

ROBUST NONSTATIONARY REGRESSION

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This paper provides a robust statistical approach to nonstationary time series regression and inference. Fully modified extensions of traditional robust statistical procedures are developed that allow for endogeneities in the nonstationary regressors and serial dependence in the shocks that drive the regressors and the errors that appear in the equation being estimated. The suggested estimators involve semiparametric corrections to accommodate these possibilities, and they belong to the same family as the fully modified least-squares (FM-OLS) estimator of Phillips and Hansen (1990, *Review of Economic Studies* 57, 99-125). Specific attention is given to fully modified least absolute deviation (FM-LAD) estimation and fully modified M (FM-M) estimation. The criterion function for LAD and some M -estimators is not always smooth, and this paper develops generalized function methods to cope with this difficulty in the asymptotics. The results given here include a strong law of large numbers and some weak convergence theory for partial sums of generalized functions of random variables. The limit distribution theory for FM-LAD and FM-M estimators that is developed includes the case of finite variance errors and the case of heavy-tailed (infinite variance) errors. Some simulations and a brief empirical illustration are reported.

1. INTRODUCTION

Many recent empirical applications of nonstationary regression methods have involved financial data sets. Examples include econometric tests of the purchasing power parity theory (Johansen and Juselius, 1993), which use exchange rate data; tests of forward exchange market unbiasedness (Corbae, Lim, and Ouliaris, 1993), which use spot and forward exchange rates; and tests of uncovered interest rate parity (Hunter, 1992), which use interest rate and exchange rate data. A well-documented characteristic of such financial data is their non-Gaussianity. The leptokurtosis and heavy-tailed features of

My thanks go to a referee and Katsuto Tanaka for comments on an earlier draft, to Vassilis Hajivassiliou for the use of his optimization routine "gmaxim," to the NSF for research support under grant SES 9122142, and to Glenna Ames for her skill and effort in keyboarding the manuscript. All computations and graphics reported in the paper were performed by the author in programs written in GAUSS 3.0 on a 486-50 PC. Address correspondence to: Peter C.B. Phillips, Cowles Foundation for Research in Economics, Yale University, P.O. Box 208281, New Haven, CT 06520-8281, USA.

exchange rate returns are especially notable, and these features are usually accentuated when the data are sampled more frequently.

For illustration, Figure 1a shows daily data for returns (i.e., differences in logarithms) of the Australian dollar spot exchange rate measured against the U.S. dollar over the period January 1984 to April 1991. Outlier activity is a fairly prominent characteristic of this data set. Figure 1b graphs a nonparametric estimate of the density of these data against those of a normal distribution whose mean and variance are fitted to those of the data. The leptokurtosis and heavy tails of the nonparametric density are evident in comparison with the fitted normal.

Two commonly used regression methods for analyzing such data in levels or log levels form are reduced rank regression (RRR) (Johansen, 1988;

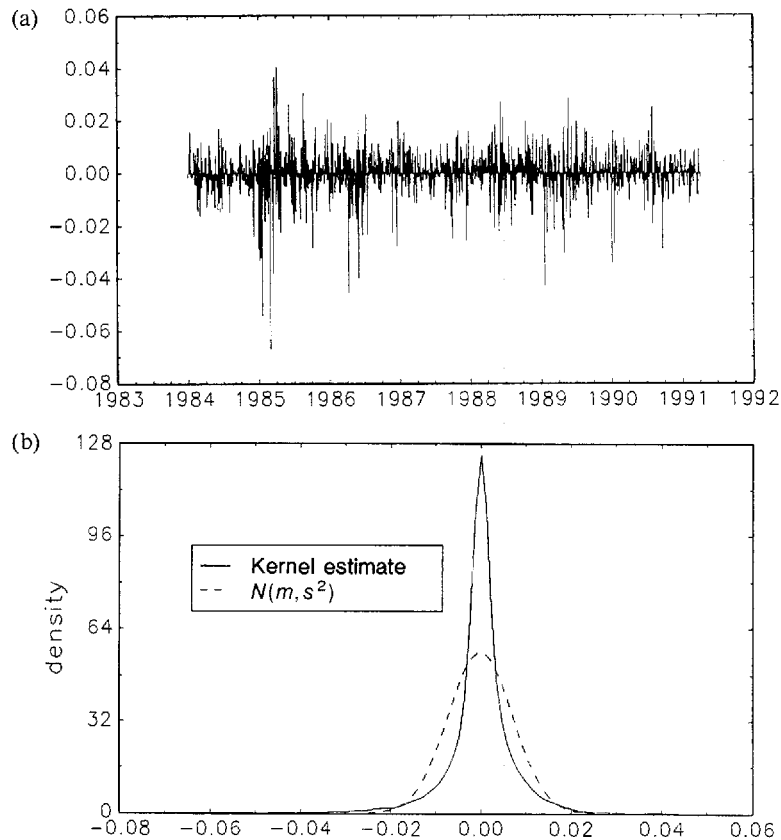


FIGURE 1. (a) Australian dollar spot exchange rate returns. (b) Density estimates spot rate returns.

Ahn and Reinsel, 1990) and fully modified least squares (FM-OLS) (Phillips and Hansen, 1990). Both these procedures are Gaussian in the sense that they can be deduced as maximum likelihood estimators under certain conditions when the data are Gaussian, and in this event they also deliver optimal estimates in nonstationary cointegrating regression (Phillips, 1991a). These techniques were designed to deal with nonstationarity in the data but, like other least-squares and Gaussian methods, they were not designed to deal specifically with data where there is prominent outlier activity. In such cases, there would seem to be a need for estimators that are more resistant to the presence of outliers than Gaussian estimators while at the same time being able to cope with data nonstationarity and endogenous regressors.

This need is addressed in the present paper. We develop extensions of robust regression procedures that allow for data nonstationarity and endogeneities in the regressors and serial dependence in the shocks that drive the regressors and in the errors that appear in the regression equation. Our suggested estimators involve semiparametric corrections to accommodate these possibilities, and they belong to the same family as the FM-OLS estimator of Phillips and Hansen (1990). Specifically, we develop a fully modified least absolute deviation (FM-LAD) estimator and a fully modified M (FM-M) estimator from the corresponding LAD- and M -estimators of ordinary regression. These estimators are designed to combine the features of nonstationary regression estimators like FM-OLS with the outlier resistant features of the common robust estimators.

Because the criterion function for the LAD estimator and for some common M -estimators is not smooth, we cannot rely on usual Taylor expansion methods to do the asymptotics. Recently, convex function approximations and stochastic equicontinuity arguments have been used to deal with this type of difficulty (for some discussion and illustration of these methods, see Pollard, 1990, 1991). The approach used here is rather different, although it does retain a convexity argument like that of Knight (1989) to assist in establishing the weak convergence of extremum estimators. Our approach is to treat the objective function in an extremum estimation problem as a generalized function and use generalized Taylor series expansions to extract the asymptotics. To facilitate this process, we introduce the concept of a generalized function of a random variable and give a strong law of large numbers and some weak convergence theory for partial sums of generalized functions of random variables.

The paper is organized as follows. Section 2 gives the model, our main assumptions, and the preliminary limit theory. Section 3 introduces the idea of a generalized function of a random variable by means of a class of suitable approximating sequences of ordinary functions of random variables. Some limit theory for generalized functions that is used later in the paper is given in this section. The FM-LAD estimator is constructed and its asymptotic theory is derived in Section 4. Section 5 deals with the FM-M estima-

tor and its asymptotics. Extensions of the asymptotic theory to cover the case of heavy-tailed (infinite variance) errors are given in Section 6. Some simulation results and a brief empirical illustration are reported in Section 7. The paper concludes in Section 8 by mentioning some further extensions of robust nonstationary regression. Proofs are given in Section 9.

2. THE MODEL, ASSUMPTIONS, AND PRELIMINARY LIMIT THEORY

We will work with the model

$$y_t = x_t' \beta + u_{0t}, \quad (1a)$$

$$\Delta x_t = u_{xt}, \quad (1b)$$

where x_t is a k -vector of full rank (i.e., not cointegrated) integrated regressors. The error vector $u_t = (u_{0t}, u_{xt})'$ in (1) is possibly temporally dependent and is required to satisfy Assumption EC, below. This assumption is convenient for our purposes here but could be replaced by a variety of similar conditions without materially affecting our subsequent results, provided the finite second moment requirement is retained. If that condition is relaxed, then the limit theory and, indeed, some rates of convergence will change. We will discuss this possibility later in the paper. Model (1) can also be extended by the inclusion of deterministic trends, and this extension affects our results in the usual way (see Park and Phillips, 1988) provided the finite error variance condition holds.

Assumption EC (Error Condition).

- (a) u_t is a strictly stationary and strong mixing sequence with mixing numbers α_m that satisfy

$$\sum_1^\infty \alpha_m^{(p-\beta)/p\beta} < \infty \quad (2)$$

for some $p > \beta > 2$.

- (b) $\|u_t\|_p < \infty$.
 (c) The probability density $h(\cdot)$ of u_{0t} is symmetric and is positive and continuous in a neighborhood $(-b, b)$ of the origin for some $b > 0$.

Mixing condition (2) and moment condition (b) in EC are sufficient to ensure the functional weak convergence of partial sum processes of u_{xt} , u_{0t} and bounded functions of u_{0t} , as will be needed later. These conditions will also validate the weak convergence to stochastic integrals of sample covariances between the regressors x_t and the errors u_{0t} and bounded functions of u_{0t} . A requirement like part (c) is conventional in the development of an asymptotic theory for the LAD estimator, whose limit theory depends on the value of $h(\cdot)$ at the origin, $h(0)$. However, the symmetry condition on $h(\cdot)$

is stronger than usual and could be relaxed, but it will be convenient in our generalized function proofs.

