2) \end{pmatrix} C' \\
&= (\Omega_{0b}\Omega_{bb}^{-1} + o_p(1) + O_p(f_{KT}), O_p(1) + O_p(f_{KT})O_p(K^2))C' \\
&= (\Omega_{0b}\Omega_{bb}^{-1} + o_p(1), O_p(1) + O_p(K^{3/2}T^{-1/2}))C' \\
&= (\Omega_{0b}\Omega_{bb}^{-1}(G'G)^{-1/2} + o_p(1), O_p(1) + O_p(K^{3/2}T^{-1/2}))C',
\end{aligned}$$

because

$$\begin{aligned}
\Omega_{0b}\Omega_{bb}^{-1} &= \Omega_{0b}(G'G)^{1/2}((G'G)^{1/2}\Omega_{bb}(G'G)^{1/2})^{-1} \\
&= \Omega_{0b}\Omega_{bb}^{-1}(G'G)^{-1/2}.
\end{aligned}$$

This establishes the required result as the restriction $k < \frac{1}{3}$ imposed to derive the explicit representation of the convergence rates is absorbed into the $O_p(K^{3/2}T^{-1/2})$ term in the preceding second block of $\hat{\Omega}_{u_0 v_h}\hat{\Omega}_{v_h v_h}^{-1}$. Note that we are not being explicit about the $O_p(1)$ term in the same block because it will later be scaled out by a factor of $o_p(1)$ (see the proofs of parts (a) and (b) of Lemma 6, later).

(d) We rewrite the expression for $\hat{v}_t$ given in (16) as follows:

$$\begin{aligned}
\hat{v}_t &= u_{ht}^* + \hat{G}_h u_{ht-1} + O_p(T^{-1/2}) \\
&= \{u_{ht}^* + G_h u_{ht-1}\} + (\hat{G}_h - G_h)u_{ht-1} + O_p(T^{-1/2}) \\
&= v_{ht} + (G_{h1} - \hat{G}_{h1})\Delta u_{1t-1} + (G_{h2} - \hat{G}_{h2})u_{2t-1} + O_p(T^{-1/2}) \\
&= v_{ht} + R_1\Delta u_{1t-1} + R_2 u_{2t-1} + O_p(T^{-1/2}),
\end{aligned}$$

where

$$R_1 := G_{h1} - \hat{G}_{h1} = O_p(1),$$
$$R_2 := G_{h2} - \hat{G}_{h2} = O_p(T^{-1/2}).$$

Then, $\hat{\Gamma}_{\hat{v}_h\hat{v}_h}(j) = T^{-1}\sum'\hat{v}_{ht}\hat{v}_{ht-j}'$ can be written as

$$\begin{aligned}
&T^{-1} \sum{}' \{v_{ht} + R_1 \Delta u_{1t-1} + R_2 u_{2t-1} + O_p(T^{-1/2})\} \\
&\qquad \times \{v'_{ht-j} + \Delta u'_{1t-1-j} R'_1 + u'_{2t-1-j} R'_2 + O_p(T^{-1/2})\} \\
&\quad = T^{-1} \sum{}' v_{ht} \{v'_{ht-j} + \Delta u'_{1t-1-j} R'_1 + u'_{2t-1-j} R'_2\} \\
&\qquad + T^{-1} \sum{}' R_1 \Delta u_{1t-1} \{v'_{ht-j} + \Delta u'_{1t-1-j} R'_1 + u'_{2t-1-j} R'_2\} \\
&\qquad + T^{-1} \sum{}' R_2 u_{2t-1} \{v'_{ht-j} + \Delta u'_{1t-1-j} R'_1 + u'_{2t-1-j} R'_2\} \\
&\qquad + (1 - |j|/T) O_p(T^{-1/2}) \\
&\quad = \hat{\Gamma}_{v_h v_h}(j) + \hat{\Gamma}_{v_h \Delta u_1}(j+1) R'_1 + \hat{\Gamma}_{v_h u_2}(j+1) R'_2 \\
&\qquad + R_1 \hat{\Gamma}_{\Delta u_1 v_h}(j-1) + R_1 \hat{\Gamma}_{\Delta u_1 \Delta u_1}(j) R'_1 + R_1 \hat{\Gamma}_{\Delta u_1 u_2}(j) R'_2 \\
&\qquad + R_2 \hat{\Gamma}_{u_2 v_h}(j-1) + R_2 \hat{\Gamma}_{u_2 \Delta u_1}(j) R'_1 + R_2 \hat{\Gamma}_{u_2 u_2}(j) R'_2 \\
&\qquad + (1 - |j|/T) O_p(T^{-1/2}).
\end{aligned}$$

We now write the kernel estimate of $\Omega_{\hat{v}_h \hat{v}_h}$ using the preceding expression, as follows:

$$\begin{aligned}
\hat{\Omega}_{\hat{v}_h \hat{v}_h} &= \sum_{j=-K+1}^{K-1} w(j/K) \hat{\Gamma}_{\hat{v}_h \hat{v}_h}(j) \\
&= \hat{\Omega}_{v_h v_h} + {}^{+}\hat{\Omega}_{v_h \Delta u_1} R'_1 + {}^{+}\hat{\Omega}_{v_h u_2} R'_2 + R_1 {}^{-}\hat{\Omega}_{\Delta u_1 v_h} \\
&\quad + R_1 \hat{\Omega}_{\Delta u_1 \Delta u_1} R'_1 + R_1 \hat{\Omega}_{\Delta u_1 u_2} R'_2 + R_2 {}^{-}\hat{\Omega}_{u_2 v_h} + R_2 \hat{\Omega}_{u_2 \Delta u_1} R'_1 \\
&\quad + R_2 \hat{\Omega}_{u_2 u_2} R'_2 + \nu_K K(1 - |j|/T) O_p(T^{-1/2}),
\end{aligned}$$

where we employ the notation ν_K, which is defined as

$$\nu_k := K^{-1} \sum_{j=-K+1}^{K-1} w(j/K) \to \int_{-1}^{1} w(s)\, ds := \nu, \tag{A.20}$$

as $K \to \infty$. Note that $\nu < \infty$ under Assumption 2. Hence, $\nu_K = O_p(1)$ and the last element on the right-hand side of the preceding equation is simply $O_p(KT^{-1/2})$, as $|j|/T \le K/T \to 0$ under Assumption 3. Next,

$$\begin{aligned}
{}^{-}\hat{\Omega}_{\Delta u_1 v_h} &= \sum_{j=-K+1}^{K-1} w(j/K) \hat{\Gamma}_{\Delta u_1 v_h}(j-1) \\
&= \sum_{j=-K+1}^{K-1} w(j/K) T^{-1} \sum{}' \Delta u_{1t} v'_{ht-(j-1)} \\
&= \sum_{j=-K+1}^{K-1} w(j/K) T^{-1} \sum{}' \{\Delta u_{1t} u^{*\prime}_{ht-(j-1)} + \Delta u_{1t} u'_{ht-1-(j-1)} G'_h\} \\
&= \sum_{j=-K+1}^{K-1} w(j/K) \{\hat{\Gamma}_{\Delta u_1 u_h^*}(j-1) + \hat{\Gamma}_{\Delta u_1 u_h}(j) G'_h\} \\
&= {}^{-}\hat{\Omega}_{\Delta u_1 u_h^*} + \hat{\Omega}_{\Delta u_1 u_h} G'_h \\
&= O_p(K^{-2}) + O_p(1/\sqrt{TK}),
\end{aligned}$$

because

$$
\begin{aligned}
{}^{-}\hat{\Omega}_{\Delta u_1 u_h^*} &= ({}^{-}\hat{\Omega}_{\Delta u_1 \Delta u_1}, {}^{-}\hat{\Omega}_{\Delta u_1 u_b}) = O_p(K^{-2}) + O_p(1/\sqrt{TK}),\\
\hat{\Omega}_{\Delta u_1 u_h} &= (\hat{\Omega}_{\Delta u_1 \Delta u_1}, \hat{\Omega}_{\Delta u_1 u_2}) = O_p(K^{-2}) + O_p(1/\sqrt{TK}),
\end{aligned}
$$

by the results in parts (a), (c), (f), and (g) of Lemma 2. We also have, by the results in parts (a), (b), (e), and (f) in Lemma 2, that

$$
{}^{+}\hat{\Omega}_{v_h \Delta u_1} = O_p(K^{-2}) + O_p(1/\sqrt{TK}).
$$

Moreover,

$$
\begin{aligned}
{}^{+}\hat{\Omega}_{v_h u_2} &= {}^{+}\hat{\Omega}_{u_h^* u_2} + G_h \hat{\Omega}_{u_h u_2}\\
&= \begin{pmatrix} {}^{+}\hat{\Omega}_{\Delta u_1 u_2} \\ {}^{+}\hat{\Omega}_{u_b u_2} \end{pmatrix} + G_h \begin{pmatrix} \hat{\Omega}_{\Delta u_1 u_2} \\ \hat{\Omega}_{u_2 u_2} \end{pmatrix}\\
&= \begin{pmatrix} 0 \\ \hat{\Omega}_{u_b u_2} \end{pmatrix} + G_h \begin{pmatrix} 0 \\ \hat{\Omega}_{u_2 u_2} \end{pmatrix} + O_p(K^{-2}) + O_p(1/\sqrt{TK}),
\end{aligned}
$$

and

$$
\begin{aligned}
{}^{-}\hat{\Omega}_{u_2 v_h} &= {}^{-}\hat{\Omega}_{u_2 u_h^*} + \hat{\Omega}_{u_2 u_h} G_h'\\
&= \begin{pmatrix} {}^{-}\hat{\Omega}_{u_2 \Delta u_1} \\ \hat{\Omega}_{u_2 u_b} \end{pmatrix} + \begin{pmatrix} \hat{\Omega}_{u_2 \Delta u_1} \\ \hat{\Omega}_{u_2 u_2} \end{pmatrix} G_h'\\
&= \begin{pmatrix} 0 \\ \hat{\Omega}_{u_2 u_b} \end{pmatrix} + \begin{pmatrix} 0 \\ \hat{\Omega}_{u_2 u_2} \end{pmatrix} G_h' + O_p(K^{-2}) + O_p(1/\sqrt{TK}).
\end{aligned}
$$

We now deduce from the preceding results that

$$
\hat{\Omega}_{\hat{v}_h \hat{v}_h} = \hat{\Omega}_{v_h v_h} + O_p(KT^{-1/2}) + O_p(K^{-2}),
$$

which is required for part (d).

(e) Recall that $\hat{u}_{0t}$ are the residuals from the regression on $y_t = Ax_t + u_{0t}$. Then, $\hat{\Gamma}_{\hat{u}_0 \hat{v}_h}(j) = T^{-1} \sum' \hat{u}_{0t} \hat{v}_{ht-j}'$ is expressed as

$$
\begin{aligned}
T^{-1} &\textstyle\sum' \{u_{0t} + (A - \hat{A})x_t\}\{v_{ht-j}' + \Delta u_{1t-1-j}' R_1' + u_{2t-1-j} R_2' + O_p(T^{-1/2})\}\\
&= T^{-1} \textstyle\sum' u_{0t}\{v_{ht-j}' + \Delta u_{1t-1-j}' R_1' + u_{2t-1-j}' R_2' + O_p(T^{-1/2})\}\\
&\quad + T^{-1} \textstyle\sum' (A - \hat{A})x_t\{v_{ht-j}' + \Delta u_{1t-1-j}' R_1' + u_{2t-1-j}' R_2' + O_p(T^{-1/2})\}\\
&= \hat{\Gamma}_{u_0 v_h}(j) + \hat{\Gamma}_{u_0 \Delta u_1}(j+1) R_1' + \hat{\Gamma}_{u_0 u_2}(j+1) R_2'\\
&\quad + (A - \hat{A})\hat{\Gamma}_{x v_h}(j) + (A - \hat{A})\hat{\Gamma}_{x \Delta u_1}(j+1) R_1' + (A - \hat{A})\hat{\Gamma}_{x u_2}(j+1) R_2'\\
&\quad + (1 - |j|/T) O_p(T^{-1/2}) + (1 - |j|/T) O_p(T^{-1}),
\end{aligned}
$$

because $(A - \hat{A})x_t = O_p(T^{-1/2})$. We decompose $\hat{\Omega}_{\hat{u}_0 \hat{v}_h}$ as

$$
\begin{aligned}
&\hat{\Omega}_{u_0 v_h} + {}^{+}\hat{\Omega}_{u_0 \Delta u_1} R_1' + {}^{+}\hat{\Omega}_{u_0 u_2} R_2' + (A - \hat{A})\hat{\Omega}_{x v_h}\\
&\quad + (A - \hat{A})^{+}\hat{\Omega}_{x \Delta u_1} R_1' + (A - \hat{A})^{+}\hat{\Omega}_{x u_2} R_2' + \nu_K K (1 - |j|/T) O_p(T^{-1/2}),
\end{aligned}
$$

which, by the results in Lemmas 2(f) and 2(i), becomes

$$\hat{\Omega}_{u_0 v_h} + (A - \hat{A})\hat{\Omega}_{x v_h} + (A - \hat{A})^{+}\hat{\Omega}_{x\Delta u_1} R_1' + (A - \hat{A})^{+}\hat{\Omega}_{x u_2} R_2' + \{O_p(K^{-2}) + O_p(1/\sqrt{TK})\} + O_p(1)O_p(T^{-1/2}) + O_p(KT^{-1/2}). \quad \textbf{(A.21)}$$

Next, we consider the terms on the right-hand side of equation (A.21). First, by Theorem 3.1 in Phillips (1991b), it follows that

$$K^{-1}\hat{\Omega}_{x_2 u_b} = K^{-1}\sum_{j=-K+1}^{K-1} w(j/K)\hat{\Gamma}_{x_2 u_b}(j) \xrightarrow{\mathfrak{D}} \nu\int_0^1 B_2\, dB_b' + \Omega_{2b}, \quad \textbf{(A.22)}$$

where $\Omega_{2b} = \sum_{j=-\infty}^{\infty} E(u_{2j} u_{b0}')$ and with ν as defined in (A.20), and that

$$K^{-1}T^{-1}\hat{\Omega}_{x_3 u_b} = K^{-1}\sum_{j=-K+1}^{K-1} w(j/K)T^{-1}\hat{\Gamma}_{x_3 u_b}(j) \xrightarrow{\mathfrak{D}} \nu\int_0^1 \bar{B}_3\, dB_b'. \quad \textbf{(A.23)}$$

Next we analyze $\hat{\Omega}_{x_2\Delta u_1}$ as follows:

$$\begin{aligned}
\hat{\Omega}_{x_2\Delta u_1} &= \sum_{j=-K+1}^{K-1} w(j/K)\hat{\Gamma}_{x_2\Delta u_1}(j) \\
&= \sum_{j=-K+1}^{K-1} w(j/K)\{\hat{\Gamma}_{x_2 u_1}(j) - \hat{\Gamma}_{x_2 u_1}(j+1)\} \\
&= \sum_{j=-K+2}^{K-1} \{w(j/K) - w((j-1)/K)\}\hat{\Gamma}_{x_2 u_1}(j) \\
&\quad + w((K-1)/K)\hat{\Gamma}_{x_2 u_1}(K) + w((-K+1)/K)\hat{\Gamma}_{x_2 u_1}(-K+1) \\
&= \sum_{j=-K+2}^{K-1} \Delta w(j/K)\hat{\Gamma}_{x_2 u_1}(j) + O_p(K^{-2}),
\end{aligned}$$

because $w((K-1)/K)$, $w((-K+1)/K) = O(K^{-2})$ by Assumption 2 and $\hat{\Gamma}_{x_2 u_1}(i) = T^{-1}\sum' x_{2t} u_{1t-i}' = O_p(1)$, for $i = K, -K+1$. For the sum in the preceding expression, we can follow the analysis given in the proof of Lemma 1(a) but now with the sample autocovariances defined with I(1) variable x_{2t}. We similarly write

$$\sum_{j=-K+2}^{K-1} \Delta w(j/K)\hat{\Gamma}_{x_2 u_1}(j) = \{\Sigma_{\mathfrak{B}_*} + \Sigma_{\mathfrak{B}^*}\}\Delta w(j/K)\hat{\Gamma}_{x_2 u_1}(j), \quad \textbf{(A.24)}$$

as in the proof of Lemma 1(a). The first sum in (A.24) is

$$\sum_{|j|\le K^*} \Delta w(j/K)\hat{\Gamma}_{x_2 u_1}(j) = K^{-1}w''(0)\sum_{|j|\le K^*} K^{-1}(j - \tfrac{1}{2})\hat{\Gamma}_{x_2 u_1}(j)(1 + o(1)),$$

upon the second-order Taylor expansion of $\Delta w(j/K)$. We now use the result established in the proof of Theorem 3.1 of Phillips (1991b, p. 431) to show that the limit of the preceding expression is

$$w''(0)\left\{\int_0^1 r\, dr\int_0^1 B_2\, dB_1' + \sum_{j=-\infty}^{\infty} (j - 1/2)\Omega_{21}(j)\right\},$$

where $\Omega_{21}(j) = \sum_{i=0}^{\infty} E(u_{20}u_{1j+i}') = \sum_{l=i}^{\infty} E(u_{20}u_{1l}')$, and this implies that the first sum in (A.24) is $O_p(1)$. We can similarly analyze the second sum in (A.24), for $(j-1)/K < \theta_j < j/K$, as

$$\begin{aligned}\sum_{|j|>K^*} \Delta w(j/K)\hat{\Gamma}_{x_2u_1}(j) &= K^{-1}\sum_{|j|>K^*} w'(\theta_j)\hat{\Gamma}_{x_2u_1}(j)\\ &\le K^{-1}\left\{\sup \sum_{|j|<K} w'(\theta_j)\right\}\sum_{|j|>K^*}\hat{\Gamma}_{x_2u_1}(j)\\ &= O_p(K^{-1}),\end{aligned}$$

using again the aforementioned result in Phillips (1991b). We deduce that

$$\hat{\Omega}_{x_2\Delta u_1} = O_p(1). \qquad \textbf{(A.25)}$$

It follows from Lemma 2(j), (A.22), and (A.25) that

$$^+\hat{\Omega}_{x_2u_b} = \hat{\Omega}_{x_2u_b} - \hat{\Omega}_{x_2\Delta u_b} = O_p(K). \qquad \textbf{(A.26)}$$

Similarly,

$$\begin{aligned}\hat{\Omega}_{x_3\Delta u_1} &= \sum_{j=-K+1}^{K-1} w(j/K)\hat{\Gamma}_{x_3\Delta u_1}(j)\\ &= \sum_{j=-K+1}^{K-1} w(j/K)\{\hat{\Gamma}_{x_3u_1}(j) - \hat{\Gamma}_{x_3u_1}(j+1)\}\\ &= \sum_{j=-K+2}^{K-1}\{w(j/K) - w((j-1)/K)\}\hat{\Gamma}_{x_3u_1}(j)\\ &\quad + w((K-1)/K)\hat{\Gamma}_{x_3u_1}(K) + w((-K+1)/K)\hat{\Gamma}_{x_3u_1}(-K+1)\\ &= \sum_{j=-K+2}^{K-1}\Delta w(j/K)\hat{\Gamma}_{x_3u_1}(j) + O_p(K^{-2}T),\end{aligned}$$

because $T^{-1}\hat{\Gamma}_{x_3u_1}(i) = T^{-2}\sum' x_{3t}u_{1t-i}' = O_p(1)$ for $i = K, -K+1$. The sum in the preceding equation now involves sample autocovariances that are defined with I(2) variables. We can follow the preceding analysis for $\hat{\Omega}_{x_2\Delta u_1}$. As in (A.24), we first decompose the sum as

$$\sum_{j=-K+2}^{K-1}\Delta w(j/K)\hat{\Gamma}_{x_3u_1}(j) = \{\Sigma_{\mathcal{B}_*} + \Sigma_{\mathcal{B}^*}\}\Delta w(j/K)\hat{\Gamma}_{x_3u_1}(j). \qquad \textbf{(A.27)}$$

The first sum in (A.27) times T^{-1} is

$$\begin{aligned}&\sum_{|j|\le K^*}\Delta w(j/K)\underbrace{T^{-1}\hat{\Gamma}_{x_3u_1}(j)}\\ &\quad = K^{-1}w''(0)\sum_{|j|\le K^*}K^{-1}(j-\tfrac{1}{2})\underbrace{T^{-1}\hat{\Gamma}_{x_3u_1}(j)}(1+o(1))\\ &\quad \xrightarrow{\mathcal{D}} w''(0)\int_0^1 r\,dr\int_0^1 \bar{B}_3\,dB_1',\end{aligned}$$

where we use the analytic framework employed in the proof of Theorem 3.1 of Phillips (1991b) to find the limit behavior of the underbraced term. Next, we consider the second sum in (A.27) by expanding $\Delta w(j/K)$ around $\theta_j \in ((j-1)/K, j/K)$ and using the preceding asymptotic result as follows:

$$\begin{aligned}\sum_{|j|>K^*} \Delta w(j/K)\hat{\Gamma}_{x_3u_1}(j) &= K^{-1} \sum_{|j|>K^*} w'(\theta_j)\hat{\Gamma}_{x_3u_1}(j) \\ &\le TK^{-1}\left\{\sup \sum_{|j|<K} w'(\theta_j)\right\} \sum_{|j|>K^*} T^{-1}\hat{\Gamma}_{x_3u_1}(j) \\ &= O_p(TK^{-1}).\end{aligned}$$

Combining the preceding results leads to

$$\hat{\Omega}_{x_3\Delta u_1} = O_p(T). \tag{A.28}$$

By using the results in Lemma 2(j), (A.23), and (A.28), we can similarly show that

$$^+\hat{\Omega}_{x_3u_b} = \hat{\Omega}_{x_3u_b} - \hat{\Omega}_{x_3\Delta u_b} = O_p(TK) \tag{A.29}$$

and

$$^+\hat{\Omega}_{x_2\Delta u_1} = O_p(1), \qquad ^+\hat{\Omega}_{x_3\Delta u_1} = O_p(T). \tag{A.30}$$

Now we consider

$$(A-\hat{A})\hat{\Omega}_{xv_h} = (A_1-\hat{A}_1)\hat{\Omega}_{u_1v_h} + (A_2-\hat{A}_2)\hat{\Omega}_{x_2v_h} + (A_3-\hat{A}_3)\hat{\Omega}_{x_3v_h},$$

where

$$\begin{aligned}\hat{\Omega}_{u_1v_h} &= \hat{\Omega}_{u_1u_h^*} + {}^+\hat{\Omega}_{u_1u_h}G_h' \\ &= \begin{pmatrix}\hat{\Omega}_{u_1\Delta u_1}\\ \hat{\Omega}_{u_1u_b}\end{pmatrix} + \begin{pmatrix}\hat{\Omega}_{u_1\Delta u_1}\\ \hat{\Omega}_{u_1u_2}\end{pmatrix}G_h' \\ &= \begin{pmatrix}0\\ \hat{\Omega}_{u_1u_b}\end{pmatrix} + \begin{pmatrix}0\\ \hat{\Omega}_{u_1u_2}\end{pmatrix}G_h' + O_p(K^{-2}) + O_p(1/\sqrt{TK}),\\ \hat{\Omega}_{x_2v_h} &= \hat{\Omega}_{x_2u_h^*} + {}^+\hat{\Omega}_{x_2u_h}G_h' \\ &= \begin{pmatrix}\hat{\Omega}_{x_2\Delta u_1}\\ \hat{\Omega}_{x_2u_b}\end{pmatrix} + \begin{pmatrix}^+\hat{\Omega}_{x_2\Delta u_1}\\ ^+\hat{\Omega}_{x_2u_2}\end{pmatrix}G_h' \\ &= \begin{pmatrix}O_p(1)\\ O_p(K)\end{pmatrix} + \begin{pmatrix}O_p(1)\\ O_p(K)\end{pmatrix}G_h',\\ \hat{\Omega}_{x_3v_h} &= \hat{\Omega}_{x_3u_h^*} + {}^+\hat{\Omega}_{x_3u_h}G_h' \\ &= \begin{pmatrix}\hat{\Omega}_{x_3\Delta u_1}\\ \hat{\Omega}_{x_3u_b}\end{pmatrix} + \begin{pmatrix}^+\hat{\Omega}_{x_3\Delta u_1}\\ ^+\hat{\Omega}_{x_3u_2}\end{pmatrix}G_h' \\ &= \begin{pmatrix}O_p(T)\\ O_p(TK)\end{pmatrix} + \begin{pmatrix}O_p(T)\\ O_p(TK)\end{pmatrix}G_h',\end{aligned}$$

by the results in (A.21), (A.26), and (A.28)–(A.30), and this finally gives

$$(A - \hat{A})\hat{\Omega}_{xv_h} = O_p(T^{-1/2}) + O_p(K/T). \qquad \textbf{(A.31)}$$

From the results in parts (e) and (i) of Lemma 2, (A.29), and (A.30), we also have

$$\begin{aligned}(A - \hat{A})^+\hat{\Omega}_{x\Delta u_1} &= (A_1 - \hat{A}_1)^+\hat{\Omega}_{u_1\Delta u_1} + (A_2 - \hat{A}_2)^+\hat{\Omega}_{x_2\Delta u_1} \\ &\quad + (A_3 - \hat{A}_3)^+\hat{\Omega}_{x_3\Delta u_1} \\ &= O_p(K^{-2}T^{-1/2}) + O_p(T^{-1})\end{aligned}$$

and

$$\begin{aligned}(A - \hat{A})^+\hat{\Omega}_{xu_2} &= (A_1 - \hat{A}_1)^+\hat{\Omega}_{u_1u_2} + (A_2 - \hat{A}_2)^+\hat{\Omega}_{x_2u_2} + (A_3 - \hat{A}_3)^+\hat{\Omega}_{x_3u_2} \\ &= O_p(T^{-1/2}) + O_p(K/T).\end{aligned}$$

We then deduce from results in (A.21) and (A.31) and those in the preceding equations that

$$\hat{\Omega}_{\hat{u}_0\hat{v}_h} = \hat{\Omega}_{u_0v_h} + O_p(K^{-2}) + O_p(KT^{-1/2}),$$

as required for part (e). ∎

Proof of Lemma 4.