Under Assumption EC, the long-run covariance matrix of u_t exists, and we partition this matrix conformably with u_t as

$$\Omega_{uu} = \sum_{k=-\infty}^{\infty} E(u_0 u_k') = \begin{bmatrix} \Omega_{00} & \Omega_{0x} \\ \Omega_{x0} & \Omega_{xx} \end{bmatrix}.$$

We also use the transformed error process $v_t = \text{sgn}(u_{0t}) = 1, -1$ for $u_{0t} \geq 0$, $u_{0t} < 0$, respectively, and define $w_t = (v_t, u_{xt}')'$. Because v_t is a bounded function of u_{0t} , the long-run covariance matrix of w_t also exists under Assumption EC, and we partition this matrix conformably with w_t as follows

$$\Omega_{ww} = \sum_{k=-\infty}^{\infty} E(w_0 w_k') = \begin{bmatrix} \Omega_{vv} & \Omega_{vx} \\ \Omega_{xv} & \Omega_{xx} \end{bmatrix}.$$

In a similar way, we define and partition the one-sided long-run covariance matrices of u_t and w_t , respectively, as

$$\Delta_{uu} = \sum_{k=0}^{\infty} E(u_0 u_k') = \begin{bmatrix} \Delta_{00} & \Delta_{0x} \\ \Delta_{x0} & \Delta_{xx} \end{bmatrix},$$

and

$$\Delta_{ww} = \sum_{k=0}^{\infty} E(w_0 w_k') = \begin{bmatrix} \Delta_{vv} & \Delta_{vx} \\ \Delta_{xv} & \Delta_{xx} \end{bmatrix}.$$

Under Assumption EC, a multivariate invariance principle for w_t holds, namely,

$$T^{-1/2} \sum_1^{[Tr]} w_t \rightarrow_d B_w(r) \equiv BM(\Omega_{ww}), \quad 0 < r \leq 1, \quad (3)$$

as shown in Phillips and Durlauf (1986). We partition the limit Brownian motion B in (3) conformably with w_t and Ω using the notation $B_w(r)' = (B_v(r), B_x(r)')$. A similar invariance principle holds for partial sums of u_t , namely,

$$T^{-1} \sum_1^{[Tr]} u_t \rightarrow_d B_u(r) \equiv BM(\Omega_{uu}), \quad 0 < r \leq 1, \quad (4)$$

where the limit process is partitioned as $B_u(r)' = (B_0(r), B_x(r)')$ conformably with $u_t' = (u_{0t}, u_{xt}')$. In addition, Assumption EC ensures that sample covariances between the regressors x_t and the error vectors have limits that can be expressed as stochastic integrals with drift. In particular,

$$T^{-1} \sum_1^{[Tr]} x_t v_t \rightarrow_d \int_0^r B_x dB_v + r \Delta_{xv}, \quad 0 < r \leq 1, \quad (5)$$

and

$$T^{-1} \sum_1^{[Tr]} x_t u_t' \rightarrow_d \int_0^r B_x dB_u + r \Delta_{xu}, \quad 0 < r \leq 1, \quad (6)$$

where $\Delta_{xu} = [\Delta_{x0} \quad \Delta_{xx}]$ (see Phillips, 1988; Hansen, 1992).

3. GENERALIZED FUNCTIONS OF RANDOM VARIABLES AND GENERALIZED LIMIT THEORY

Our approach is to treat nonsmooth objective criteria like those that appear in LAD estimation as generalized functions and use generalized Taylor series expansions to represent their local behavior. The basic ideas behind this approach and an application to LAD estimation in a stationary regression were laid out by the author (Phillips, 1991b). We will follow those ideas here and develop some additional concepts to make the approach rigorous.

Our main concerns will involve generalized functions of random variables and stochastic limit operations with partial sums of these generalized functions of random variables. The concept of a generalized function of a random variable is different from the idea of a generalized random process, as it appears in the existing literature on generalized functions (see, e.g., Gel'fand and Vilenkin, 1964, Ch. III), wherein such a process is defined as a mapping from a given space of test functions into a random variable. An example of the latter is the continuous linear functional $B(\varphi) = \int_0^1 \varphi(t) dW(t)$, which is here expressed as a stochastic integral of the Wiener process $W(t)$ on $C[0,1]$.

Instead, our need is to give a meaning to objects such as $\delta(u_t)$, where u_t is a real valued random variable (indexed by discrete time t) and $\delta(\cdot)$ is the Dirac delta generalized function, which has the property that $\int_{-\infty}^{\infty} \delta(x) F(x) dx = F(0)$ for any continuous function $F(x)$. There are, in fact, several ways in which this can be done. In defining generalized functions like $\delta(\cdot)$ (i.e., before we deal with such "functions" of random variables), we will use the "regular sequence" approach given in Lighthill (1958). Associated with (and, in fact, defining) any generalized function $f(x)$ is a sequence $f_m(x)$ of *good functions* (i.e., functions that are continuously differentiable any number of times with derivatives of $O(|x|^{-N})$ as $|x| \rightarrow \infty$ for any N ; the set of such functions is denoted as GF) with the property that

$$\lim_{m \rightarrow \infty} \int_{-\infty}^{\infty} f_m(x) F(x) dx \quad (7)$$

exists for any $F \in GF$. The integral of $f(x)$ is then defined by the equation

$$\int_{-\infty}^{\infty} f(x) F(x) dx := \lim_{m \rightarrow \infty} \int_{-\infty}^{\infty} f_m(x) F(x) dx.$$

A sequence such as $f_m(x)$ with this property is called a *regular sequence* for $f(x)$.

Since the sequence $f_m(\cdot)$ is measurable, $f_m(u_t)$ has a meaning as an ordinary random variable on the probability space where u_t is itself defined. The generalized function $f(u_t)$ of the random variable u_t is then defined by the associated regular sequence $f_m(u_t)$ or, more precisely, the class of all regular sequences that are equivalent to $f_m(\cdot)$ in the sense that (7) is the same for each sequence. It follows that if $\text{pdf}(u) \in GF$ is the density of u_t then we can define the expectation of the generalized function $f(\cdot)$ of u_t by

$$E(f(u_t)) := \lim_{m \rightarrow \infty} E(f_m(u_t)) = \lim_{m \rightarrow \infty} \int_{-\infty}^{\infty} f_m(u) \text{pdf}(u) du. \quad (8)$$

Provided the limit on the right side of (8) exists, we can relax the requirement that $\text{pdf}(u) \in GF$.

Now suppose we wish to establish a weak law of large numbers (WLLN) or strong law of large numbers (SLLN) for partial sums of the generalized function of random variables $f(u_t)$. Because $f(\cdot)$ is defined in terms of the regular sequence $f_m(\cdot)$, we can define a WLLN and SLLN for $f(u_t)$, that is,

$$T^{-1} \sum_1^T f(u_t) \rightarrow_{p, \text{a.s.}} E\{f(u_t)\}, \quad (9)$$

by the corresponding weak and strong laws for partial sums of the regular sequence $f_m(u_t)$ of ordinary random variables, that is, by

$$T^{-1} \sum_1^T f_m(u_t) \rightarrow_{p, \text{a.s.}} E\{f_m(u_t)\}, \quad \forall m, \quad (10)$$

and the limit that appears on the right side of (10) is given by (8). This definition is, in fact, compatible with that of a WLLN or SLLN for ordinary functions of u_t .

LEMMA 3.1. (SLLN for Ordinary Random Variables as Generalized Functions of Random Variables). *Suppose u_t is strictly stationary and ergodic and $f(u_t)$ is an ordinary (measurable) function of u_t . Then, (9) holds in the sense of ordinary random sequences iff it holds in the sense of generalized functions of random sequences, that is, iff (10) holds.*

Proof. To prove necessity, suppose $f(u_t)$ is an ordinary function of u_t satisfying (9) and $E(f(u_t))$ is finite. We need to demonstrate (10). We construct the following regular sequence of good functions to approximate $f(\cdot)$ (cf. Lighthill, 1958, p. 22):

$$f_m(u) = \int_{-\infty}^{\infty} f(v) S\{m(v - u)\} m e^{-v^2/m^2} dv. \quad (11)$$

In (11) the function $S(\cdot)$ is a "smudge function," whose role in $f_m(u)$ is to smudge out $f(v)$ when v is outside the interval $(u - m^{-1}, u + m^{-1})$. $S(\cdot)$ is defined as

$$S(y) = s(y) / \int_{-1}^1 s(y) dy, \quad (12)$$

where

$$s(y) = \begin{cases} e^{-1/(1-y^2)} & |y| < 1 \\ 0 & |y| \geq 1 \end{cases}$$

and

$$\begin{aligned} \int_{-1}^1 s(y) dy &= 2 \int_0^1 e^{-1/(1-y^2)} dy = (2/e) \int_0^1 e^{-y^2/(1-y^2)} dy \\ &= (1/e) \int_0^\infty e^{-z} z^{-1/2} (1+z)^{-3/2} dz, \quad \text{with } z = y^2/(1-y^2) \\ &= (\pi^{1/2}/e) \Psi(\tfrac{1}{2}, 0; 1), \end{aligned}$$

where Ψ is the confluent hypergeometric function of the second kind (Erdelyi, 1953, p. 255). Note that $S(y)$ and all of its derivatives are 0 at $y = \pm 1$.

Now, $S\{m(v - u_t)\}$ is a measurable and integrable function of u_t and therefore constitutes an ergodic sequence, so that

$$T^{-1} \sum_1^T S\{m(v - u_t)\} \rightarrow_{a.s.} E[S\{m(v - u_t)\}] = \int_{-\infty}^\infty S\{m(v - u)\} \text{pdf}(u) du.$$

Hence,

$$\begin{aligned} T^{-1} \sum_1^T f_m(u_t) &= \int_{-\infty}^\infty f(v) T^{-1} \sum_1^T S\{m(v - u_t)\} m e^{-v^2/m^2} dv \\ &\rightarrow_{a.s.} \int_{-\infty}^\infty f(v) E[S\{m(v - u_t)\}] m e^{-v^2/m^2} dv \\ &= E\{f_m(u_t)\}, \quad \forall m, \end{aligned}$$

giving (10) as a necessary condition for (9) in the case of ordinary functions $f(u_t)$ of the random sequence u_t .

To show sufficiency of (10) in this case (i.e., when $f(u_t)$ is an ordinary function of u_t), note first that, because $f_m(\cdot)$ is a regular sequence for $f(\cdot)$, $E\{f(u_t)\}$ is finite and is given by the limit shown in (8). Equation (9) then follows directly by the ergodic theorem because $f(u_t)$ is an ordinary function and measurable (as the limit of a sequence of ordinary measurable functions) and $E\{f(u_t)\}$ exists. ■

Example 3.2

Let $\delta(u_t)$ be the Dirac delta generalized function of the strictly stationary and ergodic time series u_t with continuous marginal density $\text{pdf}(u)$. A corresponding regular sequence for $\delta(u_t)$ is

$$\delta_m(u_t) = (m/\pi)^{1/2} e^{-mu_t^2}. \quad (13)$$

We have from (9)

$$T^{-1} \sum_1^T \delta(u_t) \rightarrow_{p,\text{a.s.}} E\{\delta(u_t)\} = \int_{-\infty}^{\infty} \delta(u) \text{pdf}(u) du = \text{pdf}(0). \quad (14)$$

The corresponding result for the sequence $\delta_m(u_t)$ is

$$\begin{aligned} T^{-1} \sum_1^T \delta_m(u_t) &\rightarrow_{p,\text{a.s.}} E\{\delta_m(u_t)\} = (m/\pi)^{1/2} \int_{-\infty}^{\infty} e^{-mu^2} \text{pdf}(u) du \\ &= \text{pdf}(0)\{1 + O(m^{-1})\}, \end{aligned}$$

where the last equality follows by virtue of the Laplace approximation.

Example 3.3

Let x_t be the integrated process given in (1b) and suppose u_{0t} satisfies Assumption EC. We wish to show that

$$T^{-2} \sum_1^T \delta(u_{0t}) x_t x'_t \rightarrow_d \text{pdf}(0) \int_0^1 B_x B'_x. \quad (15)$$

Note that by changing the probability space this can be written as an almost sure convergence result, in which case we can invoke the earlier definition of a.s. convergence of generalized functions of random variables in terms of regular sequences. The corresponding condition in the original probability space is then

$$T^{-2} \sum_1^T \delta_m(u_{0t}) x_t x'_t \rightarrow_d E\{\delta_m(u_{0t})\} \int_0^1 B_x B'_x, \quad \forall m, \quad (16)$$

where $\delta_m(\cdot)$ is the regular sequence for the delta function given in (13).