(a) We start by considering

$$\begin{aligned}\hat{\Delta}_{\hat{u}_0\Delta u_1} &= \sum_{j=0}^{K-1} w(j/K)\hat{\Gamma}_{\hat{u}_0\Delta u_1}(j) \\ &= \sum_{j=0}^{K-1} w(j/K)\{\hat{\Gamma}_{\hat{u}_0u_1}(j) - \hat{\Gamma}_{\hat{u}_0u_1}(j+1)\} \\ &= \sum_{j=1}^{K-1} \{w(j/K) - w((j-1)/K)\}\hat{\Gamma}_{\hat{u}_0u_1}(j) \\ &\quad + w((K-1)/K)\hat{\Gamma}_{\hat{u}_0u_1}(K) + w(0)\hat{\Gamma}_{\hat{u}_0u_1}(0).\end{aligned}$$

Note that

$$\hat{\Gamma}_{\hat{u}_0u_1}(0) = T^{-1}\hat{U}_0'U_1 = T^{-1}\hat{U}_0'X_1 = 0$$

by least-squares orthogonality. We also have $w(0) = 1$ and $w((K-1)/K) = O(K^{-2})$ by Assumption 2. In addition, we have

$$\hat{\Gamma}_{\hat{u}_0u_1}(K) = O_p(T^{-1/2}) + E(\hat{\Gamma}_{\hat{u}_0u_1}(K)) = O_p(T^{-1/2}),$$

by (A.2) and the preceding result. Then,

$$\hat{\Delta}_{\hat{u}_0\Delta u_1} = \sum_{j=1}^{K-1} \Delta w(j/K)\hat{\Gamma}_{\hat{u}_0u_1}(j) + O_p(K^{-2}T^{-1/2}), \qquad \textbf{(A.32)}$$

where

$$\hat{\Gamma}_{\hat{u}_0u_1}(j) = \hat{\Gamma}_{u_0u_1}(j) + (A - \hat{A})\hat{\Gamma}_{xu_1}(j).$$

The first term in (A.32) is

$$\sum_{j=1}^{K-1} \Delta w(j/K)\hat{\Gamma}_{u_0u_1}(j) + \sum_{j=1}^{K-1} \Delta w(j/K)(A-\hat{A})HH'\hat{\Gamma}_{xu_1}(j). \tag{A.33}$$

We use the result in Lemma 1(c) to express the first term in (A.33) as

$$K^{-2}w''(0)\sum_{j=1}^{\infty}(j-\tfrac{1}{2})\Gamma_{u_0u_1}(j) + O_p(1/\sqrt{TK}) = 0 + O_p(1/\sqrt{TK}), \tag{A.34}$$

where the mean in the preceding expression is 0 because $\Gamma_{u_0u_1}(j)=0$, $\forall j \geq 0$ by Assumption 1(c). The second term in (A.33) can be decomposed as

$$(A_1-\hat{A}_1)\sum_{j=1}^{K-1}\Delta w(j/K)\hat{\Gamma}_{u_1u_1}(j) + (A_2-\hat{A}_2)\sum_{j=1}^{K-1}\Delta w(j/K)\hat{\Gamma}_{x_2u_1}(j) + (A_3-\hat{A}_3)\sum_{j=1}^{K-1}\Delta w(j/K)\hat{\Gamma}_{x_3u_1}(j). \tag{A.35}$$

The sum in the first term in (A.35) is $\{O_p(K^{-2})+O_p(1/\sqrt{TK})\}$ by Lemma 1(c). By following exactly the same line of the analyses as in the proof of Lemma 3(e) in deriving (A.25) and (A.28), we can establish that the one-sided sums in the second and the third terms in (A.35) are of $O_p(1)$ and of $O_p(T)$, respectively. Then, (A.35) reduces to

$$O_p(T^{-1/2}K^{-2}) + O_p(T^{-1}K^{-1/2}) + O_p(T^{-1}).$$

We finally deduce from (A.32)–(A.35) and the preceding result that

$$\hat{\Delta}_{\hat{u}_0\Delta u_1} = O_p(1/\sqrt{TK}),$$

as required.

(b)

$$\begin{aligned}
\hat{\Delta}_{\hat{u}_0u_2} &= \sum_{j=0}^{K-1} w(j/K)\hat{\Gamma}_{\hat{u}_0u_2}(j)\\
&= \sum_{j=0}^{K-1} w(j/K)\hat{\Gamma}_{u_0u_2}(j) + \sum_{j=0}^{K-1} w(j/K)(A-\hat{A})HH'\hat{\Gamma}_{xu_2}(j)\\
&= \hat{\Delta}_{u_0u_2} + (A_1-\hat{A}_1)\sum_{j=0}^{K-1} w(j/K)\hat{\Gamma}_{u_1u_2}(j)\\
&\quad + (A_2-\hat{A}_2)\sum_{j=0}^{K-1} w(j/K)\hat{\Gamma}_{x_2u_2}(j)\\
&\quad + (A_3-\hat{A}_3)\sum_{j=0}^{K-1} w(j/K)\hat{\Gamma}_{x_3u_2}(j)\\
&= \hat{\Delta}_{u_0u_2} + (A_1-\hat{A}_1)\hat{\Delta}_{u_1u_2} + (A_2-\hat{A}_2)\hat{\Delta}_{x_2u_2} + (A_3-\hat{A}_3)\hat{\Delta}_{x_3u_2}.
\end{aligned}$$

Now,

$$\hat{\Delta}_{u_0 u_2} = \sum_{j=0}^{K-1} w(j/K)\hat{\Gamma}_{u_0 u_2}(j) = \Delta_{02} + O_p((K/T)^{1/2}) \tag{A.36}$$

because its mean is

$$\sum_{j=0}^{K-1} w(j/K)(1-|j|/T)\Gamma_{u_0 u_2}(j) \to \sum_{j=0}^{\infty} \Gamma_{u_0 u_2}(j) = \Delta_{02}$$

and its variance matrix is of order $O(K/T)$. Similarly, we have

$$\hat{\Delta}_{u_1 u_2} = \sum_{j=0}^{K-1} w(j/K)\hat{\Gamma}_{u_1 u_2}(j) = \Delta_{12} + O_p((K/T)^{1/2}). \tag{A.37}$$

We also have from Theorem 3.1 in Phillips (1991b)

$$K^{-1}\hat{\Delta}_{x_2 u_2} = K^{-1}\sum_{j=0}^{K-1} w(j/K)\hat{\Gamma}_{x_2 u_2}(j) \xrightarrow{\mathfrak{D}} \nu^* \int_0^1 B_2\, dB_2' + \Delta_{22}, \tag{A.38}$$

where $\nu^* = \int_0^1 w(s)d(s)$. Similarly,

$$\begin{aligned} T^{-1}K^{-1}\hat{\Delta}_{x_3 u_2} &= K^{-1}\sum_{j=0}^{K-1} w(j/K)T^{-1}\hat{\Gamma}_{x_3 u_2}(j) \\ &\xrightarrow{\mathfrak{D}} \nu^* \int_0^1 \bar{B}_3\, dB_2', \end{aligned}$$

giving

$$\hat{\Delta}_{x_3 u_2} = O_p(TK). \tag{A.39}$$

The results in (A.36)–(A.39) then give

$$\hat{\Delta}_{\hat{u}_0 u_2} = \Delta_{02} + O_p(K/T)^{1/2},$$

which is required.

(c)–(d)

$$\begin{aligned} \hat{\Delta}_{\hat{u}_0 \Delta x_3} &= \sum_{j=0}^{K-1} w(j/K)\hat{\Gamma}_{\hat{u}_0 \Delta x_3}(j) \\ &= \hat{\Delta}_{u_0 \Delta x_3} + \sum_{j=0}^{K-1} w(j/K)(A-\hat{A})HH'\hat{\Gamma}_{x \Delta x_3}(j) \\ &= \hat{\Delta}_{u_0 \Delta x_3} + (A_1 - \hat{A}_1)\hat{\Delta}_{u_1 \Delta x_3} + (A_2 - \hat{A}_2)\hat{\Delta}_{x_2 \Delta x_3} \\ &\quad + (A_3 - \hat{A}_3)\hat{\Delta}_{x_3 \Delta x_3}. \end{aligned}$$

From (A.38), $\hat{\Delta}_{u_0 \Delta x_3}$ and $\hat{\Delta}_{u_1 \Delta x_3}$ are $O_p(K)$ because Δx_{3t} is also I(1). Note that

$$T^{-1}K^{-1}\hat{\Delta}_{x_2\Delta x_3} = K^{-1}\sum_{j=0}^{K-1} w(j/K)\left\{T^{-2}{\sum}' x_{2t}\Delta x'_{3t-j}\right\}$$

$$\xrightarrow{\mathfrak{D}} \nu^* \int_0^1 B_2 B_3',$$

$$T^{-2}K^{-1}\hat{\Delta}_{x_3\Delta x_3} = K^{-1}\sum_{j=0}^{K-1} \left\{T^{-3}{\sum}' x_{3t}\Delta x'_{3t-j}\right\}$$

$$\xrightarrow{\mathfrak{D}} \nu^* \int_0^1 \bar{B}_3 B_3'. \qquad \textbf{(A.40)}$$

Therefore,

$$\hat{\Delta}_{\hat{u}_0\Delta x_3} = O_p(K),$$

and this establishes the result in part (c).

The result in part (d) immediately follows from (A.38).

(e)–(h) Proofs for parts (e)–(h) are exactly the same as those given for parts (e)–(g) and (j) and Lemma 8.1 in Phillips (1995).

(i) We first note that

$$T^{-1}\Delta U_1' X_3 = T^{-1}\{u_{1T}x_{3T}' - U_{1-1}'\Delta X_3\} = T^{-1}u_{1T}x_{3T}' - \hat{\Gamma}_{u_1\Delta x_3}(-1). \qquad \textbf{(A.41)}$$

Then, consider

$$\hat{\Delta}_{\Delta u_1\Delta x_3} = \sum_{j=0}^{K-1} w(j/K)\hat{\Gamma}_{\Delta u_1\Delta x_3}(j)$$

$$= \sum_{j=0}^{K-1} w(j/K)\{\hat{\Gamma}_{u_1\Delta x_3}(j) - \hat{\Gamma}_{u_1\Delta x_3}(j-1)\}$$

$$= \sum_{j=0}^{K-2}\{w(j/K) - w((j+1)/K)\}\hat{\Gamma}_{u_1\Delta x_3}(j)$$

$$+ w((K-1)/K)\hat{\Gamma}_{u_1\Delta x_3}(K-1) - w(0)\hat{\Gamma}_{u_1\Delta x_3}(-1)$$

$$= -\sum_{j=0}^{K-2}\Delta w((j+1)/K)\hat{\Gamma}_{u_1\Delta x_3}(j) - \hat{\Gamma}_{u_1\Delta x_3}(-1) + O_p(K^{-2}),$$

because

$$w((K-1)/K)\hat{\Gamma}_{u_1\Delta x_3}(K-1) = w((K-1)/K)T^{-1}{\sum}' u_{1t}\Delta x'_{3t-(K-1)}$$

$$= O(K^{-2})O_p(1) = O_p(K^{-2}).$$

Then we have

$$T^{-1}\Delta U_1' X_3 - \hat{\Delta}_{\Delta u_1\Delta x_3} = T^{-1}u_{1T}x_{3T}' + \sum_{j=0}^{K-2}\Delta w((j+1)/K)\hat{\Gamma}_{u_1\Delta x_3}(j)$$

$$+ O_p(K^{-2}).$$

Note that the second term in the preceding expression is $O_p(1)$ as discussed in part (a). This then leads to the required result as

$$T^{-2}\Delta U_1' X_3 - T^{-1}\hat{\Delta}_{\Delta u_1 \Delta x_3} = O_p(T^{-1/2}).$$

(j) The result in part (j) follows directly from (A.38) and the results in Lemma 2.1 of Park and Phillips (1989). ■

Proof of Lemma 5.

(a) We do the following analysis just as in part (c) of Lemma 4 but now with the one-sided sum as

$$\begin{aligned} {}^{-}\hat{\Delta}_{\Delta u_1 \Delta u_1} &= \sum_{j=0}^{K-1} w(j/K)\hat{\Gamma}_{\Delta u_1 \Delta u_1}(j-1) \\ &= -\sum_{j=0}^{K-1} w(j/K)\Delta^2 \hat{\Gamma}_{u_1 u_1}(j) \\ &= -\sum_{j=0}^{K-3} \Delta^2 w((j+2)/K)\hat{\Gamma}_{u_1 u_1}(j) + \eta, \end{aligned}$$

where

$$\begin{aligned} \eta = &-w((K-1)/K)\{-2\hat{\Gamma}_{u_1 u_1}(K-2) + \hat{\Gamma}_{u_1 u_1}(K-1)\} \\ &- w((K-2)/K)\hat{\Gamma}_{u_1 u_1}(K-2) - w(1/K)\hat{\Gamma}_{u_1 u_1}(-1) \\ &- w(0)\{\hat{\Gamma}_{u_1 u_1}(-2) - 2\hat{\Gamma}_{u_1 u_1}(-1)\}. \end{aligned}$$

Note that by using (A.5) we have

$$\begin{aligned} \eta &= -\hat{\Gamma}_{u_1 u_1}(-2) + \hat{\Gamma}_{u_1 u_1}(-1) - \Delta w(1/K)\hat{\Gamma}_{u_1 u_1}(-1) + o_p(K^{-2}) \\ &= -\hat{\Gamma}_{u_1 \Delta u_1}(-2) - \Delta w(1/K)\hat{\Gamma}_{u_1 u_1}(-1) + o_p(K^{-2}). \end{aligned}$$

Also note that

$$\begin{aligned} T^{-1}\Delta U_{1-1}' U_1 &= T^{-1} u_{1T-1} u_{1T} - T^{-1} U_{1-2}' \Delta U_1 \\ &= O_p(T^{-1}) - \hat{\Gamma}_{u_1 \Delta u_1}(-2). \end{aligned}$$

Then, $\{T^{-1}\Delta U_{1-1}' U_1 - {}^{-}\hat{\Delta}_{\Delta u_1 \Delta u_1}\}$ is simply

$$\sum_{j=0}^{K-3} \Delta^2 w((j+2)/K)\hat{\Gamma}_{u_1 u_1}(j) + \Delta w(1/K)\hat{\Gamma}_{u_1 u_1}(-1) + O_p(T^{-1}) + o_p(K^{-2}). \tag{A.42}$$

Now we use the result in part (d) of Lemma 1 for the first term in (A.42) and write it as

$$K^{-2} w''(0) \sum_{j=0}^{\infty} \Gamma_{u_1 u_1}(j) + O_p(1/\sqrt{TK^3}).$$

For the second term in (A.42), we first do the following first-order Taylor expansion as

$$\Delta w(1/K) = w'(0)(1/K) + (\tfrac{1}{2})w''(0)(1/K)^2 + o(K^{-2})$$
$$= (\tfrac{1}{2})K^{-2}w''(0)(1 + o(1)),$$

where we use $w'(0) = 0$. Then, the mean of the second term is

$$(\tfrac{1}{2})K^{-2}w''(0)\Gamma_{u_1u_1}(-1),$$

as $K \to \infty$, and the variance matrix of the second term is $O_p(K^{-2}T^{-1/2})$ in view of Assumption 2(b) and (A.4). From all the preceding results, (A.42) finally reduces to

$$K^{-2}w''(0)\{\Delta_{11} + (\tfrac{1}{2})\Gamma_{u_1u_1}(-1)\} + O_p(1/\sqrt{TK^3}) + O_p(T^{-1}),$$

which is required for part (a).

(b) We start by considering

$$\begin{aligned}
{}^-\hat{\Delta}_{u_2\Delta u_1} &= \sum_{j=0}^{K-1} w(j/K)\hat{\Gamma}_{u_2\Delta u_1}(j-1) \\
&= \sum_{j=0}^{K-1} w(j/K)\{\hat{\Gamma}_{u_2u_1}(j-1) - \hat{\Gamma}_{u_2u_1}(j)\} \\
&= \sum_{j=0}^{K-2} \{w((j+1)/K) - w(j/K)\}\hat{\Gamma}_{u_2u_1}(j) \\
&\quad - w((K-1)/K)\hat{\Gamma}_{u_2u_1}(K-1) + \hat{\Gamma}_{u_2u_1}(-1) \\
&= \sum_{j=0}^{K-2} \Delta w((j+1)/K)\hat{\Gamma}_{u_2u_1}(j) + \hat{\Gamma}_{u_2u_1}(-1) + o_p(K^{-2}).
\end{aligned}$$

We then apply the result in Lemma 1(c) with $l(j) = j+1$ to the sum in the last line, and doing this we get the following expression for ${}^-\hat{\Delta}_{u_2\Delta u_1}$:

$$K^{-2}w''(0)\sum_{j=0}^{\infty}(j+\tfrac{1}{2})\Gamma_{u_2u_1}(j) + \hat{\Gamma}_{u_2u_1}(-1) + O_p(1/\sqrt{TK}) + o_p(K^{-2}). \tag{A.43}$$

Then, it immediately follows that $\{T^{-1}U'_{2-1}U_1 - {}^-\hat{\Delta}_{u_2\Delta u_1}\}$ is

$$\begin{aligned}
&-\sum_{j=0}^{K-2}\Delta w((j+1)/K)\hat{\Gamma}_{u_2u_1}(j) + o_p(K^{-2}) \\
&\quad = -K^{-2}w''(0)\sum_{j=0}^{\infty}(j+\tfrac{1}{2})\Gamma_{u_2u_1}(j) + O_p(1/\sqrt{TK}) + o_p(K^{-2}),
\end{aligned}$$

as required.

(c) To prove part (c), we first open the difference operator in the following expression as in (A.41), that is,

$$T^{-1}\Delta U'_{1-1}X_2 = T^{-1}\{u_{1T-1}x'_{2T} - U'_{1-2}\Delta X_2\} = T^{-1}u_{1T-1}x'_{2T} - \hat{\Gamma}_{u_1u_2}(-2);$$

and note that

$$
\begin{aligned}
{}^{-}\hat{\Delta}_{\Delta u_1 u_2} &= \sum_{j=0}^{K-1} w(j/K)\hat{\Gamma}_{\Delta u_1 u_2}(j-1) \\
&= \sum_{j=0}^{K-1} w(j/K)\{\hat{\Gamma}_{u_1 u_2}(j-1) - \hat{\Gamma}_{u_1 u_2}(j-2)\} \\
&= \sum_{j=-1}^{K-3} \{w((j+1)/K) - w((j+2)/K)\}\hat{\Gamma}_{u_1 u_2}(j) \\
&\quad + w((K-1)/K)\hat{\Gamma}_{u_1 u_2}(K-2) - \hat{\Gamma}_{u_1 u_2}(-2).
\end{aligned}
$$

Then, $\{T^{-1}\Delta U'_{1-1}X_2 - {}^{-}\hat{\Delta}_{\Delta u_1 u_2}\}$ is

$$
T^{-1}u_{1T-1}x'_{2T} + \sum_{j=-1}^{K-3} \Delta w((j+2)/K)\hat{\Gamma}_{u_1 u_2}(j) + o_p(K^{-2}).
$$

Applying the result in Lemma 1(c) with $l(j) = j+2$ to the second term, the preceding expression becomes

$$
T^{-1}u_{1T-1}x'_{2T} + \sum_{j=-1}^{\infty} (j+3/2)\Gamma_{u_1 u_2}(j) + o_p(K^{-2}).
$$

This, together with the fact that $T^{-1}u_{1T-1}x'_{2T} = O_p(T^{-1/2})$, establishes the required result.

(d) We begin by considering

$$
\begin{aligned}
{}^{-}\hat{\Delta}_{u_2 u_2} &= \sum_{j=0}^{K-1} w(j/K)\hat{\Gamma}_{u_2 u_2}(j-1) \\
&= \sum_{j=0}^{K-1} w(j/K)\{\hat{\Gamma}_{u_2 u_2}(j) - \hat{\Gamma}_{u_2 \Delta u_2}(j-1)\} \\
&= \hat{\Delta}_{u_2 u_2} + {}^{-}\hat{\Delta}_{u_2 \Delta u_2},
\end{aligned}
$$

just as in the analysis given in the proof of Lemma 2(k). Then, by the result in (A.43), we have

$$
{}^{-}\hat{\Delta}_{u_2 u_2} = \hat{\Delta}_{u_2 u_2} + \hat{\Gamma}_{u_2 u_2}(-1) + O_p(K^{-2}) + O_p(1/\sqrt{TK}). \tag{A.44}
$$

Next, we consider

$$
\begin{aligned}
T^{-1}U'_{2-1}X_2 &= T^{-1}U'_2X_2 - T^{-1}\Delta U'_2 X_2 \\
&= T^{-1}U'_2X_2 - T^{-1}\{u_{2T}x'_{2T} - U'_{2-1}\Delta X_2\} \\
&= T^{-1}U'_2X_2 - T^{-1}u_{2T}x'_{2T} - \hat{\Gamma}_{u_2 u_2}(-1),
\end{aligned}
$$

as $\Delta x_{2t} = u_{2t}$. Then it follows that

$$
\begin{aligned}
T^{-1}U'_{2-1}X_2 - {}^{-}\hat{\Delta}_{u_2 u_2} &= N_{22T} - \hat{\Delta}_{u_2 u_2} - T^{-1}u_{2T}x'_{2T} + O_p(K^{-2}) \\
&\quad + O_p(1/\sqrt{TK}),
\end{aligned}
$$

where

$$N_{22T} := T^{-1}U_2'X_2 - \hat{\Delta}_{u_2u_2} \xrightarrow{\mathfrak{D}} \int_0^1 dB_2B_2'.$$

This proves the result in part (d).

(e) We note that

$$T^{-1}U_{1-1}'X_3 = T^{-1}\{u_{1T-1}x_{3T}' - U_{1-2}'\Delta X_3\} = T^{-1}u_{1T-1}x_{3T}' - \hat{\Gamma}_{u_1\Delta x_3}(-2)$$

and

$$\begin{aligned}\hat{\Delta}_{\Delta u_1\Delta x_3} &= \sum_{j=0}^{K-1} w(j/K)\hat{\Gamma}_{\Delta u_1\Delta x_3}(j-1)\\ &= \sum_{j=0}^{K-1} w(j/K)\{\hat{\Gamma}_{u_1\Delta x_3}(j-1) - \hat{\Gamma}_{u_1\Delta x_3}(j-2)\}\\ &= \sum_{j=-1}^{K-3}\{w((j+1)/K) - w((j+2)/K)\}\hat{\Gamma}_{u_1\Delta x_3}(j)\\ &\quad + w((K-1)/K)\hat{\Gamma}_{u_1\Delta x_3}(K-2) - \hat{\Gamma}_{u_1\Delta x_3}(-2).\end{aligned}$$

Then we have

$$T^{-1}U_{1-1}'X_3 - \hat{\Delta}_{\Delta u_1\Delta x_3} = \underbrace{\sum_{j=-1}^{K-3}\Delta w((j+2)/K)\hat{\Gamma}_{u_1\Delta x_3}(j)} + O_p(\sqrt{T})$$

because

$$\begin{aligned}&T^{-1}u_{1T-1}x_{3T}' - w((K-1)/K)\hat{\Gamma}_{u_1\Delta x_3}(K-2)\\ &\quad = \sqrt{T}\{u_{1T-1}T^{-3/2}x_{3T}\} + w(1-1/K)T^{-1}\sum{}' u_{1t}\Delta x_{3t-j}'\\ &\quad = O_p(\sqrt{T}) + O(K^{-2})O_p(1) = O_p(\sqrt{T}).\end{aligned}$$

We can follow the analysis given for the derivation of (A.25) in the proof of Lemma 3(e), with the lower limit of summation replaced with $j=-1$, to show that the preceding underbraced term is $O_p(1)$; this then gives

$$T^{-2}\Delta U_{1-1}'X_3 - T^{-1}\,{}^{-}\hat{\Delta}_{\Delta u_1\Delta x_3} = O_p(T^{-1/2}),$$

as required.

(f) Following the analysis given in the proof for part (d), we can decompose $T^{-1}U_{2-1}'X_3$ and ${}^{-}\hat{\Delta}_{u_2\Delta x_3}$ as follows:

$$\begin{aligned}T^{-1}U_{2-1}'X_3 &= T^{-1}U_2'X_3 - T^{-1}\Delta U_2'X_3\\ &= T^{-1}U_2'X_3 - T^{-1}\{u_{2T}x_{3T}' - U_{2-1}'\Delta X_3\}\\ &= T^{-1}U_2'X_3 - T^{-1}u_{2T}x_{3T}' + \hat{\Gamma}_{u_2\Delta x_3}(-1)\end{aligned}$$

and

$$^{-}\hat{\Delta}_{u_2 \Delta x_3} = \hat{\Delta}_{u_2 \Delta x_3} + {}^{-}\hat{\Delta}_{u_2 \Delta(\Delta x_3)} = \hat{\Delta}_{u_2 \Delta x_3} + {}^{-}\hat{\Delta}_{u_2 u_3}.$$

By the results given in Lemma 4(d) and (A.44), we have $\hat{\Delta}_{u_2 \Delta x_3} = O_p(K)$ and ${}^{-}\hat{\Delta}_{u_2 u_3} = O_p(1)$. Then it follows that

$$\begin{aligned} &T^{-2} U_{2-1}' X_3 - T^{-1}\,{}^{-}\hat{\Delta}_{u_2 \Delta x_3} \\ &\quad = T^{-2} U_2' X_3 - T^{-2} u_{2T} x_{3T}' + T^{-1} \hat{\Gamma}_{u_2 \Delta x_3}(-1) + O_p(K/T) \\ &\quad = \bar{N}_{23T} + O_p(T^{-1/2}) + O_p(K/T), \end{aligned}$$

where we introduce the following notation:

$$\bar{N}_{23T} := T^{-2} U_2' X_3 \xrightarrow{\mathfrak{D}} \int_0^1 dB_2 \bar{B}_3'.$$

This establishes the result in part (f). ■

Proof of Lemma 6.

(a) To prove part (a), we first write $\hat{v}_{ht}$ in matrix notations as

$$\hat{V}_h' = U_h^{*\prime} + \hat{G}_h U_{h-1}'.$$

We recall that $\hat{G}_h$, defined in (16), is $O_p(1)$. Then we have

$$\begin{aligned} &T^{-1} \hat{V}_h' X_1 - \hat{\Delta}_{\hat{v}_h \Delta u_1} \\ &\quad = T^{-1}\{U_h^{*\prime} X_1 + \hat{G}_h U_{h-1}' X_1\} - \{\hat{\Delta}_{u^* \Delta u_1} + \hat{G}_h \hat{\Delta}_{u_{h-1} \Delta u_1}\} \\ &\quad = \{T^{-1} U_h^{*\prime} X_1 - \hat{\Delta}_{u_h^* \Delta u_1}\} + \hat{G}_h\{T^{-1} U_{h-1}' X_1 - {}^{-}\hat{\Delta}_{u_h \Delta u_1}\} \\ &\quad = \begin{pmatrix} T^{-1} \Delta U_1' U_1 - \hat{\Delta}_{\Delta u_1 \Delta u_1} \\ T^{-1} U_b' U_1 - \hat{\Delta}_{u_b \Delta u_1} \end{pmatrix} + \hat{G}_h \begin{pmatrix} T^{-1} \Delta U_{1-1}' U_1 - {}^{-}\hat{\Delta}_{\Delta u_1 \Delta u_1} \\ T^{-1} U_{2-1}' U_1 - {}^{-}\hat{\Delta}_{u_2 \Delta u_1} \end{pmatrix} \\ &\quad = \begin{pmatrix} O_p(K^{-2}) + O_p(1/\sqrt{TK^3}) \\ O_p(K^{-2}) + O_p(1/\sqrt{TK}) \end{pmatrix} + O_p(1) \begin{pmatrix} O_p(K^{-2}) + O_p(1/\sqrt{TK^3}) \\ O_p(K^{-2}) + O_p(1/\sqrt{TK}) \end{pmatrix} \\ &\quad = O_p(K^{-2}) + O_p(1/\sqrt{TK}). \end{aligned}$$

For the last two lines, we use the results in parts (e) and (f) of Lemma 4 and those in parts (a) and (b) of Lemma 5. This then proves the result in part (a).