To establish (16), we will show that

$$T^{-2} \sum_1^T [\delta_m(u_{0t}) - E\{\delta_m(u_{0t})\}] x_t x'_t \rightarrow_p 0. \quad (17)$$

First, in view of (3), we have $T^{-1/2} x_{[T\cdot]} \rightarrow_d B_x(\cdot)$. Next, because u_{0t} is strong mixing, $z_{mt} = \delta_m(u_{0t}) - E\{\delta_m(u_{0t})\}$ is also (with the same mixing numbers) and

$$\omega_m = \text{lrvar}(z_{mt}) = \sum_{j=-\infty}^{\infty} E(z_{mt} z_{mt+j}),$$

which is finite for all m . Note, however, that $\omega_m = O(m^{1/2})$ as $m \rightarrow \infty$, as is apparent from the fact that

$$\begin{aligned} \text{var}\{\delta_m(u_t)\} &= E\{\delta_m(u_t)^2\} - E\{\delta_m(u_t)\}^2 \\ &= (m/\pi) \int_{-\infty}^{\infty} e^{-2mu^2} \text{pdf}(u) du \\ &\quad - \left\{ (m/\pi)^{1/2} \int_{-\infty}^{\infty} e^{-mu^2} \text{pdf}(u) du \right\}^2 \\ &= (m/\pi)^{1/2} \text{pdf}(0)\{1 + O(m^{-1})\} - \text{pdf}(0)^2\{1 + O(m^{-1})\} \\ &= O(m^{1/2}). \end{aligned} \quad (18)$$

(Note that higher order covariances, that is, $E\{\delta_m(u_t)\delta_m(u_{t+j})\}$ for $j \geq 1$, are of $O(1)$ as $m \rightarrow \infty$.) Thus, ω_m is unbounded as $m \rightarrow \infty$. But for all finite m , ω_m exists, and we have the functional law

$$T^{-1/2} \sum_1^{[Tr]} z_{mt} \rightarrow_d B_{z_m}(r) \equiv BM(\omega_m). \quad (19)$$

To prove (17), we simply note that

$$\begin{aligned} T^{-2} \sum_1^T [\delta_m(u_{0t}) - E\{\delta_m(u_{0t})\}] x_t x_t' \\ = T^{-1/2} \sum_1^T (T^{-1/2} z_{mt})(T^{-1/2} x_t)(T^{-1/2} x_t') = O_p(T^{-1/2}), \quad \forall m. \end{aligned}$$

In fact, it is not difficult to establish the explicit limit

$$T^{-3/2} \sum_1^T z_{mt} x_t x_t' \rightarrow_d \int_0^1 dB_{z_m} B_x B_x' + \Delta_{zx} \int_0^1 B' + \int_0^1 B \Delta_{zx}', \quad \forall m, \quad (20)$$

where $\Delta_{zx} = \sum_{j=0}^{\infty} E(z_{mj} x_0)$, which is a limit result that is related to one given in Hansen (1992, Theorem 4.2). Thus, (17) holds and this gives (16) and, thus, the required limit of (15).

Example 3.4

Under Assumption EC, we have the functional CLT

$$T^{-1/2} \sum_1^{[Tr]} \text{sgn}(u_{0t}) \rightarrow_d B_v(r) = BM(\Omega_{vv}) \quad (21)$$

(see (3)) with

$$\Omega_{vv} = \text{lrvar}\{\text{sgn}(u_{0t})\} = \sum_{j=-\infty}^{\infty} E\{\text{sgn}(u_{0t})\text{sgn}(u_{0t+j})\}. \quad (22)$$

If we treat the ordinary function $\text{sgn}(u_{0t})$ of u_{0t} as a generalized function of u_{0t} , result (21) can be viewed as a functional law for partial sums of generalized functions of random variables. The limit process $B_v(r)$ can then be interpreted as a generalized process, although of course it also has meaning as an ordinary random process, namely, a Brownian motion with variance Ω_{vv} .

A regular sequence for $\text{sgn}(u_{0t})$ can be constructed as in (11). We get

$$\text{sgn}_m(u_{0t}) = \int_{-\infty}^{\infty} \text{sgn}(v) S(m(v-u)) m e^{-v^2/m^2} dv. \quad (23)$$

Note that with this construction we have

$$\begin{aligned} \text{sgn}_m(-u_{0t}) &= \int_{-\infty}^{\infty} \text{sgn}(v) S(m(v+u_{0t})) m e^{-v^2/m^2} dv \\ &= - \int_{+\infty}^{-\infty} \text{sgn}(-w) S(m(u_{0t}-w)) m e^{-w^2/m^2} dw \\ &= - \int_{-\infty}^{\infty} \text{sgn}(w) S(m(u_{0t}-w)) m e^{-w^2/m^2} dw = -\text{sgn}_m(u_{0t}), \end{aligned}$$

so that $\text{sgn}_m(u)$ is an odd function of u , just like $\text{sgn}(u)$. As a consequence,

$$E\{\text{sgn}_m(u_{0t})\} = 0,$$

because the density of u_{0t} is symmetric.

Being a regular sequence, $\text{sgn}_m(u)$ tends to 0 faster than any negative power of $|u|$ as $|u| \rightarrow \infty$ (see Lighthill, 1958, p. 22). Indeed, recognizing that for large m the dominant part of integral (23) comes from integrating in the neighborhood of $v = u$, we have from the Laplace approximation

$$\begin{aligned} \text{sgn}_m(u_{0t}) &= \int_{-1}^1 \text{sgn}(u_{0t} + y/m) e^{-(u_{0t} + y/m)^2/m^2} S(y) dy \\ &= \text{sgn}(u_{0t}) e^{-u_{0t}^2/m^2} \{1 + O(m^{-1})\}. \end{aligned} \quad (24)$$

In view of this behavior for large $|u_{0t}|$, all moments of $\text{sgn}_m(u_{0t})$ exist. Also, $\text{sgn}_m(u_{0t})$ is a measurable function of u_{0t} and is therefore mixing (with the same mixing numbers as u_{0t}). It follows that

$$\Omega_m = \text{lrvar}\{\text{sgn}_m(u_{0t})\} = \sum_{j=-\infty}^{\infty} E\{\text{sgn}_m(u_{0t}) \text{sgn}_m(u_{0t+j})\} < \infty,$$

and we have the functional law

$$T^{-1/2} \sum_1^{[Tr]} \text{sgn}_m(u_{0t}) \rightarrow_d B_m(r) \equiv BM(\Omega_m), \quad \forall m. \quad (25)$$

Moreover, in view of (24), $E\{\text{sgn}_m(u_{0t})\text{sgn}_m(u_{0t+j})\} \rightarrow E\{\text{sgn}(u_{0t})\text{sgn}(u_{0t+j})\}$ and $\Omega_m \rightarrow \Omega_{vv}$ as $m \rightarrow \infty$, so that

$$\lim_{m \rightarrow \infty} B_m(r) \equiv BM(\Omega_{vv}). \quad (26)$$

Thus, (25) describes a regular sequence of functional laws whose limit (26) is equivalent to the limit of (21). In this sense, (25) and (26) give an alternative representation of functional law (21), with the difference that the ordinary random variable $\text{sgn}(u_{0t})$ is treated as a generalized function of u_{0t} (by virtue of the regular sequence $\text{sgn}_m(u_{0t})$). Because $\text{sgn}(u_{0t})$ is an ordinary random variable and the limit process in (26) is an ordinary random process, the weak convergence results are equivalent.

Example 3.5

Assumption EC also validates weak convergence to stochastic integrals, as in (5) and (6). Repeating (5) for $r = 1$ gives us

$$T^{-1} \sum_1^T x_t \text{sgn}(u_{0t}) \rightarrow_d \int_0^1 B_x dB_v + \Delta_{xv}. \quad (27)$$

As in the last example, we can again treat $\text{sgn}(u_{0t})$ as a generalized function of u_{0t} , using the regular sequence $\text{sgn}_m(u_{0t})$ given in (23). In the same way as we derived functional law (25) for $\text{sgn}_m(u_{0t})$, we obtain

$$T^{-1} \sum_1^T x_t \text{sgn}_m(u_{0t}) \rightarrow_d \int_0^1 B_x dB_m + \Delta_{xm}, \quad \forall m, \quad (28)$$

where $B_m \equiv BM(\Omega_m)$ and $\Delta_{xm} = \sum_{j=0}^{\infty} E\{u_{xt} \text{sgn}_m(u_{0t+j})\}$. Now, $\Delta_{xm} \rightarrow \Delta_{xv} = \sum_{j=0}^{\infty} E\{u_{xt} \text{sgn}(u_{0t+j})\}$ as $m \rightarrow \infty$ and, thus, in view of (26), we have

$$\lim_{m \rightarrow \infty} \left(\int_0^1 B_x dB_m + \Delta_{xm} \right) \equiv \int_0^1 B_x dB_v + \Delta_{xv}. \quad (29)$$

It follows that (28) describes a regular sequence of weak convergence results whose limit, from (29), is distributionally equivalent to the limit of (27). Thus, (28) and (29) give a generalized function characterization of limit law (27).

4. LAD AND FM-LAD ESTIMATION

The LAD estimator of β in model (1) is defined as the solution of the extremum problem

$$\beta_{\text{LAD}} = \arg\min \left[\sum_1^T |y_t - x_t' \beta| \right]. \quad (30)$$

We examine the asymptotic behavior of the estimator β_{LAD} and use this theory to suggest suitable modifications to the estimator that lead to

improved asymptotic performance in nonstationary regression situations. Our approach to the development of the asymptotic theory uses generalized functions of random variables and the limit theory for such functions developed in Section 3 to deal with the fact that the objective criterion in (30) is not differentiable as an ordinary function of β .

We start with β_{LAD} and give its asymptotic distribution in the following result.

THEOREM 4.1. *Under Assumption EC,*

$$T(\beta_{\text{LAD}} - \beta) \rightarrow_d \left[2h(0) \int_0^1 B_x B_x' \right]^{-1} \left[\int_0^1 B_x dB_v + \Delta_{xv} \right]. \quad (31)$$

Remark 4.2.

- (i) Theorem 4.1 shows that β_{LAD} is consistent at the usual $O(T)$ rate for a nonstationary regression estimator. But like OLS, β_{LAD} suffers from second-order asymptotic bias arising from the presence of Δ_{xv} in the second factor of (31) and the fact that the limit Brownian motions B_v and B_x are, in general, correlated (i.e., $\Omega_{vx} \neq 0$ in Ω_{ww}). In fact, formula (31) is very similar to the limit result for the OLS estimator $\hat{\beta}$, namely,

$$T(\hat{\beta} - \beta) \rightarrow_d \left(\int_0^1 B_x B_x' \right)^{-1} \left(\int_0^1 B_x dB_0 + \Delta_{x0} \right)$$

(from Phillips and Durlauf, 1986).