(b) We recall that $\hat{G}_{h2}$ is the OLS estimator for the coefficient on Δx_{2t-1}, which is stationary, in the regression given in (16). Hence, we have the following result for the estimation error in $\hat{G}_{h2}$:

$$\sqrt{T}(\hat{G}_{h2} - G_{h2}) = O_p(1).$$

We also use the results in parts (g)–(j) of Lemma 4 and parts (c)–(f) of Lemma 5 in the following analysis. Now we consider

$$
\begin{aligned}
&(\hat{V}_h' X_b - T\hat{\Delta}_{\hat{v}_h \Delta x_b}) D_T^{-1} \\
&= (T^{-1}\hat{V}_h' X_2 - \hat{\Delta}_{\hat{v}_h u_2}, T^{-2}\hat{V}_h' X_3 - T^{-1}\hat{\Delta}_{\hat{v}_h \Delta x_3}) \\
&= \begin{pmatrix} T^{-1}\Delta U_1' X_2 - \hat{\Delta}_{\Delta u_1 u_2} & T^{-2}\Delta U_1' X_3 - T^{-1}\hat{\Delta}_{\Delta u_1 \Delta x_3} \\ T^{-1} U_b' X_2 - \hat{\Delta}_{u_b u_2} & T^{-2} U_b' X_3 - T^{-1}\hat{\Delta}_{u_b \Delta x_3} \end{pmatrix} \\
&\quad + \hat{G}_h \begin{pmatrix} T^{-1}\Delta U_{1-1}' X_2 - {}^{-}\hat{\Delta}_{\Delta u_1 u_2} & T^{-2}\Delta U_{1-1}' X_3 - T^{-1}{}^{-}\hat{\Delta}_{\Delta u_1 \Delta x_3} \\ T^{-1} U_{2-1}' X_2 - {}^{-}\hat{\Delta}_{u_2 u_2} & T^{-2} U_{2-1}' X_3 - T^{-1}{}^{-}\hat{\Delta}_{u_2 \Delta x_3} \end{pmatrix} \\
&= \begin{pmatrix} O_p(K^{-2}) + O_p(T^{-1/2}) & O_p(T^{-1/2}) \\ N_{b2T} & \bar{N}_{b3T} + O_p(K/T) \end{pmatrix} \\
&\quad + \hat{G}_h \begin{pmatrix} O_p(K^{-2}) + O_p(1/\sqrt{TK}) & O_p(T^{-1/2}) \\ N_{22T} + O_p(K^{-2}) + O_p(T^{-1/2}) & \bar{N}_{23T} + O_p(K/T) + O_p(T^{-1/2}) \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 \\ N_{b2T} & \bar{N}_{b3T} \end{pmatrix} + (\hat{G}_{h1}, \hat{G}_{h2}) \begin{pmatrix} 0 & 0 \\ N_{22T} & \bar{N}_{23T} \end{pmatrix} + O_p(K^{-2}) + O_p(K/T) \\
&\quad + O_p(T^{-1/2}) \\
&= \begin{pmatrix} 0 \\ \bar{N}_{bbT} \end{pmatrix} + \{G_{h2} + \overbrace{\hat{G}_{h2} - G_{h2}}\} \bar{N}_{2bT} + O_p(K^{-2}) + O_p(K/T) + O_p(T^{-1/2}) \\
&= \begin{pmatrix} 0 \\ \bar{N}_{bbT} \end{pmatrix} + \{G_{h2} + O_p(T^{-1/2})\} \bar{N}_{2bT} + O_p(K^{-2}) + O_p(K/T) + O_p(T^{-1/2}) \\
&= \begin{bmatrix} 0 & 0 \\ I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} 0 \\ \bar{N}_{2bT} \\ \bar{N}_{3bT} \end{bmatrix} + \begin{bmatrix} L_2 \\ M_2 \\ N_2 \end{bmatrix} \bar{N}_{2bT} + O_p(K^{-2}) + O_p(K/T) + O_p(T^{-1/2}) \\
&= \begin{bmatrix} L_2 & 0 \\ I + M_2 & 0 \\ N_2 & I \end{bmatrix} \begin{pmatrix} \bar{N}_{2bT} \\ \bar{N}_{3bT} \end{pmatrix} + O_p(K^{-2}) + O_p(K/T) + O_p(T^{-1/2}) \\
&= G\bar{N}_{bbT} + O_p(K^{-2}) + O_p(K/T) + O_p(T^{-1/2}),
\end{aligned}
$$

where we employ the following notation:

$$
\bar{N}_{bbT} := \begin{pmatrix} \bar{N}_{2bT} \\ \bar{N}_{3bT} \end{pmatrix} := \begin{pmatrix} N_{22T} & \bar{N}_{23T} \\ N_{32T} & \bar{N}_{33T} \end{pmatrix} \xrightarrow{\mathfrak{D}} \begin{pmatrix} \int_0^1 dB_2 B_2' & \int_0^1 dB_2 \bar{B}_3' \\ \int_0^1 dB_3 B_2' & \int_0^1 dB_3 \bar{B}_3' \end{pmatrix}. \tag{A.45}
$$

This establishes the result in part (b).

(c) Using the results in Lemma 3(c) and the result just established in part (a), we have

$$
\begin{aligned}
&T^{-1/2}\hat{\Omega}_{u_0 v_h}\hat{\Omega}_{v_h v_h}^{-1}(\hat{V}_h' U_1 - T\hat{\Delta}_{\hat{v}_h \Delta u_1}) \\
&\quad = T^{1/2}\hat{\Omega}_{u_0 v_h}\hat{\Omega}_{v_h v_h}^{-1}(T^{-1}\hat{V}_h' U_1 - \hat{\Delta}_{\hat{v}_h \Delta u_1}) \\
&\quad = T^{1/2}(O_p(1), O_p(1) + O_p(K^{3/2}T^{-1/2}))C'\begin{pmatrix} O_p(K^{-2}) + O_p(1/\sqrt{TK^3}) \\ O_p(K^{-2}) + O_p(1/\sqrt{TK}) \end{pmatrix} \\
&\quad = O_p(T^{1/2}K^{-2}) + O_p(KT^{-1/2}) + O_p(K^{-1/2}),
\end{aligned}
$$

as required.

(d) To prove part (d), we similarly use the results in Lemma 3(c) and in part (b) of this lemma as

$$
\begin{aligned}
&\hat{\Omega}_{u_0 v_h}\hat{\Omega}_{v_h v_h}^{-1}(\hat{V}_h' X_b - T\hat{\Delta}_{\hat{v}_h \Delta x_b})D_T^{-1} \\
&\quad = (\Omega_{0b}\Omega_{bb}^{-1}(G'G)^{-1/2} + o_p(1), O_p(1) + O_p(K^{3/2}T^{-1/2}))C' \\
&\qquad \times \{G\bar{N}_{bbT} + O_p(K/T) + O_p(K^{-2}) + O_p(T^{-1/2})\} \\
&\quad = \{(\Omega_{0b}\Omega_{bb}^{-1}(G'G)^{-1/2} + o_p(1), O_p(1) + O_p(K^{3/2}T^{-1/2}))C'G\bar{N}_{bbT}\} \\
&\qquad + O_p(T^{-1/2}) + O_p(K^{-2}) + O_p(K^{5/2}T^{-3/2}) + O_p(K^{3/2}/T).
\end{aligned}
$$

Note that

$$
C'G = \begin{pmatrix} \underset{\sim}{G}' \\ \underset{\sim}{G}_\perp' \end{pmatrix} \underset{\sim}{G}(G'G)^{1/2} = \begin{pmatrix} \underset{\sim}{G}'\underset{\sim}{G} \\ \underset{\sim}{G}_\perp'\underset{\sim}{G} \end{pmatrix}(G'G)^{1/2} = \begin{pmatrix} I \\ 0 \end{pmatrix}(G'G)^{1/2},
$$

directly from the definitions of $\underset{\sim}{G}$ given in (A.19) and $C = (\underset{\sim}{G}, \underset{\sim}{G}_\perp)$. Then, the term in braces in the last equation is

$$
\begin{aligned}
&(\Omega_{0b}\Omega_{bb}^{-1}(G'G)^{-1/2} + o_p(1), O_p(1) + O_p(K^{3/2}T^{-1/2}))\begin{pmatrix} (G'G)^{1/2} \\ 0 \end{pmatrix}\bar{N}_{bbT} \\
&\quad = (\Omega_{0b}\Omega_{bb}^{-1}(G'G)^{-1/2}(G'G)^{1/2} + o_p(1))\bar{N}_{bbT} \\
&\quad = \Omega_{0b}\Omega_{bb}^{-1}\bar{N}_{bbT} + o_p(1),
\end{aligned}
$$

and this proves the result in part (d). ■

APPENDIX B: PROOFS OF THE MAIN THEOREMS

Proof of Proposition 1. The estimation error in the OLS estimator $\hat{A}$ is $\hat{A} - A = U_0'X(X'X)^{-1}$, and its component submatrices are expressed as

$$
\begin{aligned}
(\hat{A} - A)H &= (\hat{A}_1 - A_1, \hat{A}_b - A_b) \\
&= U_0'X(X'X)^{-1}H \\
&= U_0'XH(H'X'XH)^{-1}H'H \\
&= U_0'X_h(X_h'X_h)^{-1}H'H,
\end{aligned}
$$

where $U_0'X_h = (U_0'X_1, U_0'X_b)$, and

$$(X_h'X_h)^{-1} = \begin{pmatrix} X_1'X_1 & X_1'X_b \\ X_b'X_1 & X_b'X_b \end{pmatrix}^{-1}$$
$$= \begin{pmatrix} (X_1'Q_bX_1)^{-1} & -(X_1'X_1)^{-1}X_1'X_b(X_b'Q_1X_b)^{-1} \\ -(X_b'X_b)^{-1}X_b'X_1(X_1'Q_bX_1)^{-1} & (X_b'Q_1X_b)^{-1} \end{pmatrix}, \tag{A.46}$$

by the partitioned matrix inversion formula. Note that

$$H'H = (H'H_1, H'H_b)$$
$$= \begin{array}{cc} \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} & \begin{array}{c} m_1 \\ (m_2+m_3) \end{array} \end{array}.$$

Then, from Lemma 2.1 of Park and Phillips (1989), we have

$$U_0'X_b^D = (T^{-1}U_0'X_2, T^{-2}U_0'X_3)$$
$$\xrightarrow{\mathfrak{D}} \left(\int_0^1 dB_0B_2' + \Delta_{02}, \int_0^1 dB_0\bar{B}_3'\right)$$
$$\equiv \int_0^1 dB_0\bar{B}_b' + \Delta_{0b},$$
$$X_1'X_b^D = (T^{-1}X_1'X_2, T^{-2}X_1'X_3)$$
$$\xrightarrow{\mathfrak{D}} \left(\int_0^1 dB_1B_2' + \Delta_{12}, \int_0^1 dB_1\bar{B}_3'\right)$$
$$\equiv \int_0^1 dB_1\bar{B}_b' + \Delta_{1b},$$

where $\Delta_{0b} = (\Delta_{02}, 0)$, $\Delta_{1b} = (\Delta_{12}, 0)$, and $\bar{B}_b' = (B_2', \bar{B}_3')$. Also,

$$(X_b^{D\prime}X_b^D)^{-1} = \begin{pmatrix} T^{-2}X_2'X_2 & T^{-3}X_2'X_3 \\ T^{-3}X_3'X_2 & T^{-4}X_3'X_3 \end{pmatrix}^{-1}$$
$$\xrightarrow{\mathfrak{D}} \begin{pmatrix} \int_0^1 B_2B_2' & \int_0^1 B_2\bar{B}_3' \\ \int_0^1 \bar{B}_3B_2' & \int_0^1 \bar{B}_3\bar{B}_3' \end{pmatrix}^{-1}$$
$$\equiv \left(\int_0^1 \bar{B}_b\bar{B}_b'\right)^{-1},$$

by the continuous mapping theorem.

(a) We can express $\sqrt{T}(\hat{A}-A)H_1$ as

$$
\begin{aligned}
&U_0'X_h(X_h'X_h)^{-1}H'H_1\\
&\quad = \sqrt{T}U_0'X_1(X_1'Q_bX_1)^{-1} - \sqrt{T}U_0'X_b(X_b'X_b)^{-1}X_b'X_1(X_1'Q_bX_1)^{-1}\\
&\quad = \left(\frac{U_0'X_1}{\sqrt{T}} - \frac{1}{\sqrt{T}}U_0'X_b^D(X_b^{D\prime}X_b^D)^{-1}X_b^{D\prime}X_1\right)\\
&\quad\quad \times \left(\frac{X_1X_1}{T} - \frac{1}{T}X_1'X_b^D(X_b^{D\prime}X_b^D)^{-1}X_b^{D\prime}X_1\right)^{-1}\\
&\quad = \frac{U_0'X_1}{\sqrt{T}}\left(\frac{X_1'X_1}{T}\right)^{-1} + o_p(1).
\end{aligned}
$$

We then derive the limit distribution of the first term in the preceding expression by considering

$$
\begin{aligned}
\operatorname{vec}(\sqrt{T}(\hat{A}-A)H_1) &= \operatorname{vec}\left(\frac{U_0'X_1}{\sqrt{T}}\left(\frac{X_1'X_1}{T}\right)^{-1}\right)\\
&= \left(I\otimes\left(\frac{X_1'X_1}{T}\right)^{-1}\right)\frac{1}{\sqrt{T}}\sum_{t=1}^{T}(u_{0t}\otimes u_{1t})\\
&\xrightarrow{\mathfrak{D}} (I\otimes\Sigma_{11}^{-1})Z_\varphi,
\end{aligned}
$$

where $Z_\varphi \equiv \mathcal{N}(0,\Omega_{\varphi\varphi})$ as given in (3), and this establishes the required result in part (a).