- (ii) Limit distribution (31) depends on the value at the origin of the probability density of u_{0t} , that is, $h(0)$. In this respect, (31) is similar to the usual limit theory for the LAD estimator that applies in the stationary or linear regression case. However, because (31) is not mixed normal in general, the scale effects of $h(0)$ affect more than just the dispersion of the estimator.
- (iii) When $v_t = \text{sgn}(u_{0t})$ is a martingale difference sequence with respect to $\mathcal{F}_{t-1} = \sigma(v_{s-1}, u_{xp} : s = t-1, \dots; p = \dots, t+1, t, t-1, \dots)$, then $\Delta_{xv} = 0$, $\Omega_{xv} = 0$ and (31) specializes to

$$\begin{aligned} T(\beta_{\text{LAD}} - \beta) &\rightarrow_d \left[2h(0) \int_0^1 B_x B_x' \right]^{-1} \left[\int_0^1 B_x dB_v \right] \\ &\equiv MN \left(0, (2h(0))^{-2} \left(\int_0^1 B_x B_x' \right)^{-1} \right) \end{aligned} \quad (32)$$

(because B_v and B_x are independent), which is a mixed normal limit that is comparable in form to the normal limit theory for LAD in stationary models. In this special case, x_t is exogenous and the system has no feedback between v_t and u_{xt} .

4.1. The FM-LAD Estimator

Our purpose is to modify the LAD estimator so that we obtain a mixed normal limit theory like (32) even when x_t is not exogenous. To do so, we need

to adjust for serial dependence to eliminate the one-sided long-run covariance Δ_{xv} and adjust for the endogeneity of x_t that is manifested in the long-run covariance Ω_{xv} . Our construction is based on the idea of the FM-OLS estimator developed by Phillips and Hansen (1990). However, in the present case we need to take into account the following: (i) the extremum estimator properties of LAD (i.e., unlike OLS, there is no explicit formula for LAD), and (ii) the fact that the limit theory for LAD, as given in Theorem 4.1, relies on the robust function $v_t = \text{sgn}(u_{0t})$ of the equation errors rather than the errors themselves.

We define the FM-LAD estimator of β in (1) as the following corrected version of β_{LAD} :

$$\beta_{\text{LAD}}^+ = \beta_{\text{LAD}} - [2\hat{h}(0)X'X]^{-1}[X'\Delta X\hat{\Omega}_{xx}^{-1}\hat{\Omega}_{xv} + T\hat{\Delta}_{xv}^+]. \quad (33)$$

In (33), $X'X = \sum_1^T x_t x_t'$, $X'\Delta X = \sum_1^T x_t \Delta x_t'$, $\hat{h}(0)$ is a (nonparametric) consistent estimator of $h(0)$, the probability density of u_{0t} at the origin, $\hat{\Omega}_{xx}$ and $\hat{\Omega}_{xv}$ are consistent estimates of the long-run variance submatrices Ω_{xx} and Ω_{xv} , and $\hat{\Delta}_{xv}^+$ is a consistent estimate of the one-sided long-run covariance matrix

$$\Delta_{xv}^+ = \sum_{j=0}^{\infty} E(u_{x0} v_j^+) = \Delta_{xy} - \Delta_{xx} \Omega_{xx}^{-1} \Omega_{xv}, \quad (34)$$

where

$$v_t^+ = v_t - \Omega_{vx} \Omega_{xx}^{-1} \Delta x_t.$$

To estimate Δ_{xv}^+ , we need first to estimate error v_t^+ , which in turn involves the estimation of v_t . This is achieved by a first-stage LAD regression that produces the error estimate $\hat{u}_{0t} = y_t - \beta'_{\text{LAD}} x_t$ and consequently $\hat{v}_t = \text{sgn}(\hat{u}_{0t})$. We then construct

$$\hat{v}_t^+ = \hat{v}_t - \hat{\Omega}_{\hat{v}x} \hat{\Omega}_{xx}^{-1} \Delta x_t, \quad (35)$$

using conventional kernel estimates of the long-run covariance matrices Ω_{vx} and Ω_{xx} , whereupon we can estimate $\hat{\Delta}_{xv}^+$ as given by (34) directly by using a kernel estimate of the one-sided long-run covariance of u_{xt} and \hat{v}_t^+ (for more details on kernel estimation of long-run covariance matrices, see Park and Phillips, 1989; Andrews, 1991; Phillips, 1995). Note from (34) that the estimation of $\hat{\Delta}_{xv}^+$ effectively involves the estimation of the four submatrices Δ_{xv} , Δ_{xx} , Ω_{xx} , and Ω_{xv} . We use the notation $\hat{\Omega}_{\hat{v}x}$ in (35) to make it clear that our estimate of Ω_{vx} (and Δ_{vx} , for that matter) relies on \hat{v}_t rather than on v_t , which is unobserved.

We can also write (33) in the form

$$\beta_{\text{LAD}}^+ = \beta_{\text{LAD}} - [2\hat{h}(0)X'X]^{-1} T\hat{\Delta}_{xv}^{++},$$

where

$$\hat{\Delta}_{xv}^{++} = (T^{-1}X'\Delta X - \hat{\Delta}_{xx})\hat{\Omega}_{xx}^{-1}\hat{\Omega}_{xv} + \hat{\Delta}_{xv}.$$

In this formula for $\hat{\Delta}_{xv}^{++}$, the first expression on the right side is an endogeneity correction. This term adjusts the regression estimate for potential endogeneity in the regressor x_t . In LAD estimation, what is important is the correlation between Δx_t (the shocks in x_t) and the signed equation error function $v_t = \text{sgn}(u_{0t})$. Because there is persistence in the shocks to x_t we measure this correlation by means of Ω_{xv} . The variable $\Delta x_t' \Omega_{xx}^{-1} \Omega_{xv}$ then adjusts the regression coefficient for the conditional mean of the signed error v_t given Δx_t . The term involving $\hat{\Delta}_{xx}$ adjusts for the effects of serial dependence in Δx_t on the covariance $T^{-1} X' \Delta X$ in the limit. Finally, the second term in $\hat{\Delta}_{xv}^{++}$ is $\hat{\Delta}_{xv}$, and this adjusts for serial covariance between the past history of shocks Δx_t and the signed error v_t . In all these cases, we make the corrections by nonparametric (kernel) density estimation. Thus, β_{LAD}^+ is a semiparametric LAD estimator with nonparametric corrections for endogeneity in the regressor x_t and serial dependence in the equation errors and shocks to x_t .

THEOREM 4.3. *Under Assumption EC,*

$$\begin{aligned} T(\beta_{\text{LAD}}^+ - \beta) &\rightarrow_d \left[2h(0) \int_0^1 B_x B_x' \right]^{-1} \left[\int_0^1 B_x dB_{v \cdot x} \right] \\ &\equiv MN \left(0, (2h(0))^{-2} \omega_{vv \cdot x} \left[\int_0^1 B_x B_x' \right]^{-1} \right), \end{aligned} \quad (36)$$

where $B_{v \cdot x} = B_v - \Omega_{vx} \Omega_{xx}^{-1} B_x \equiv BM(\omega_{vv \cdot x})$ and $\omega_{vv \cdot x} = \Omega_{vv} - \Omega_{vx} \Omega_{xx}^{-1} \Omega_{xv} = \text{lrvar}(v_t^+)$.

Remark 4.4.

- (i) The limit theory of FM-LAD is similar to that of the FM-OLS estimator $\beta^+ = (X'X)^{-1}(X'y^+ - T\hat{\Delta}_{x0}^+)$, where $y^+ = y - \Delta X' \hat{\Omega}_{xx}^{-1} \hat{\Omega}_{x0}$. This is given by

$$T(\beta^+ - \beta) \rightarrow_d \left(\int_0^1 B_x B_x' \right)^{-1} \left(\int_0^1 B_x dB_{0 \cdot x} \right) \equiv MN \left(0, \omega_{00 \cdot x} \left(\int_0^1 B_x B_x' \right)^{-1} \right), \quad (37)$$

where $B_{0 \cdot x} = B_0 - \Omega_{0x} \Omega_{xx}^{-1} B_x \equiv BM(\omega_{00 \cdot x})$ and $\omega_{00 \cdot x} = \Omega_{00} - \Omega_{0x} \Omega_{xx}^{-1} \Omega_{x0}$. The relative asymptotic efficiency of the two estimators depends on the ratio $\omega_{vv \cdot x} / (2h(0))^2 \omega_{00 \cdot x}$, so that FM-LAD is more efficient than FM-OLS when

$$h(0)^2 > \omega_{vv \cdot x} / 4\omega_{00 \cdot x}. \quad (38)$$

In the case where x_t is exogenous and u_{0t} is i.i.d. $(0, \sigma_{00}^2)$, we have $\omega_{vv \cdot x} = \omega_{vv} = 1$, $\omega_{00 \cdot x} = \omega_{00} = \sigma_{00}^2$, and (38) reduces to

$$h(0)^2 > 1/4\sigma_{00}^2,$$

which corresponds to the criterion for the asymptotic superiority of LAD over OLS in linear regression.

- (ii) Wald statistics for testing restrictions on β can be constructed in the usual way from the limit theory in Theorem 4.3. For instance, consider the restrictions

$$H_0: \varphi(\beta) = 0,$$

where φ is a $q \times 1$ vector function with $\Phi(\beta) = \partial\varphi/\partial\beta'$ of full row rank q . The Wald statistic for testing H_0 based on FM-LAD is

$$W^+ = \varphi(\beta_{\text{LAD}}^+)' \{ \Phi^+ [4\hat{h}(0)^2 X'X]^{-1} \Phi^{+'} \}^{-1} \varphi(\beta_{\text{LAD}}^+) / \hat{\omega}_{vv \cdot x}, \quad (39)$$

where $\hat{\omega}_{vv \cdot x} = \hat{\Omega}_{vv} - \hat{\Omega}_{vx} \hat{\Omega}_{xx}^{-1} \hat{\Omega}_{xv}$ is a consistent estimate of the conditional long-run variance $\omega_{vv \cdot x}$, $\hat{h}(0)$ is a (nonparametric) consistent estimate of $h(0)$, and $\Phi^+ = \Phi(\beta_{\text{LAD}}^+)$. In view of (36), we have the limit $W^+ \rightarrow \chi_q^2$ under H_0 by a simple deduction. Thus, the statistic W^+ can be used for testing H_0 in the usual way.

- (iii) Fully modified standard errors for the β_{LAD}^+ estimator can be constructed from (the square roots of)

$$s_i^2 = (\hat{\omega}_{vv \cdot x} / 4\hat{h}(0)^2) [(X'X)^{-1}]_{ii} \quad (i = 1, \dots, k), \quad (40)$$

where $\hat{\omega}_{vv \cdot x} = \hat{\Omega}_{vv} - \hat{\Omega}_{vx} \hat{\Omega}_{xx}^{-1} \hat{\Omega}_{xv}$. Variance estimate (40) is based directly on the (conditional) asymptotic variance matrix that appears in (36). Correspondingly, we have the fully modified LAD t -ratios $t_i = (\beta_{i\text{LAD}}^+ - \beta_i) / s_i$, which are asymptotically $N(0, 1)$. These statistics simplify the statistical reporting of FM-LAD regressions—in effect, we report the estimated coefficients, standard errors, and t -ratios in the usual way. The modifications that are built into these statistics mean that they can be interpreted as in conventional stationary linear regression.