(b) Similarly, we write $(\hat{A}-A)H_bD_T$ as

$$
\begin{aligned}
&U_0'X_h(X_h'X_h)^{-1}H'H_bD_T\\
&\quad = (U_0'X_b^D - T^{-1/2}(T^{-1/2}u_0'X_1)(T^{-1}X_1'X_1)^{-1}X_1'X_b^D)\\
&\quad\quad \times (X_b^{D\prime}X_b^D - T^{-1}X_b^{D\prime}X_1(T^{-1}X_1'X_1)^{-1}X_1'X_b^D)^{-1}\\
&\quad = U_0'X_b^D(X_b^{D\prime}X_b^D)^{-1} + o_p(1)\\
&\quad \xrightarrow{\mathfrak{D}} \left(\int_0^1 dB_0\bar{B}_b' + \Delta_{0b}\right)\left(\int_0^1 \bar{B}_b\bar{B}_b'\right)^{-1},
\end{aligned}
$$

as required. ■

Proof of Theorem 2. Notice that

$$(\hat{A}^+ - A)H = (U_0^{+\prime}X - T\hat{\Delta}^+)H(H'X'XH)^{-1}H'H,$$

where

$$
\begin{aligned}
U_0^{+\prime}X - T\hat{\Delta}^+ &= U_0'X - T\hat{\Delta}_{\hat{u}_0\Delta x} - \hat{\Omega}_{\hat{u}_0\hat{v}}H(H'\hat{\Omega}_{\hat{v}\hat{v}}H)^{-1}H'(\hat{V}'X - T\hat{\Delta}_{\hat{v}\Delta x})\\
&= U_0'X - T\hat{\Delta}_{\hat{u}_0\Delta x} - \hat{\Omega}_{\hat{u}_0\hat{v}_h}\hat{\Omega}_{\hat{v}_h\hat{v}_h}^{-1}(\hat{V}_h'X - T\hat{\Delta}_{\hat{v}_h\Delta x}),
\end{aligned}
$$

and

$$H(H'X'XH)^{-1}H'H_1 = (H_1 - H_b(X_b'X_b)^{-1}X_b'X_1)(X_1'Q_bX_1)^{-1},$$

$$H(H'X'XH)^{-1}H'H_b = (-H_1(X_1'X_1)^{-1}X_1'X_b + H_b)(X_b'Q_1X_b)^{-1},$$

from the partitioned matrix inversion given in (A.46). We can similarly transform the serial correlation correction term $\hat{\Delta}^+$ defined in (20) as

$$\hat{\Delta}^+H = (\hat{\Delta}^+_{\hat{u}_0\Delta u_1}, \hat{\Delta}^+_{\hat{u}_0u_2}, \hat{\Delta}^+_{\hat{u}_0\Delta x_3}).$$

The results in parts (d) and (e) of Lemma 3 imply that $\hat{\Omega}_{\hat{u}_0\hat{v}}$ and $\hat{\Omega}_{\hat{v}\hat{v}}$ are consistent estimates of $\hat{\Omega}_{u_0v_h}$ and $\hat{\Omega}_{v_hv_h}$ for $K = O_e(T^k)$ with $k \in (0, \frac{1}{2})$, because $O_p(KT^{-1/2}) = o_p(1)$ for that range of k. Thus, we will carry out the later asymptotic analyses with the FM correction terms defined with $\hat{\Omega}_{u_0v_h}$ and $\hat{\Omega}_{v_hv_h}$, that is,

$$U_0^{+\prime}X - T\hat{\Delta}^+ = U_0'X - T\hat{\Delta}_{\hat{u}_0\Delta x} - \hat{\Omega}_{u_0v_h}\hat{\Omega}^{-1}_{v_hv_h}(\hat{V}_h'X - T\hat{\Delta}_{\hat{v}_h\Delta x}). \tag{A.47}$$

For $k \in (0, \frac{1}{2})$, it follows from the results in (A.47), Lemma 4(a), and parts (a) and (c) of Lemma 6 that

$$\begin{aligned} &T^{-1/2}(U_0^{+\prime}X_1 - T\hat{\Delta}^+_{\hat{u}_0\Delta u_1}) \\ &\quad = T^{-1/2}U_0'X_1 + O_p(T^{1/2}K^{-2}) + O_p(KT^{-1/2}) + O_p(K^{-1/2}). \end{aligned} \tag{A.48}$$

By (A.47), we can similarly write

$$\begin{aligned} &(U_0^{+\prime}X_b - T\hat{\Delta}^+_{\hat{u}_0\Delta x_b})D_T^{-1} \\ &\quad = \underbrace{(U_0'X_b - T\hat{\Delta}_{\hat{u}_0\Delta x_b})D_T^{-1}} - \hat{\Omega}_{u_0v_h}\hat{\Omega}^{-1}_{v_hv_h}(\hat{V}_h'X_b - T\hat{\Delta}_{\hat{v}_h\Delta x_b})D_T^{-1}, \end{aligned} \tag{A.49}$$

where the underbraced term is

$$(T^{-1}U_0'X_2 - \hat{\Delta}_{\hat{u}_0u_2}, T^{-2}U_0'X_3 - T^{-1}\hat{\Delta}_{\hat{u}_0\Delta x_3}) = (N_{02T}, \overline{N}_{03T}) + O_p((K/T)^{1/2})$$

by the results in parts (b) and (c) of Lemma 4. Then, (A.49) becomes

$$\overline{N}_{0bT} - \Omega_{0b}\Omega_{bb}^{-1}\overline{N}_{bb} + O_p(K^{5/2}T^{-3/2}) + O_p(K^{3/2}/T) + o_p(1). \tag{A.50}$$

(a) Using the results in (A.47)–(A.50), we may deduce that

$$\begin{aligned}\sqrt{T}(\hat{A}^+ - A)H_1 &= T^{-1/2}U_0'X_1(T^{-1}X_1'X_1)^{-1} + O_p(T^{1/2}K^{-2}) + O_p(KT^{-1/2}) \\ &\quad + O_p(K^{-1/2}) + O_p(K^{5/2}T^{-2}) + O_p(K/T)^{3/2} \\ &= T^{-1/2}U_0'X_1(T^{-1}X_1'X_1)^{-1} + o_p(1)\end{aligned}$$

for a bandwidth parameter $K = O_e(T^k)$ with $k \in (\frac{1}{4}, \frac{1}{2})$. From this, the stated result in part (a) follows immediately.

(b) To prove part (b), we again use the results in (A.47)–(A.50) to get

$$\begin{aligned}&(\hat{A}^+ - A)H_bD_T \\ &\quad = (\overline{N}_{0bT} - \Omega_{0b}\Omega_{bb}^{-1}\overline{N}_{bbT})(X_b^{D\prime}X_b^D)^{-1} + o_p(1) + O_p(K^{5/2}T^{-3/2}) \\ &\quad\quad + O_p(K^{3/2}/T),\end{aligned}$$

in which all the error terms are $o_p(1)$ for $k \in (0, \frac{3}{5})$. Now recall that

$$\bar{N}_{0bT} \xrightarrow{\mathfrak{D}} \int_0^1 dB_0 \bar{B}_b',$$

$$\bar{N}_{bbT} \xrightarrow{\mathfrak{D}} \int_0^1 dB_b \bar{B}_b',$$

$$(X_b^{D\prime} X_b^D)^{-1} \xrightarrow{\mathfrak{D}} \left(\int_0^1 \bar{B}_b \bar{B}_b' \right)^{-1},$$

from their definitions given in (A.45) and by the asymptotics following (A.46). The result in part (b) now easily follows. ■

Proof of Corollary 3. Note that there are now no I(2) variables in the model since $H = (H_1, H_2)$. We define a subscript coupling notation a as $a =$ "1,2." Following the analysis given in the proof of Lemma 3(c), we can show that

$$\hat{\Omega}_{u_0 v_a} \hat{\Omega}_{v_a v_a}^{-1} = (\Omega_{02}\Omega_{22}^{-1}(G_a' G_a)^{-1/2} + o_p(1), O_p(1) + O_p(K^{3/2}T^{-1/2}))C_a',$$

where

$$G_a = \begin{pmatrix} L_2 \\ I + M_2 \end{pmatrix},$$

$$C_a = (\underset{\sim}{G}_a, \underset{\sim}{G}_{a\perp}) \in O(m_1 + m_2),$$

and

$$\underset{\sim}{G}_a = G_a (G_a' G_a)^{-1/2}.$$

It follows directly from the proofs for parts (a) and (b) of Lemma 6 that

$$T^{-1} \hat{V}_a' X_1 - \hat{\Delta}_{\hat{v}_a \Delta u_1} = O_p(K^{-2}) + O_p(1/\sqrt{TK}),$$

$$T^{-1} \hat{V}_a' X_2 - \hat{\Delta}_{\hat{v}_a u_2} = G_a N_{22T} + O_p(K^{-2}) + O_p(T^{-1/2}).$$

We finally note from the proofs of parts (d) and (e) of Lemma 3 that

$$\hat{\Omega}_{\hat{u}_0 \hat{v}_a} \hat{\Omega}_{\hat{v}_a \hat{v}_a}^{-1} = \hat{\Omega}_{u_0 v_a} \hat{\Omega}_{v_a v_a}^{-1} + o_p(1),$$

for all $k \in (0,1)$, and, thus, using $\hat{\Omega}_{\hat{u}_0 \hat{v}_a} \hat{\Omega}_{\hat{v}_a \hat{v}_a}^{-1}$ in the place of $\hat{\Omega}_{u_0 v_a} \hat{\Omega}_{v_a v_a}^{-1}$ will not affect the following asymptotic results. Combining the preceding results together gives

$$\begin{aligned} &T^{-1/2}(U_0^{+\prime} X_1 - T\hat{\Delta}_{\hat{u}_0 \Delta u_1}^{+}) \\ &\quad = T^{-1/2} U_0' X_1 - \sqrt{T}\hat{\Delta}_{\hat{u}_0 \Delta u_1}^{+} - \sqrt{T}\hat{\Omega}_{u_0 v_a} \hat{\Omega}_{v_a v_a}^{-1}(T^{-1} \hat{V}_a' X_1 - \hat{\Delta}_{\hat{v}_a \Delta u_1}) \\ &\quad = T^{-1/2} U_0' X_1 + O_p(K^{-1/2}) + O_p(T^{1/2}K^{-2}) + O_p(KT^{-1/2}). \end{aligned} \tag{A.51}$$

We also have

$$\begin{aligned} &T^{-1} U_0^{+\prime} X_2 - \hat{\Delta}_{\hat{u}_0 \Delta x_2}^{+} \\ &\quad = T^{-1} U_0' X_2 - \hat{\Delta}_{\hat{u}_0 \Delta u_2} - \hat{\Omega}_{\hat{u}_0 \hat{v}_a} \hat{\Omega}_{\hat{v}_a \hat{v}_a}^{-1}(T^{-1} \hat{V}_a' X_2 - \hat{\Delta}_{\hat{v}_a u_2}) \\ &\quad = N_{02T} - \Omega_{02}\Omega_{22}^{-1} N_{22T} + O_p(K^{-2}) + O_p(T^{-1/2}) + O_p(K^{3/2}/T). \end{aligned} \tag{A.52}$$

(a) We follow the analysis given in the proof of Theorem 2(a) with the results in (A.51) and (A.52) to get

$$\sqrt{T}(\hat{A}^{+} - A)H_1 = T^{-1/2} U_0' X_1 (T^{-1} X_1' X_1)^{-1} + o_p(1),$$

as $T \to \infty$ for $k \in (\frac{1}{4}, \frac{1}{2})$, and this has the required normal distribution as shown in the proof of Proposition 1(a).