- (iv) In applications (such as the empirical illustration given in Section 7.2), regression model (1a) will often include a fitted intercept. In this case, the theory leading to Theorem 4.3 continues to apply after some modifications to the limits of the terms in the generalized Taylor series expansion of the LAD objective function. These modifications have the effect that the Brownian motion $B_x(r)$ that appears in the mixed normal limit in (36) is replaced by the demeaned Brownian motion $\underline{B}_x = B_x - \int_0^1 B_x$. In a similar way, when there is a polynomial deterministic trend in (1a) the limiting functions in (36) involve detrended Brownian motion. The derivations follow in the same fashion as those described in Park and Phillips (1988). Inference is unaffected by these changes because the limit theory is still mixed normal. Cases where x_t itself has a deterministic component (for instance, where (1b) has a nonzero constant) can also be considered along the lines of the Park and Phillips (1988) analysis. In such cases one needs to distinguish the directions where the deterministic component of x_t dominates and the residual directions in which the stochastic trend dominates. The limit theory can be constructed from these constituent pieces just as in Park and Phillips (1988). Finally on this matter, we note that there is no need to make endogeneity corrections for deterministic components, and the formula for the FM-LAD estimator β_{LAD}^+ can be correspondingly adjusted. Phillips (1993) gave a detailed discussion of this matter in the context of FM-OLS and FM-VAR estimation. Because the same considerations apply here, the reader is referred to that paper for a full treatment.

5. FM-M ESTIMATION

A more general class of robust procedures is that of M -estimators. In the present case, these estimators can be defined by the extremum problem

$$\beta_M = \operatorname{argmin} \left[\sum_1^T \rho(y_t - x_t' \beta) \right], \quad (41)$$

for some function ρ . When $\rho(u) = |u|$, this includes the LAD estimator. Other common choices are $\rho(u) = |u|^\delta$ for $\delta \in [1, 2]$, thereby including OLS when $\delta = 2$, and the Huber (1964) loss function

$$\rho_c(u) = \begin{cases} (\frac{1}{2})u^2 & \text{for } |u| \leq c, \\ c|u| - (\frac{1}{2})c^2 & \text{for } |u| > c, \end{cases} \quad (42)$$

which combines the OLS criterion for deviations bounded by the parameter c with the LAD criterion for bigger deviations.

The estimator β_M can also be defined as a solution to the equation

$$\sum_1^T x_t \psi(y_t - x_t' \beta_M) = 0, \quad (43)$$

and when ρ is differentiable $\psi = \rho'$ and (43) are the first-order conditions. Definitions (41) and (43) are equivalent when ρ is convex and differentiable because in that case there is only one solution to (43). A scale estimate can also be employed in criteria (41) and (43), and this can be obtained using the residuals of a preliminary consistent regression (possibly by OLS), as discussed by Huber (1981).

Like the LAD and OLS estimators, β_M needs some modification before it has good asymptotic properties in nonstationary regressions. We will construct a fully modified M -estimator β_M^+ to improve the asymptotic behavior of β_M , and the construction is similar to that of β_{LAD}^+ . As in the LAD case, we first need the limit theory for the unmodified estimator β_M . This calls for some additional conditions that relate to the properties of the functions that appear in (41) and (43).

Assumption ML (M -Estimator Loss Function Conditions).

- (a) $\psi(u_t)$ has mean zero and $\|\psi(u_t)\|_p < \infty$,
- (b) ψ' is Lipschitz continuous and $\|\psi'(u_t)\|_p < \infty$, for some $p > \beta > 2$, as in (2).

Conditions of this type are fairly standard in the development of M -estimator asymptotics. The p th moment conditions (which relate to the strong mixing condition in (2) in Assumption EC) on ψ and ψ' in parts (a) and (b) are helpful because of the allowance for serial dependence in u_t (cf. Knight, 1991) and because of the need to establish results on weak convergence for sample covariances such as $T^{-1} \sum_1^T x_t \psi(u_t)$ to stochastic integrals with drift. However, for many ψ functions, these conditions will be implied by the cor-

responding conditions on u_t , and often ψ and ψ' are bounded, in which case they hold automatically. The centering condition $E\{\psi(u_t)\} = 0$ in Assumption ML(a) is the analog for M -estimation of the zero mean and zero median conditions for OLS and LAD estimation.

Some M -estimators are excluded by the differentiability condition of Assumption ML(b). When ψ' fails to exist at a finite number of points, we can proceed by treating ψ and ψ' as generalized functions. The asymptotic results given later will then continue to hold under some additional conditions on the probability density $h(u)$ of u_t , so that, for instance, we could write

$$E\{\psi'(u_t)\} = \int_{-\infty}^{\infty} \psi'(u)h(u) du = - \int_{-\infty}^{\infty} \psi(u)h'(u) du;$$

that is, this linear functional of the generalized function $\psi'(u_t)$ of the random variable u_t is equivalent to $-\int_{-\infty}^{\infty} \psi(u)h'(u) du$, which exists as an ordinary function. In the addendum to the proof of Theorem 5.1 (see Section 9), we will outline how this particular extension of the theory proceeds. The development follows our analysis of LAD asymptotics using generalized functions of random variables and generalized Taylor series.

Here we will focus attention on the nonstationary regression M -estimator asymptotics and the construction of the FM- M estimator.

THEOREM 5.1. *Let Assumptions EC and ML hold. Suppose also that either of the following two conditions apply:*

- (a) ρ is convex, $\psi = \rho'$, and β_M satisfies (41).
- (b) β_M is a solution of (43), and $T^{1/2}(\beta_M - \beta) = o_p(1)$.

Then,

$$T(\beta_M - \beta) \rightarrow_d \left[E\{\psi'(u_{0t})\} \int_0^1 B_x B_x' \right]^{-1} \left[\int_0^1 B_x dB_{\psi} + \Delta_{x\psi} \right], \quad (44)$$

where

$$B_{\psi} \equiv BM(\Omega_{\psi\psi}), \quad \Omega_{\psi\psi} = \sum_{j=-\infty}^{\infty} E\{\psi(u_{0t})\psi(u_{0t+j})\}$$

and

$$\Delta_{x\psi} = \sum_{j=0}^{\infty} E\{u_{xt}\psi(u_{0t+j})\}.$$

5.1. FM- M Estimation

As with the construction of the FM-LAD estimator, our purpose is to modify the M -estimator β_M so that the second-order bias effects in limit theory (44) are removed and the limit distribution is mixed normal. The required

corrections are similar to those used in the LAD case, and we define the FM-M estimator as

$$\beta_M^+ = \beta_M - \left[\left\{ T^{-1} \sum_1^T \psi'(\hat{u}_{0t}) \right\} X'X \right]^{-1} [X'\Delta X \hat{\Omega}_{xx}^{-1} \hat{\Omega}_{x\psi} + T\hat{\Delta}_{x\psi}^+], \quad (45)$$

where $\hat{\Omega}_{x\psi}$ is a consistent estimator of

$$\Omega_{x\psi} = \sum_{j=-\infty}^{\infty} E\{u_{xt} \psi(u_{0t+j})\}$$

and $\hat{\Delta}_{x\psi}^+$ is a consistent estimator of

$$\Delta_{x\psi}^+ = \Delta_{x\psi} - \Delta_{xx} \Omega_{xx}^{-1} \Omega_{x\psi}.$$

Again, all of these component matrices can be estimated using kernel techniques. But we do need a preliminary consistent estimate of β , say β_M , to construct the residuals \hat{u}_{0t} from which we can form the function $\psi(\hat{u}_{0t})$, which is required for the estimation of $\Omega_{x\psi}$ and $\Delta_{x\psi}$.

THEOREM 5.2. *Under the conditions of Theorem 5.1*

$$\begin{aligned} T(\beta_M^+ - \beta) &\rightarrow_d \left[E\{\psi'(u_{0t})\} \int_0^1 B_x B_x' \right]^{-1} \left[\int_0^1 B_x dB_{\psi \cdot x} \right] \\ &= MN \left(0, \omega_{\psi \cdot x} [E\{\psi'(u_{0t})\}]^{-2} \left[\int_0^1 B_x B_x' \right]^{-1} \right), \end{aligned} \quad (46)$$

where

$$B_{\psi \cdot x} \equiv BM(\omega_{\psi \cdot x}), \quad \omega_{\psi \cdot x} = \Omega_{\psi\psi} - \Omega_{\psi x} \Omega_{xx}^{-1} \Omega_{x\psi}.$$

Remark 5.3.

- (i) In the case where x_t is exogenous and u_{0t} is i.i.d. $(0, \sigma_{00}^2)$, the limit theory given in (46) reduces to

$$T(\beta_M^+ - \beta) \rightarrow_d \text{var}(\psi(u_{0t}))^{1/2} \left[E\{\psi'(u_{0t})\} \int_0^1 B_x B_x' \right]^{-1} \left[\int_0^1 B_x dW \right], \quad (47)$$

where W is standard Brownian motion independent of B_x . Observe that limit (47) depends on ψ only through the factor

$$\text{var}(\psi(u_{0t}))^{1/2} / E(\psi'(u_{0t})). \quad (48)$$

Consequently, the efficiency of the estimator β_M^+ depends on this factor also, just as it does in the case of linear regression (see, e.g., Huber, 1981, p. 173). If the density $h(u)$ of u_{0t} is continuously differentiable, then the M estimators β_M, β_M^+ will be asymptotically efficient in this case (note that these two estimators are asymptotically equivalent under the conditions of this remark) if $\psi(\cdot)$ is chosen to satisfy

$$\psi(u) = -ch'(u)/h(u), \quad \text{for } c \neq 0 \quad (49)$$

- (cf. Huber, 1981, pp. 70, 176). When the density $h(\cdot)$ is unknown, there is the possibility of adaptive estimation, as recently discussed by Jeganathan (1995).
- (ii) In the general case, we can write limit (46) as

$$T(\beta_M^+ - \beta) \rightarrow_d \frac{\omega_{\psi\psi \cdot x}^{1/2}}{[E\{\psi'(u_{0t})\}]} \left(\int_0^1 B_x B_x' \right)^{-1} \left(\int_0^1 B_x dW \right), \quad (50)$$

where $W \equiv BM(1)$ is independent of B_x . So the limit distribution of the class of all FM-M estimators depends on the $\psi(\cdot)$ function only through the factor

$$\omega_{\psi\psi \cdot x}^{1/2}/E\{\psi'(u_{0t})\} = \text{lrvar}\{\psi(u_{0t}) | u_{xt}\}^{1/2}/E\{\psi'(u_{0t})\}.$$

It will be interesting to consider the issue of an optimal estimator in this class. Note that FM-M estimation is semiparametric and $\psi(\cdot)$ depends on the equation error u_{0t} . Maximum likelihood estimation, in contrast, involves the complete specification of the system, including the transient dynamics of the vector error process $u_t = (u_{0t}, u_{xt})'$. If the latter is parametric, like the linear process $u_t = C(L; \theta)\varepsilon_t = \sum_{j=0}^{\infty} C_j(\theta)\varepsilon_{t-j}$ with $\sum_{j=0}^{\infty} j^{1/2} \|C_j\| < \infty$, then the likelihood can be constructed using a form of innovations algorithm (as when u_t is ARMA). For the Gaussian case, the results in Phillips (1991, Theorem 1') confirm that the limit distribution of the maximum likelihood estimator (MLE) of β is of the form given in (50). Indeed, FM-M estimation is optimal with $\psi(u) = u$ in this case; that is, the optimal FM-M estimator is just FM-OLS. It will be interesting to try to extend this theory to the non-Gaussian case and to develop a theory of optimal semiparametric M -estimation. This task will be left for later work.