(b) Again by using the results in (A.49) and (A.50), we similarly have

$$\begin{aligned} T(\hat{A}^+ - A)H_2 &= (N_{02T} - \Omega_{02}\Omega_{22}^{-1}N_{22T})(T^{-2}X_2'X_2)^{-1} + o_p(1) \\ &\xrightarrow{\mathfrak{D}} \left(\int_0^1 dB_0B_2' - \Omega_{0b}\Omega_{bb}^{-1}\int_0^1 dB_2B_2'\right)\left(\int_0^1 B_2B_2'\right)^{-1} \\ &\equiv \int_0^1 dB_{0\cdot 2}B_2'\left(\int_0^1 B_2B_2'\right)^{-1}, \end{aligned}$$

as $T \to \infty$ for all $k \in (0, \frac{2}{3})$, as required for part (b). ■

Proof of Corollary 4. For the given specification of $H = (H_1, H_3)$, the regressors are now rotated in I(0) and I(2) directions in the following proof. Define a subscript coupling notation c as $c =$ "1,3." Now there are no I(1) variables in the model, and thus we have the following directly from the analysis for parts (a) and (b) of Lemma 3:

$$\hat{\Omega}_{u_0v_c} = (O_p(f_{KT}), \hat{\Omega}_{u_0u_3})$$

and

$$\begin{aligned} \hat{\Omega}_{v_cv_c} &= \begin{pmatrix} \hat{\Omega}_{\Delta u_1\Delta u_1} & \hat{\Omega}_{\Delta u_1u_3} \\ \hat{\Omega}_{u_3\Delta u_1} & \hat{\Omega}_{u_3u_3} \end{pmatrix} + G_{hc}(-\hat{\Omega}_{\Delta u_1\Delta u_1} \quad -\hat{\Omega}_{\Delta u_1u_3}) \\ &\quad + \begin{pmatrix} +\hat{\Omega}_{\Delta u_1\Delta u_1} \\ +\hat{\Omega}_{u_3\Delta u_1} \end{pmatrix} G_{hc}' + G_{hc}\hat{\Omega}_{\Delta u_1\Delta u_1}G_{hc}' \\ &= \begin{pmatrix} 0 & 0 \\ 0 & \hat{\Omega}_{u_3u_3} \end{pmatrix} + K^{-2}w''(0)\hat{\Omega}_{Rc}^* \\ &\quad + \begin{pmatrix} O_p(1/\sqrt{TK^3}) + o_p(K^{-2}) & O_p(1/\sqrt{TK}) + o_p(K^{-2}) \\ O_p(1/\sqrt{TK}) + o_p(K^{-2}) & O_p(1/\sqrt{TK}) + o_p(K^{-2}) \end{pmatrix}, \end{aligned}$$

where

$$\hat{\Omega}_{Rc}^* = \begin{pmatrix} -\Omega_{11} & -\Phi_{13} \\ \Phi_{31} & 0 \end{pmatrix} - \begin{pmatrix} L_1 \\ N_1 \end{pmatrix}(\Omega_{11}, \Phi_{13}^-) + \begin{pmatrix} -\Omega_{11} \\ \Phi_{31}^+ \end{pmatrix}(L_1', N_1'),$$

from the results in parts (a)–(c), (e), and (g) of Lemma 2. To find the inverse of $\hat{\Omega}_{v_cv_c}$, we conveniently denote its component submatrices as

$$\hat{\Omega}_{v_cv_c} := \begin{pmatrix} {}_c\Theta_{11} & {}_c\Theta_{12} \\ {}_c\Theta_{21} & {}_c\Theta_{22} \end{pmatrix}$$

and those of its inverse by

$$\hat{\Omega}_{v_cv_c}^{-1} := \begin{pmatrix} (\hat{\Omega}_{v_cv_c})_{11}^{-1} & (\hat{\Omega}_{v_cv_c})_{12}^{-1} \\ (\hat{\Omega}_{v_cv_c})_{21}^{-1} & (\hat{\Omega}_{v_cv_c})_{22}^{-1} \end{pmatrix}.$$

We first note that

$${}_c\Theta_{11} = K^{-2}(w''(0)(\hat{\Omega}_{Rc}^*)_{11} + o_p(1)),$$

for all $k \in (0,1)$, where

$$(\hat{\Omega}^*_{Rc})_{11} = -(I + L_1)\Omega_{11} + (\Phi^+_{31} - \Omega_{11})L'_1 =: {}_c\Theta^*_{11},$$

from the expression for $\hat{\Omega}^*_{Rc}$ already given. We assume that the limit matrix ${}_c\Theta^*_{11}$ is invertible, and, consequently, we have the following explicit representation of the limit behavior of Θ^{-1}_{11}:

$$K^{-2}{}_c\Theta^{-1}_{11} \xrightarrow{\mathcal{P}} (1/w''(0))_c\Theta^{*-1}_{11}.$$

Then it can be deduced from applying the partitioned matrix inversion formula that

$$(\hat{\Omega}_{v_c v_c})^{-1}_{22} = \hat{\Omega}^{-1}_{33} + o_p(1),$$
$$(\hat{\Omega}_{v_c v_c})^{-1}_{11} = O_p(K^2),$$
$$(\hat{\Omega}_{v_c v_c})^{-1}_{12} = O_p(1) + O_p(K^{3/2}T^{-1/2}).$$

We now have

$$\begin{aligned}
&\hat{\Omega}_{u_0 v_c}\hat{\Omega}^{-1}_{v_c v_c} \\
&= (O_p(f_{KT}), \Omega_{03} + o_p(1)) \begin{pmatrix} O_p(K^2) & O_p(1) + O_p(K^{3/2}T^{-1/2}) \\ O_p(1) + O_p(K^{3/2}T^{-1/2}) & \Omega^{-1}_{33} + o_p(1) \end{pmatrix} \\
&= (O_p(1) + O_p(K^{3/2}T^{-1/2}), \Omega_{03}\Omega^{-1}_{33} + O_p(f_{KT}) + O_p(K/T)).
\end{aligned}$$

Also, if follows directly from the analyses given in the proofs for parts (a) and (b) of Lemma 6 that

$$T^{-1}\hat{V}'_c X_1 - \hat{\Delta}_{\hat{v}_c \Delta u_1} = \begin{pmatrix} O_p(K^{-2}) + O_p(1/\sqrt{TK^3}) \\ O_p(K^{-2}) + O_p(1/\sqrt{TK}) \end{pmatrix},$$

$$T^{-2}\hat{V}'_c X_3 - T^{-1}\hat{\Delta}_{\hat{v}_c \Delta x_3} = \begin{pmatrix} O_p(T^{-1/2}) \\ \bar{N}_{33T} + O_p(K/T) + O_p(T^{-1/2}) \end{pmatrix}.$$

Then, by using the results already established, we have

$$\hat{\Omega}_{u_0 v_c}\hat{\Omega}^{-1}_{v_c v_c}(T^{-1}\hat{V}'_c X_1 - \hat{\Delta}_{\hat{v}_c \Delta u_1}) = O_p(K^{-2}) + O_p(1/\sqrt{TK})$$

and

$$\begin{aligned}
\hat{\Omega}_{u_0 v_c}\hat{\Omega}^{-1}_{v_c v_c}(T^{-2}\hat{V}'_c X_3 - T^{-1}\hat{\Delta}_{\hat{v}_c \Delta x_3}) &= \Omega_{03}\Omega^{-1}_{33}\bar{N}_{33T} + O_p(T^{-1/2}) + O_p(K^{3/2}/T) \\
&\quad + O_p(K^{-2}).
\end{aligned}$$

We note that the result in parts (d) and (e) of Lemma 3 holds as it is stated because the error term of the largest order, $O_p(KT^{-1/2})$, results from the presence of I(2) variables. Hence, for $k \in (0, \frac{1}{2})$, we can use $\hat{\Omega}_{u_0 v_c}$ and $\hat{\Omega}_{v_c v_c}$ in the places of $\hat{\Omega}_{\hat{u}_0 \hat{v}_c}$ and $\hat{\Omega}_{\hat{v}_c \hat{v}_c}$ without affecting the following asymptotic results.

From all the preceding considerations, we have, for $k \in (0, \frac{1}{2})$,

$$T^{-1/2}(U_0^{+\prime}X_1 - T\hat{\Delta}^+_{\hat{u}_0 \Delta u_1}) = T^{-1/2}U'_0 X_1 + O_p(K^{-1/2}) + O_p(T^{1/2}K^{-2}), \qquad \textbf{(A.53)}$$

and

$$\begin{aligned}
T^{-2}U_0^{+\prime}X_3 - T^{-1}\hat{\Delta}^+_{\hat{u}_0 \Delta x_3} &= \bar{N}_{03T} - \Omega_{03}\Omega^{-1}_{33}\bar{N}_{33T} + O_p(T^{-1/2}) \\
&\quad + O_p(K^{3/2}/T) + O_p(K^{-2}). \qquad \textbf{(A.54)}
\end{aligned}$$

(a) It follows directly from the results established in (A.51) and (A.52) that

$$\sqrt{T}(\hat{A}^{+} - A)H_1 = T^{-1/2}U_0'X_1(T^{-1}X_1'X_1)^{-1} + o_p(1),$$

as $T \to \infty$ for $k \in (\frac{1}{4},1)$, and this, together with the earlier restriction $k \in (0,\frac{1}{2})$, establishes the result in part (a).

(b) We similarly analyze the limit distribution of the estimator in the I(2) direction by using the results in (A.53) and (A.54) as follows:

$$\begin{aligned} T^2(\hat{A}^{+} - A)H_3 &= (\bar{N}_{03T} - \Omega_{03}\Omega_{33}^{-1}\bar{N}_{33T})(T^{-4}X_3'X_3)^{-1} + o_p(1) \\ &\xrightarrow{\mathfrak{D}} \left(\int_0^1 dB_0\bar{B}_3' - \Omega_{03}\Omega_{33}^{-1}\int_0^1 B_3\bar{B}_3'\right)\left(\int_0^1 \bar{B}_3\bar{B}_3'\right)^{-1} \\ &\equiv \int_0^1 dB_{0\cdot 3}\bar{B}_3'\left(\int_0^1 \bar{B}_3\bar{B}_3'\right)^{-1}, \end{aligned}$$

as $T \to \infty$ for $k \in (0,\frac{1}{2}) \cap (0,\frac{2}{3}) = (0,\frac{1}{2})$, as required for part (b). ■

Proof of Corollary 5. We start by reconsidering the error terms in parts (d) and (e) of Lemma 3 as, with $H = H_1$, the model includes only stationary regressors. From (15), we have

$$v_{ht} = v_{1t} = \Delta u_{1t} + L_1\Delta u_{1t-1},$$

and it is easy to see from the proof of Lemma 3(d) that

$$\hat{v}_{1t} = v_{1t} + R_1\Delta u_{1t-1}.$$

Therefore,

$$\begin{aligned} \hat{\Omega}_{\hat{v}_1\hat{v}_1} &= \hat{\Omega}_{v_1v_1} + {}^{+}\hat{\Omega}_{v_1\Delta u_1}R_1' + R_1{}^{-}\hat{\Omega}_{\Delta u_1 v_1} + R_1\hat{\Omega}_{\Delta u_1\Delta u_1}R_1' \\ &= O_p(K^{-2}) + O_p(1/\sqrt{TK^3}), \end{aligned} \tag{A.55}$$

because

$$\begin{aligned} \hat{\Omega}_{v_1v_1} &= \hat{\Omega}_{\Delta u_1\Delta u_1} + {}^{+}\hat{\Omega}_{\Delta u_1\Delta u_1}L_1' + L_1{}^{-}\hat{\Omega}_{\Delta u_1\Delta u_1} + L_1\hat{\Omega}_{\Delta u_1\Delta u_1}L_1' \\ &= O_p(K^{-2}) + O_p(1/\sqrt{TK^3}), \\ {}^{+}\hat{\Omega}_{v_1\Delta u_1} &= {}^{+}\hat{\Omega}_{\Delta u_1\Delta u_1} + L_1\hat{\Omega}_{\Delta u_1\Delta u_1} \\ &= O_p(K^{-2}) + O_p(1/\sqrt{TK^3}), \\ {}^{-}\hat{\Omega}_{\Delta u_1\Delta v_1} &= {}^{-}\hat{\Omega}_{\Delta u_1\Delta u_1} + \hat{\Omega}_{\Delta u_1\Delta u_1}L_1' \\ &= O_p(K^{-2}) + O_p(1/\sqrt{TK^3}), \end{aligned}$$

by the results in parts (a)–(c) of Lemma 2.

We also have from the proof of Lemma 3(e) that

$$\hat{u}_{0t} = u_{0t} + (A - \hat{A})H_1H_1'x_t = u_{0t} + (A_1 - \hat{A}_1)x_{1t}$$

and that

$$\hat{\Gamma}_{\hat{u}_0\hat{v}_1}(j) = \hat{\Gamma}_{u_0v_1}(j) + \hat{\Gamma}_{u_0\Delta u_1}(j+1)R_1' + O_p(T^{-1/2})\{\hat{\Gamma}_{u_1v_1}(j) + \hat{\Gamma}_{u_1\Delta u_1}(j+1)R_1'\}.$$

Then we use this result to obtain

$$\hat{\Omega}_{\hat{u}_0\hat{v}_1} = \hat{\Omega}_{u_0v_1} + {}^{+}\hat{\Omega}_{u_0\Delta u_1}R_1' + O_p(T^{-1/2})\hat{\Omega}_{u_1v_1} + O_p(T^{-1/2})^{+}\hat{\Omega}_{u_1\Delta u_1}R_1'$$
$$= O_p(K^{-2}) + O_p(1/\sqrt{TK}), \qquad \textbf{(A.56)}$$

because

$$\hat{\Omega}_{u_0v_1} = \hat{\Omega}_{u_0\Delta u_1} + {}^{+}\hat{\Omega}_{u_0\Delta u_1}L_1' = O_p(K^{-2}) + O_p(1/\sqrt{TK}),$$
$$\hat{\Omega}_{u_1v_1} = \hat{\Omega}_{u_1\Delta u_1} + {}^{+}\hat{\Omega}_{u_1\Delta u_1}L_1' = O_p(K^{-2}) + O_p(1/\sqrt{TK}),$$

from Lemma 2(d). From (A.55) and (A.56), it follows that

$$\hat{\Omega}_{\hat{u}_0\hat{v}_1}\hat{\Omega}_{\hat{v}_1\hat{v}_1} = O_p(K^{-2}) + O_p(1/\sqrt{TK}). \qquad \textbf{(A.57)}$$