- (iii) Theorem 5.2 can be used as a basis for inference using the FM-M estimator β_M in the same way as FM-LAD (refer to parts (ii) and (iii) of Remark 4.4). Thus, to test H_0 as in Remark 4.4(ii), we can use the Wald statistic

$$W_M^+ = \varphi(\beta_M^+)' \left\{ \Phi_M^+ \left[T^{-1} \sum_1^T \psi'(\hat{u}_{0t}) \right]^{-2} (X'X)^{-1} \Phi_M^{+'} \right\}^{-1} \varphi(\beta_M^+)/\hat{\omega}_{\psi\psi \cdot x},$$

where $\Phi_M^+ = \Phi(\beta_M^+)$, $\hat{\omega}_{\psi\psi \cdot x} = \hat{\Omega}_{\psi\psi} - \hat{\Omega}_{\psi x} \hat{\Omega}_{xx}^{-1} \hat{\Omega}_{x\psi}$ is an estimate of the conditional long-run variance of $\psi(u_{0t})$ given Δx_t , and $\hat{u}_{0t} = y_t - \beta_M^{+'} x_t$ is the residual from the FM-M regression. The latter quantity is used in the sample estimate $T^{-1} \sum_1^T \psi'(\hat{u}_{0t})$ of $E\{\psi'(u_{0t})\}$ and in the construction of the $\hat{\omega}_{\psi\psi \cdot x}$, which relies on the sample values $\hat{\psi}_t = \psi(\hat{u}_{0t})$. In light of Theorem 5.2, we have $W_M^+ \rightarrow \chi_q^2$ under the null H_0 . FM-M coefficient standard errors and t -ratios are constructed in the same way as in Remark 4.4(iii) for FM-LAD estimation.

6. EXTENSIONS TO MODELS WITH INFINITE VARIANCE ERRORS

This section outlines some extensions of the theory to the case where the errors in model (1) have infinite variance. Our purpose is to sketch the development and indicate some interesting points of departure from the earlier theory.

It is simplest to suppose that the tail behavior of the errors in (1a) and (1b) is of the same form. It is especially helpful to require that the components of u_{xt} have distributions with the same tail shape, for then the normalizing constant in central limit theory for partial sums of u_{xt} is a scalar. The general case of an operator stable law when the components of u_{xt} have different tail shapes (e.g., follow asymptotic Pareto laws with different slope coefficients) does not, to the author's knowledge anyway, seem to have been worked out. However, because one of the main applications of a regression theory in the infinite variance case is to series like spot and forward exchange rates, the restriction of comparable tail behavior does not seem to be too limiting. At each point in time, spot and forward rates reflect the same information set and economic fundamentals. As a consequence, it seems reasonable to model such series with distributions that have related tail shape.

Accordingly, we will confine our attention to limit laws that are of the symmetric α -stable ($S\alpha S$) form. Thus, a k -vector ξ has an $S\alpha S$ distribution in \mathbb{R}^k if its characteristic function is of the form

$$E(e^{i\xi'p}) = \exp\left\{-\int_{S_k} |p'h|^\alpha \Gamma(dh)\right\}, \quad (51)$$

where $S_k = \{h \in \mathbb{R}^k : h'h = 1\}$ is the unit sphere in \mathbb{R}^k and $\Gamma(\cdot)$ is a probability measure (possibly discrete) on S_k . Paulauskas (1976) provides a discussion of multivariate stable distributions in this class. The most common examples (arising from discrete measures on S_k) are exponentials of powers of quadratic forms such as $\exp\{-(p'\Sigma p)^{\alpha/2}\}$, which include the multivariate normal when $\alpha = 2$.

We will assume that the following condition applies to $u_t = (u_{0t}, u'_{xt})'$ in place of Assumption EC.

Assumption EC² (Error Condition 2).

(a) u_t is generated by the linear process

$$u_t = D(L)\varepsilon_t = \sum_{j=0}^{\infty} D_j \varepsilon_{t-j}, \quad D_0 = I, \quad |D(1)| \neq 0, \quad (52)$$

where ε_t is an i.i.d. sequence of random vectors whose components have infinite variance and are each in the domain of normal attraction of a stable law of order $\alpha \in (0, 2)$. The coefficient matrices in (52) satisfy the summability condition

$$\sum_0^\infty j \|D_j\|^\delta < \infty, \quad \text{with } 0 < \delta < \alpha \wedge 1. \quad (53)$$

(b) Partial sums of the ε_t in (52) satisfy the following functional limit law in the product space $D[0, 1]^{k+1}$ of $k+1$ copies of $D[0, 1]$ with the product Skorohod topology:

$$a_T^{-1} \sum_1^{[Tr]} \varepsilon_t \rightarrow_d U_\alpha(r). \quad (54)$$

The limit process $U_\alpha(r)$ in (54) is an α -stable process in $D[0,1]^{k+1}$ whose increments are $S\alpha S$, that is, have a characteristic function of the form in (51), and $a_T = T^{1/\alpha}$ is a normalizing constant.

- (c) The sequence u_t is strong mixing with mixing numbers α_m that satisfy the summability condition

$$\sum_0^\infty \alpha_m < \infty.$$

- (d) Condition EC(c) holds.

Condition EC²(b) is a “high-level” condition. Because each of the components of ε_t is in the domain of normal attraction of a stable law with exponent α , simple sufficient conditions for a component-wise version of (54) are available (for earlier applications, see, e.g., Resnick, 1986; Chan and Tran, 1989; Knight, 1991). Condition (b) requires joint convergence and specifies the limit process to be in the $S\alpha S$ class. Condition (a) specifies that u_t has a linear process form, and this facilitates the use of arguments like those in Phillips (1991b) and Phillips and Solo (1992) for obtaining the limit distributions of certain functions of partial sums of u_t . Mixing condition (c) is useful because we need to work with and characterize the dependence properties of functions of the error process u_{0T} .

Our main result is the following.

THEOREM 6.1. *Under Assumption EC², we have the following:*

- (a) *The estimators β_{LAD} and β_{LAD}^+ have the common limit distribution*

$$\begin{aligned} T^a(\beta_{LAD} - \beta), T^a(\beta_{LAD}^+ - \beta) &\rightarrow_d \left(\int_0^1 U_{x\alpha} U'_{x\alpha} \right)^{-1} \left(\int_0^1 U_{x\alpha}^- dB_v \right) \\ &\equiv MN \left(0, (2h(0))^{-2} \Omega_{vv} \left(\int_0^1 U_{x\alpha} U'_{x\alpha} \right)^{-1} \right), \end{aligned}$$

where $a = \frac{1}{2} + 1/\alpha$, $U_{x\alpha}(r) = D'_x U_\alpha(r)$, and $U_{x\alpha}^-(r) = U_{x\alpha}(r-)$ is the left limit of the process $U_{x\alpha}$. Here, D_x is the second submatrix of $D(1)' = [D_0, D_x]$ in a partition of $D(1)'$ that is conformable with $u_t = (u_{0t}, u'_{xt})'$.

- (b) *The estimators β_M and β_M^+ have the common limit distribution*

$$\begin{aligned} T^a(\beta_M - \beta), T^a(\beta_M^+ - \beta) &\rightarrow_d \left[E\{\psi'(u_{0t})\} \int_0^1 U_{x\alpha} U'_{x\alpha} \right]^{-1} \left[\int_0^1 U_{x\alpha}^- dB_\psi \right] \\ &\equiv MN \left[0, \Omega_{\psi\psi} \left(E\{\psi'(u_{0t})\} \int_0^1 U_{x\alpha} U'_{x\alpha} \right)^{-1} \right]. \end{aligned}$$

Remark 6.2.

- (i) Theorem 6.1 shows that the robust estimators β_{LAD} and β_M are $O(T^a)$ consistent. Because $a = \frac{1}{2} + 1/\alpha > 1$ for $\alpha \in (0, 2)$, these estimators converge faster than the OLS and FM-OLS estimators, whose convergence rate is still $O(T)$ in the infinite variance case. The situation is analogous to the case of coefficient estimation in an AR(1) with a unit root. In that case, Knight (1989, 1991) showed that LAD and M -estimators of the unit root have a rate of convergence equal to $O(T^a)$, and Chan and Tran (1989) and Phillips (1991b) have shown that OLS and semiparametrically corrected OLS have convergence rates of $O(T)$. Thus, just as in the unit root case, the robust estimators β_{LAD} and β_M are infinitely more efficient than OLS-based estimation procedures when there are infinite variance errors.
- (ii) Interestingly, β_{LAD} and β_{LAD}^+ are asymptotically equivalent in the infinite variance case. Thus, there is no need to make corrections for endogeneity or serial correlation when the errors have infinite variance. Intuitively, this is because the robust estimators control the effects of outliers in the errors but retain the additional strength in the signal from x_t that arises from the presence of heavy-tailed and persistent shocks. In doing so, these estimators not only achieve a higher rate of convergence than OLS and FM-OLS, but they also remove the endogeneity effects of the regressors and the effects of dependence between the past history of the shocks that drive x_t and the equation error u_{0t} . In effect, whereas $T^{-2} \sum_1^T u_{xt} u_{0t} = O_p(1)$ (it converges weakly to the double stochastic integral or quadratic covariation process $\int_0^1 dU_{x\alpha} dU_{0\alpha}$, where $U_\alpha(r) = (U_{0\alpha}(r), U_{x\alpha}(r))'$), we have $T^{-2/\alpha} \sum_1^T u_{xt} \text{sgn}(u_{0t}) = o_p(1)$, so that the endogeneity and serial dependence effects wash out in robust estimation with heavy-tailed errors.
- (iii) Because no modifications to β_{LAD} are required in the infinite variance case, we may as well use β_{LAD} rather than β_{LAD}^+ if it were known that $\alpha < 2$. On the other hand, if we do use β_{LAD}^+ , then it follows from the theorem that nothing is lost asymptotically because the modifications in β_{LAD}^+ wash out in large samples. As we will see, however, in the simulations reported in the next section, clear evidence indicates that β_{LAD}^+ does pay a price for the modifications over β_{LAD} in terms of additional sampling dispersion.
- (iv) The mixed normality of the robust estimators in the limit means that standard errors, t -ratios, and Wald tests can be constructed in the usual way, as shown in Sections 4 and 5.

7. SOME SIMULATION RESULTS AND AN EMPIRICAL ILLUSTRATION

7.1. Simulations

A small simulation study was conducted to study the sampling performance of the new robust regression estimators. The model we used for data generation was the following:

$$\begin{aligned} y_t &= \beta x_t + u_{0t}, \quad \beta = 1, \\ \Delta x_t &= u_{xt}, \end{aligned} \tag{55}$$

where

$$\begin{aligned} u_{0t} &= \{1/(1+c^2)^{1/2}\}\varepsilon_{1t} + \{c/(1+c^2)^{1/2}\}\varepsilon_{2t}, \\ u_{xt} &= \varepsilon_{2t}, \end{aligned} \quad (56)$$

and ε_{1t} and ε_{2t} are each serially independent and independent of each other and are drawn from the following four distributions:

- D(a): $N(0,1)$,
- D(b): t distribution with 4 degrees of freedom (t_4),
- D(c): t distribution with 2 degrees of freedom (t_2), and
- D(d): standard Cauchy.