Because $H = H_1$, there is no need to rotate the regressor space to separate out the nonstationary regressors. Hence, we can establish the required result simply by considering

$$\sqrt{T}(\hat{A}^{+} - A) = (T^{-1/2}U_0'X_1 - \sqrt{T}\hat{\Delta}_{\hat{u}_0\Delta u_1} - \sqrt{T}\hat{\Omega}_{\hat{u}_0\hat{v}_1}\hat{\Omega}_{\hat{v}_1\hat{v}_1}(T^{-1}\hat{V}_1'X_1 - \hat{\Delta}_{\hat{v}_1\Delta u_1}))$$
$$\times (T^{-1}X_1'X_1)^{-1}$$
$$= T^{-1/2}U_0'X_1(X_1'X_1)^{-1} + O_p(T^{1/2}K^{-2}) + O_p(K^{-1/2}),$$

where we used the result in (A.57) and

$$T^{-1}\hat{V}_1'X_1 - \hat{\Delta}_{\hat{v}_1\Delta u_1} = O_p(K^{-2}) + O_p(1/\sqrt{TK^3}),$$

from the proof of Lemma 6(a), and this leads to the required result, as shown in earlier proofs. ∎

Proof of Corollary 6. When $H = H_2$, it follows from (13)–(15) and the proof of Lemma 3(d) that

$$v_{ht} = v_{2t} = u_{2t} + M_2u_{2t-1},$$
$$\hat{v}_{ht} = \hat{v}_{2t} = v_{2t} + (\hat{M}_2 - M_2)u_{2t-1},$$

where $\sqrt{T}(\hat{M}_2 - M_2) = O_p(1)$. Using this, we have

$$\hat{\Omega}_{\hat{v}_2\hat{v}_2} = \hat{\Omega}_{v_2v_2} + {}^{+}\hat{\Omega}_{v_2u_2}O_p(T^{-1/2}) + O_p(T^{-1/2})^{-}\hat{\Omega}_{u_2v_2} + O_p(T^{-1/2})\hat{\Omega}_{u_2u_2}O_p(T^{-1/2})$$
$$= (I + M_2)\hat{\Omega}_{u_2u_2}(I + M_2)' + O_p(K^{-2}) + O_p(1/\sqrt{TK}), \qquad \textbf{(A.58)}$$

where

$$\hat{\Omega}_{v_2v_2} = \hat{\Omega}_{u_2u_2} + {}^{+}\hat{\Omega}_{u_2u_2}M_2' + M_2{}^{-}\hat{\Omega}_{u_2u_2} + M_2\hat{\Omega}_{u_2u_2}M_2'$$
$$= (I + M_2)\hat{\Omega}_{u_2u_2}(I + M_2)' + O_p(K^{-2}) + O_p(1/\sqrt{TK}),$$
$${}^{+}\hat{\Omega}_{v_2u_2} = {}^{+}\hat{\Omega}_{u_2u_2} + M_2\hat{\Omega}_{u_2u_2}$$
$$= (I + M_2)\hat{\Omega}_{u_3u_2} + O_p(K^{-2}) + O_p(1/\sqrt{TK}),$$
$${}^{-}\hat{\Omega}_{u_2v_2} = {}^{-}\hat{\Omega}_{u_2u_2} + \hat{\Omega}_{u_2u_2}M_2'$$
$$= \hat{\Omega}_{u_2u_2}(I + M_2)' + O_p(K^{-2}) + O_p(1/\sqrt{TK}),$$

by the results in parts (j) and (k) of Lemma 2.

It also follows immediately from the proof of Lemma 3(e) that

$\hat{u}_{0t} = u_{0t} + (A_2 - \hat{A}_2)x_{2t}.$

Then we write $\hat{\Gamma}_{\hat{u}_0\hat{v}_2}(j)$ as

$\hat{\Gamma}_{u_0v_2}(j) + \hat{\Gamma}_{u_0u_2}(j+1)O_p(T^{-1/2}) + O_p(T^{-1})\hat{\Gamma}_{x_2v_2}(j) + O_p(T^{-3/2})\hat{\Gamma}_{x_2u_2}(j+1),$

and this leads to

$$\begin{aligned}\hat{\Omega}_{\hat{u}_0\hat{v}_2} &= \hat{\Omega}_{u_0v_2} + {}^{+}\hat{\Omega}_{u_0u_2}O_p(T^{-1/2}) + O_p(T^{-1})\hat{\Omega}_{x_2v_2} + O_p(T^{-3/2})^{+}\hat{\Omega}_{x_2u_2}\\ &= \hat{\Omega}_{u_0u_2}(I+M_2)' + O_p(K/T) + O_p(T^{-1/2}),\end{aligned} \tag{A.59}$$

where

$\hat{\Omega}_{u_0v_2} = \hat{\Omega}_{u_0u_2} + {}^{+}\hat{\Omega}_{u_0u_2}M_2' = \hat{\Omega}_{u_0u_2}(I+M_2)' + O_p(K^{-2}) + O_p(1/\sqrt{TK}),$

$\hat{\Omega}_{x_2v_2} = \hat{\Omega}_{x_2u_2} + {}^{+}\hat{\Omega}_{x_2u_2}M_2' = O_p(K),$

by the results in part (i) of Lemma 2, (A.23), and (A.26). It follows from (A.58), (A.59), and Lemma 6(b) that

$\hat{\Omega}_{\hat{u}_0\hat{v}_2}\hat{\Omega}^{-1}_{\hat{v}_2\hat{v}_2}(T^{-1}\hat{V}_2'X_2 - \hat{\Delta}_{\hat{v}_2u_2}) = \hat{\Omega}_{u_0u_2}\hat{\Omega}^{-1}_{u_2u_2}N_{22T} + O_p(K^{-2}) + O_p(T^{-1/2}).$

Finally we use the results in parts (b) and (h) of Lemma 4 and the preceding result to establish

$$\begin{aligned}&T(\hat{A}^{+} - A)\\ &\quad= T(U_0^{+\prime}X_2 - T\hat{\Delta}^{+}_{\hat{u}_0u_2})(X_2'X_2)^{-1}\\ &\quad= (N_{02T} - \hat{\Omega}_{u_0u_2}\hat{\Omega}^{-1}_{u_2u_2}N_{22T})(T^{-2}X_2'X_2)^{-1} + O_p(K^{-2}) + O_p((K/T)^{1/2})\\ &\quad\xrightarrow{\mathfrak{D}} \int_0^1 dB_{0\cdot 2}B_2'\left(\int_0^1 B_2B_2'\right)^{-1},\end{aligned}$$

for all $k \in (0,1)$, where $O_p(K^{-2})$, $O_p(K/T)^{1/2} = o_p(1)$, and this establishes the required result. ■

Proof of Corollary 7. With $H = H_3$, the model has only I(2) variables, and it follows from (16) that

$\hat{v}_{ht} = \hat{v}_{3t} = \Delta^2 x_{3t} + O_p(T^{-1/2}),$

giving

$$\hat{\Omega}_{\hat{v}_3\hat{v}_3} = \hat{\Omega}_{u_3u_3} + O_p(KT^{-1/2}). \tag{A.60}$$

We also have from the proof of Lemma 3(e) that

$\hat{u}_{0t} = u_{0t} + (A_3 - \hat{A}_3)x_{3t};$

therefore,

$\hat{\Gamma}_{\hat{u}_0\hat{v}_3}(j) = \hat{\Gamma}_{u_0u_3}(j) + O_p(T^{-2})\hat{\Gamma}_{x_3u_3}(j) + O_p(T^{-1/2}).$

It follows that

$$\begin{aligned}\hat{\Omega}_{\hat{u}_0\hat{v}_3} &= \hat{\Omega}_{u_0u_3} + O_p(T^{-2})\hat{\Omega}_{x_3u_3} + O_p(KT^{-1/2})\\ &= \hat{\Omega}_{u_0u_3} + O_p(K/T) + O_p(KT^{-1/2}),\end{aligned}$$

because $\hat{\Omega}_{x_3u_3} = O_p(TK)$ as shown in (A.23). We now have from (A.60), Lemma 6(b), and the preceding result that

$$\hat{\Omega}_{\hat{u}_0\hat{v}_3}\hat{\Omega}^{-1}_{\hat{v}_3\hat{v}_3}(T^{-2}\hat{V}_3X_3 - T^{-1}\hat{\Delta}_{\hat{v}_3\Delta x_3}) = \hat{\Omega}_{u_0u_3}\hat{\Omega}^{-1}_{u_3u_3}\bar{N}_{33T} + O_p(K^{-2}) + O_p(KT^{-1/2}).$$

Then, by parts (c) and (h) of Lemma 4 and the preceding result, we have

$$\begin{aligned} T^2(\hat{A}^+ - A) & \\ &= T^2(U_0^{+\prime}X_3 - T\hat{\Delta}^+_{\hat{u}_0\Delta x_3})(X_3'X_3)^{-1} \\ &= (\bar{N}_{03T} - \hat{\Omega}_{u_0u_3}\hat{\Omega}^{-1}_{u_3u_3}\bar{N}_{33T})(T^{-3}X_3'X_3)^{-1} + O_p(K^{-2}) + O_p(KT^{-1/2}) \\ &\xrightarrow{\mathcal{D}} \int_0^1 dB_{0\cdot 3}\bar{B}_3'\left(\int_0^1 \bar{B}_3\bar{B}_3'\right)^{-1}, \end{aligned}$$

for $k \in (0,\frac{1}{2})$, where the error terms $O_p(K^{-2})$ and $O_p(KT^{-1/2})$ are $o_p(1)$. This establishes the required result. ■

Proof of Corollary 8. Now the model includes no stationary variables. Then, from the proofs of parts (a) and (b) of Lemma 3, we have

$$\hat{\Omega}_{u_0v_b} = \hat{\Omega}_{u_0u_b}\begin{pmatrix} I + M_2' & N_2' \\ 0 & I \end{pmatrix} + \begin{pmatrix} O_p(f_{KT}) & O_p(f_{KT}) \\ O_p(f_{KT}) & O_p(f_{KT}) \end{pmatrix},$$

$$\hat{\Omega}_{v_bv_b} = \left(\begin{pmatrix} I + M_2 & 0 \\ N_2 & I \end{pmatrix}\hat{\Omega}_{u_bu_b}\begin{pmatrix} I + M_2' & N_2' \\ 0 & I \end{pmatrix}\right) + \begin{pmatrix} O_p(f_{KT}) & O_p(f_{KT}) \\ O_p(f_{KT}) & O_p(f_{KT}) \end{pmatrix},$$

where $f_{KT} = K^{-2} + 1/\sqrt{TK} = o(1)$ for all $k \in (0,1)$. Let

$$G_b := \begin{pmatrix} I + M_2 & 0 \\ N_2 & I \end{pmatrix},$$

which is nonsingular. It can be shown that

$$\hat{\Omega}_{u_0v_b}\hat{\Omega}^{-1}_{v_bv_b} = \Omega_{0b}\Omega^{-1}_{bb}G_b^{-1} + o_p(1).$$

By using the result in Lemma 6(b), we also have

$$(\hat{V}_b'X_b - \hat{\Delta}_{\hat{v}_b\Delta x_b})D_T^{-1} = G_b\bar{N}_{bbT} + O_p(K^{-2}) + O_p(K/T) + O_p(T^{-1/2}).$$

We finally note from the proofs of parts (d) and (e) of Lemma 3 that

$$\hat{\Omega}_{\hat{u}_0\hat{v}_b} = \hat{\Omega}_{u_0v_b} + O_p(KT^{-1/2}) + O_p(K^{-2}),$$

$$\hat{\Omega}_{\hat{v}_b\hat{v}_b} = \hat{\Omega}_{v_bv_b} + O_p(KT^{-1/2}) + O_p(K^{-2}),$$

where $(KT^{-1/2} + K^{-2}) \to 0$ as $T \to \infty$ when $K = O_e(T^k)$ with $k \in (0,\frac{1}{2})$. Hence, we can use $\hat{\Omega}_{u_0v_b}$ and $\hat{\Omega}_{v_bv_b}$ as consistent estimates for $\hat{\Omega}_{\hat{u}_0\hat{v}_b}$ and $\hat{\Omega}_{\hat{v}_b\hat{v}_b}$ for $k \in (0,\frac{1}{2})$. Combining all the preceding results together gives

$$(U_0^{+\prime}X_b - T\hat{\Delta}^+_{\hat{u}_0\Delta x_b})D_T^{-1} = \bar{N}_{0bT} - \Omega_{0b}\Omega^{-1}_{bb}G_b^{-1}G_b\bar{N}_{bbT} + o_p(1),$$

where we used the result established in the analysis following (A.48) for the underbraced term and the fact that all the error terms appearing in the preceding expression are $o_p(1)$ for all $k \in (0,1)$. Hence, the preceding statement holds for $k \in (0,\frac{1}{2}) \cap (0,1) = (0,\frac{1}{2})$.

Now, we normalize the estimation error $(\hat{A}^+ - A)$ by D_T, as the estimator is defined only in nonstationary directions, and use the preceding results to give

$$
\begin{aligned}
(\hat{A}^+ - A)D_T &= (\bar{N}_{0bT} - \Omega_{0b}\Omega_{bb}^{-1}\bar{N}_{bbT} + o_p(1))(X_b^{D\prime}X_b^D)^{-1} \\
&\overset{\mathfrak{D}}{\to} \left(\int_0^1 dB_0\bar{B}_b' - \Omega_{0b}\Omega_{bb}^{-1}\int_0^1 B_b\bar{B}_b'\right)\left(\int_0^1 \bar{B}_b\bar{B}_b'\right)^{-1} \\
&\equiv \int_0^1 dB_{0\cdot b}\bar{B}_b'\left(\int_0^1 \bar{B}_b\bar{B}_b'\right)^{-1},
\end{aligned}
$$

for $k \in (0, \frac{1}{2})$. This establishes the required result. ■