According to construction (56), the equation error u_{0t} is an orthonormal combination of the independent shocks $(\varepsilon_{1t}, \varepsilon_{2t})$. The parameter c controls the degree of association between u_{0t} and u_{xt} and, therefore, measures the amount of dependence in the regressor x_t in (55). When $c = 0$, x_t is exogenous; when $|c| = 1$, the squared correlation between u_{xt} and u_{0t} is $\frac{1}{2}$; and when $c \rightarrow \infty$, u_{xt} and u_{0t} become linearly dependent.

The parameter values chosen for our small simulation study were $c = -1, 0, 1$ and $T = 100$. We computed FM-OLS, FM-LAD, LAD, and RRR estimates of the regression coefficient in (55). All of the FM estimates were computed using a Parzen kernel with a “plug-in” optimal bandwidth, as in Andrews (1991). From 5,000 replications in each case, kernel density estimates were calculated of the sampling distributions of these estimates. The results are shown in Figures 2–5, where each figure in this sequence displays the outcome for an error distribution in aforementioned groups D(a)–D(d). The figures show the estimated densities of the LAD, FM-LAD, and FM-OLS estimates, as well as the estimates from a reduced rank regression with two lags in the regression (i.e., one lagged difference), which is denoted RRR_2 in the figure legends. The estimates are centered on the true coefficient and are scaled by the sample size, so that the given densities are those of $T(\hat{\beta} - \beta)$ for each estimator $\hat{\beta}$. In each case, we show the results for the association parameter value $c = 1$. Very similar results were obtained for $c = -1$ and $c = 0$, with the exception that LAD shows no bias in the latter case, as would be anticipated from the asymptotic theory given in Theorem 4.1 (noting that $\Delta_{xv} = 0$ and B_v and B_x are independent when $c = 0$).

Figure 2 gives the densities for normal errors. LAD is biased ($c = 1$); FM-OLS, FM-LAD, and RRR_2 are all well centered; FM-OLS shows the best concentration and, interestingly, FM-LAD has better concentration than RRR_2. Thus, although FM-OLS and RRR_2 are asymptotically optimal in this case, FM-LAD appears to do well and to be superior to RRR_2 in this finite sample case. We reran this simulation and found similar results and rankings for the RRR_1 estimator against FM-LAD. So the lag length choice does not appear to be a critical factor in these comparisons, at least for this set of parameters.

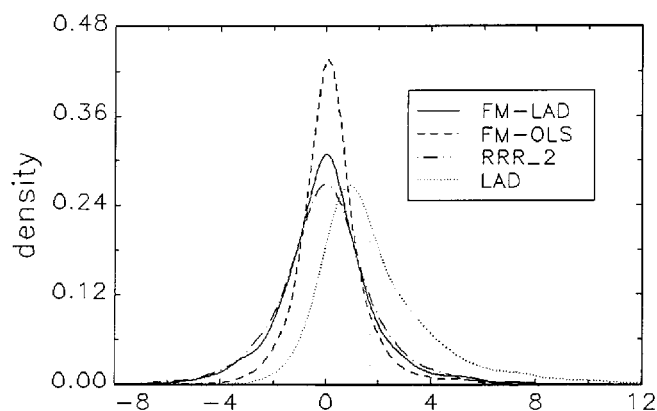
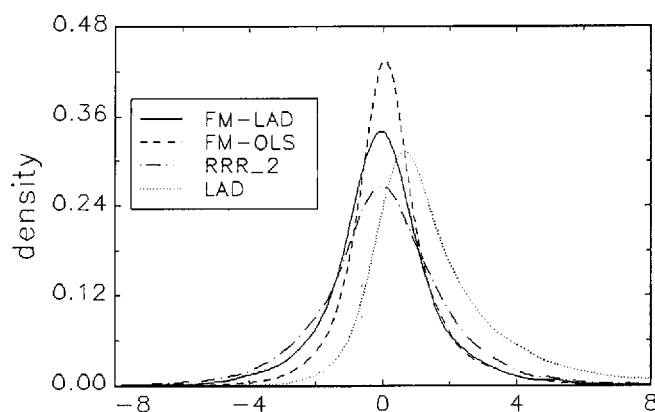


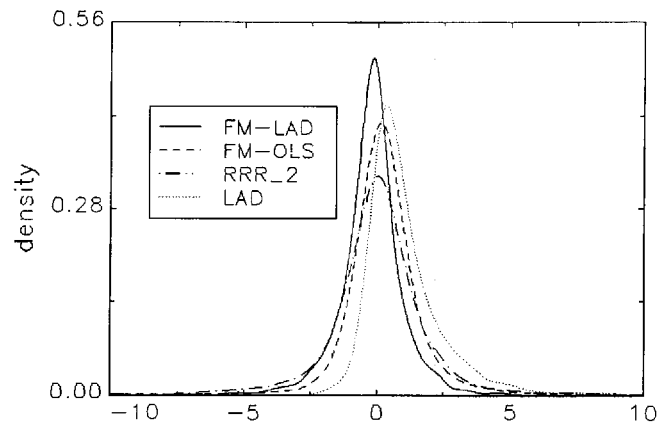
FIGURE 2. Normal errors.

Figure 3 gives the results for the case of t_4 error distributions. The outcome is very similar to the case of Gaussian errors. However, FM-LAD is now closer to FM-OLS, although FM-OLS still dominates. FM-LAD dominates RRR_2 by a wider margin than in the Gaussian error case. LAD is still biased (again $c = 1$).

Figure 4 gives the outcome for t_2 errors. Under these heavy-tailed error distributions, the rankings have changed. FM-LAD dominates both FM-OLS and RRR_2 in terms of concentration. FM-OLS continues to outperform RRR_2. LAD is much less biased in this case.

Figure 5 shows the same densities under Cauchy errors. The results are

FIGURE 3. t_4 errors.

FIGURE 4. t_2 errors.

dramatic. FM-OLS and RRR_2 are widely dispersed. FM-LAD dominates FM-OLS and RRR_2 by a wide margin, and LAD is by far the most concentrated. Note that in this case both LAD and FM-LAD have rates of convergence (here, order $T^{3/2}$) that exceed those of FM-OLS and RRR_2 (here, order T), so we expect both FM-OLS and RRR_2 to be poor in relation to the robust estimates. Although FM-LAD and LAD have the same limit distribution in this case (see Theorem 6.1), the sampling distributions are very different, with the LAD estimator showing much more concentra-

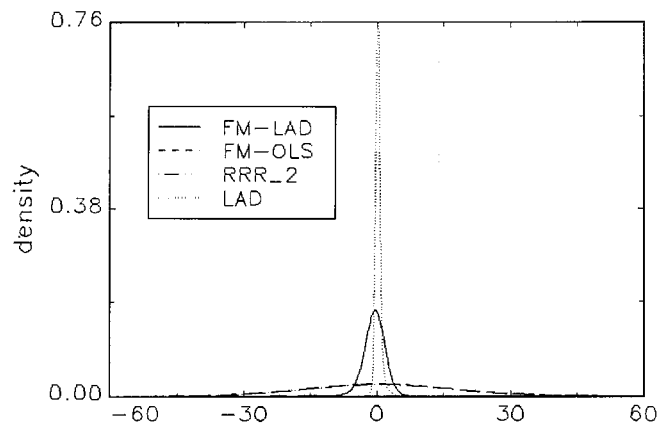


FIGURE 5. Cauchy errors.

tion. Thus, FM-LAD does pay a price in finite samples for the additional correction terms in this case of very heavy-tailed errors.

7.2. An Empirical Illustration

The robust and nonrobust regression procedures were used to estimate the foreign exchange market regression equation

$$s_{t+k} = \alpha + \beta f_{t,k} + u_{t+k} \quad (57)$$

that relates the natural logarithm of the forward exchange rate for a k -period ahead contract delivery $f_{t,k}$ to the logarithm of the future spot rate of the same currency s_{t+k} . Daily exchange rate data for the Australian dollar over the period January 1984 to April 1991 were used and the forward contract period was 3 months. There were 1,830 observations in total.

Figure 6 shows the sample data and the fitted regression lines obtained by FM-LAD, FM-OLS, and RRR_6 (reduced rank regression with six lags).

In spite of the large number of observations, there are big differences in the regression coefficients. Both FM-OLS and RRR_6 seem to be substantially affected by outlying observations (particularly the small spot rate and moderate forward rate pairs). The FM-LAD regression line seems much less affected by these outliers and seems to follow the general cluster of data more closely. The estimated coefficients and standard errors are given in Table 1, and these show that the numerical differences between the estimates are indeed substantial. Note that the FM-OLS and RRR_6 estimates of the slope coefficient are both much closer to unity than the FM-LAD estimate. Thus, inference about the forward rate unbiasedness hypothesis (under which $\beta = 1$

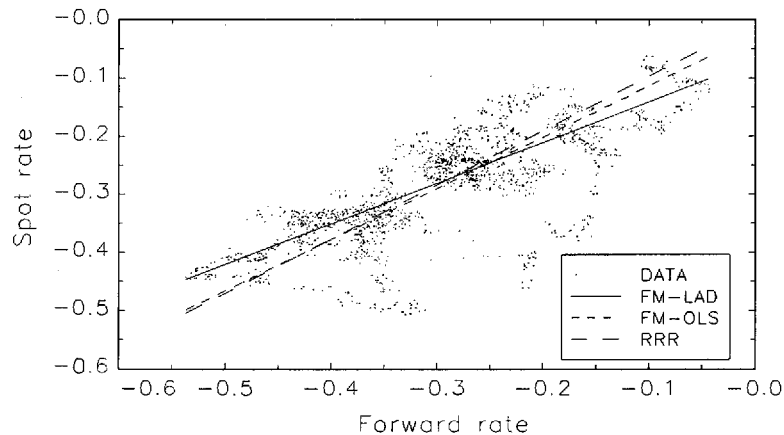


FIGURE 6. Scatter plot and regressions for equation (55).

TABLE 1. Estimates of equation (57) (standard errors in parentheses)

| | α | β |
|--------|----------------|---------------|
| FM-LAD | -0.071 (0.012) | 0.700 (0.040) |
| FM-OLS | -0.025 (0.029) | 0.883 (0.092) |
| RRR-6 | -0.003 (0.028) | 0.935 (0.089) |

in (57)) is affected by the regression procedure: the nonrobust estimates are biased in favor of this hypothesis, whereas the robust estimates do not support it. The reader is referred to the author's paper (1993) for a detailed empirical analysis of these data.

8. FURTHER USEFUL EXTENSIONS

The robust regression methods developed here are designed for use in single-equation nonstationary regression. They can be extended to multivariate regressions or subsystem cointegrating regression where there is more than one cointegrating relation. There is also the possibility of adaptive estimation, wherein the error distribution is estimated and used in the estimation of the regression coefficients. Jeganathan (1995) discusses this possibility in the context of regression models like (1) with serially independent errors and exogenous regressors. Given the extensive use of vector autoregressive models in empirical econometric research and the growing use of RRR methods in VAR models, it would seem useful to develop adaptive estimation methods for these models also.

9. PROOFS

Proof of Theorem 4.1. We start by defining the process

$$Z_T(g) = \sum_1^T \{|u_{0t} - T^{-1}x_t'g| - |u_{0t}|\}. \quad (\text{P.1})$$

The vector \hat{g} , which minimizes $Z_T(g)$, is just $\hat{g}_T = T(\beta_{\text{LAD}} - \beta)$. Because $Z_T(g)$ is convex, we can make use of the approach given by Knight (1989). In particular, by Knight's Lemma A it follows that if the finite dimensional distributions of $Z_T(g)$ converge to those of a process $Z(g)$ and $Z(g)$ has a unique minimum at \hat{g} , then the convexity of Z_T implies that $\hat{g}_T \rightarrow_d \hat{g}$. This also means that $\beta_{\text{LAD}} \rightarrow_p \beta$, and a separate argument for consistency of β_{LAD} is not required. (Pollard [1991] used a similar approach to LAD asymptot-

ics, but his Examples 1 and 2 give normal distribution limits and do not involve random quadratic elements in the limiting process.)

We will establish convergence of the unidimensional distributions of $Z_T(g)$, and then the higher dimensional distributions converge in a corresponding way by applying the Cramer-Wold device. Note that the process $Z_T(g)$ involves the ordinary random functions $|u_{0t} - T^{-1}x'_t g|$ and $|u_{0t}|$ and is itself an ordinary random process. However, it can also be treated as a generalized process (here a stochastic process defined in terms of generalized functions of random variables) by treating the function $f(\xi_t) = |\xi_t|$ of the random variable ξ_t as a generalized function of the random variable ξ_t , that is, by using the regular sequence of random variables

$$f_m(\xi_t) = \int_{-\infty}^{\infty} |v| S(m(v - \xi)) m e^{-v^2/m^2} dv$$

to represent $f(\xi_t)$ as in (11). Thus, as a generalized process, $Z_T(g)$ is defined by the following regular sequence of processes:

$$Z_{Tm}(g) = \sum_1^T \{f_m(u_{0t} - T^{-1}x'_t g) - f_m(u_{0t})\}. \quad (\text{P.2})$$

We now proceed to develop a Taylor expansion of $Z_{Tm}(g)$ and to characterize its limit behavior. Expanding $Z_{Tm}(g)$ in a Taylor series about $g = 0$, we have

$$Z_{Tm}(g) = -T^{-1} \sum_1^T f_m^{(1)}(u_{0t}) x'_t g + \left(\frac{1}{2}\right) T^{-2} \sum_1^T f_m^{(2)}(u_{0t}^*) g' x'_t x'_t g, \quad (\text{P.3})$$

where $f_m^{(1)}(\cdot)$ and $f_m^{(2)}(\cdot)$ denote the first and second derivatives of $f_m(\cdot)$ and u_{0t}^* lies between u_{0t} and $u_{0t} - T^{-1}x'_t g$. Because $f(\xi)$ has first derivative everywhere except $\xi = 0$ and $f'(\xi) = \text{sgn}(\xi)$ exists as an ordinary function, it follows that the regular sequence $f_m^{(1)}(\cdot)$ is a regular sequence for $\text{sgn}(\cdot)$ treated as a generalized function (Lighthill, 1958, Theorem 10, p. 24). Thus, $f_m^{(1)}(\cdot)$ is equivalent to the regular sequence $\text{sgn}_m(\cdot)$ given in (23). Similarly, $f_m^{(2)}(\cdot)$ is a regular sequence for the generalized function

$$d/d\xi(\text{sgn}(\xi)) = 2\delta(\xi)$$

(cf. Lighthill, 1958, p. 23) and is therefore equivalent to the regular sequence $2\delta_m(\cdot)$ given in (13).

Next, we consider the limit behavior of the two components of $Z_{Tm}(g)$ in (P.3). First, by Example 3.5, we have

$$T^{-1} \sum_1^T \text{sgn}_m(u_{0t}) x'_t g \rightarrow_d \left(\int_0^1 dB_m B'_x + \Delta'_{xm} \right) g, \quad (\text{P.4})$$

and the limit process as $m \rightarrow \infty$ is equivalent to $(\int_0^1 dB_v B'_x + \Delta)g$; that is,

$$\lim_{m \rightarrow \infty} \left(\int_0^1 dB_m B'_x + \Delta'_{xm} \right) g \equiv \left(\int_0^1 dB_v B'_x + \Delta'_{xv} \right) g, \quad (\text{P.5})$$

which is an ordinary random variable.

For the second term of (P.3), observe that the regular sequence $\delta_m(\cdot)$ is differentiable and has bounded derivative (with a bound dependent on m) for all m . Thus,

$$|\delta_m(u_{0t}^*) - \delta_m(u_{0t})| \leq K_m |T^{-1} x'_t g|, \quad \forall m.$$

and therefore

$$\begin{aligned} & \left| T^{-2} \sum_1^T \{ \delta_m(u_{0t}) - \delta_m(u_{0t}^*) \} g' x_t x'_t g \right| \\ & \leq K_m T^{-3} \sum_1^T |(x'_t g)| (g' x_t x'_t g) \rightarrow_p 0, \quad \forall m \end{aligned}$$

uniformly over g in compact sets. Now, using Example 3.3, we have

$$T^{-2} \sum_1^T \delta_m(u_{0t}) g' x_t x'_t g \rightarrow_d E\{ \delta_m(u_{0t}) \} \int_0^1 (g' B_x)^2, \quad (\text{P.6})$$

whose limit as $m \rightarrow \infty$ is $g' \int_0^1 B_x B'_x g$.

Combining (P.4) and (P.6), we deduce that

$$\begin{aligned} Z_{Tm}(g) & \rightarrow_d - \left(\int_0^1 dB_m B'_x + \Delta'_{xm} \right) g + E\{ \delta_m(u_{0t}) \} g' \left(\int_0^1 B_x B'_x \right) g \\ & = Z_m(g), \quad \text{say, } \forall m \end{aligned} \quad (\text{P.7})$$

uniformly over g in compact sets. In view of (P.5) and because $\lim_{m \rightarrow \infty} E\{ \delta_m(u_{0t}) \} = E\{ \delta(u_{0t}) \} = \text{pdf}(0)$, the limit process $Z_m(g)$ has the following equivalent representation as $m \rightarrow \infty$:

$$Z(g) = - \left(\int_0^1 dB_v B'_x + \Delta'_{xv} \right) g + \text{pdf}(0) g' \left(\int_0^1 B_x B'_x \right) g, \quad (\text{P.8})$$

which is an ordinary random variable.

Because $Z_{Tm}(g) \rightarrow_d Z_m(g)$, $\forall m$, and $\lim_{m \rightarrow \infty} Z_m(g) \equiv Z(g)$, we have established the weak convergence of $Z_T(g) \rightarrow_d Z(g)$ as generalized processes uniformly over g in compact sets. But both $Z_T(g)$ and $Z(g)$ exist as ordinary random processes so that the weak convergence applies in this sense also. The argument that we can neglect the region outside a suitable compact set for g relies on the convexity of $Z_T(g)$ and is the same as that given in Knight (1989, p. 277). Finally,

$$\hat{g} = \operatorname{argmin} Z(g) = \left[2 \operatorname{pdf}(0) \int_0^1 B_x B'_x \right]^{-1} \left[\int_0^1 B_x dB_v + \Delta_{xv} \right], \quad (\text{P.9})$$

and we deduce that $\hat{g}_T = T(\beta_{\text{LAD}} - \beta) \rightarrow_d \hat{g}$ as required. ■

Proof of Theorem 4.3. Start by writing the estimation error as

$$\begin{aligned} T(\beta_{\text{LAD}}^+ - \beta) &= T(\beta_{\text{LAD}} - \beta) \\ &\quad - (1/2\hat{h}(0))(T^{-2}X'X)^{-1}[T^{-1}X'\Delta X\hat{\Omega}_{xx}^{-1}\hat{\Omega}_{xv} + \hat{\Delta}_{xv}^+]. \end{aligned}$$

Then, using Theorem 4.1 and (6), we obtain

$$\begin{aligned} T(\beta_{\text{LAD}}^+ - \beta) &\rightarrow_d \left[2h(0) \int_0^1 B_x B'_x \right]^{-1} \left[\int_0^1 B_x dB_v + \Delta_{xv} \right] \\ &\quad - (1/2h(0)) \left(\int_0^1 B_x B'_x \right)^{-1} \left[\left(\int_0^1 B_x dB'_x + \Delta_{xx} \right) \Omega_{xx}^{-1} \Omega_{xv} + \Delta_{xv}^+ \right] \\ &= \left[2h(0) \int_0^1 B_x B'_x \right]^{-1} \left[\int_0^1 B_x (dB_v - \Omega_{vx} \Omega_{xx}^{-1} dB_x) \right] \\ &= \left[2h(0) \int_0^1 B_x B'_x \right]^{-1} \int_0^1 B_x dB_{v \cdot x} \\ &\equiv MN \left(0, (1/2h(0))^2 \omega_{vv \cdot x} \left[\int_0^1 B_x B'_x \right]^{-1} \right), \quad \blacksquare \end{aligned}$$

as required.

Proof of Theorem 5.1. The argument follows the general lines of Knight (1989, Theorem 2). Take case (a) of ρ convex and define

$$Z_T(g) = \sum_1^T \{ \rho(u_{0t} - T^{-1}x'_t g) - \rho(u_{0t}) \},$$

so that if \hat{g}_T minimizes $Z_T(g)$ we have $\hat{g}_T = T(\beta_M - \beta)$. Then, by virtue of the convexity of $Z_T(g)$, we have $\hat{g}_T \rightarrow_d g = \operatorname{argmin} Z(g)$, where $Z(g)$ is the weak limit of $Z_T(g)$. As in the proof of Theorem 4.1, we need only establish finite dimensional convergence of $Z_T(g)$ to $Z(g)$.

Taylor expansion of $Z_T(g)$ around $g = 0$ gives

$$Z_T(g) = -T^{-1} \sum_1^T \psi(u_{0t}) x'_t g + \left(\frac{1}{2} \right) T^{-2} \sum_1^T \psi'(u_{0t}^*) g' x_t x'_t g, \quad (\text{P.10})$$

where u_{0t}^* lies between u_{0t} and $u_{0t} - T^{-1}x'_t g$. Now, $|\psi'(u_{0t}) - \psi'(u_{0t}^*)| < K|T^{-1}x'_t g|$ for some $K > 0$ and therefore

$$\begin{aligned}
T^{-2} \sum_1^T |\psi'(u_{0t}) - \psi'(u_{0t}^*)| g' x_t x_t' g \\
\leq KT^{-3/2} \sum_1^T (T^{-1/2} x_t' g) (T^{-1} g' x_t x_t' g) \rightarrow_p 0
\end{aligned} \tag{P.11}$$

uniformly over g in compact sets. Next,

$$\begin{aligned}
T^{-2} \sum_1^T [\psi'(u_{0t}) - E\{\psi'(u_{0t})\}] g' x_t x_t' g \\
= T^{-1/2} g' \left[\sum_1^T T^{-1/2} [\psi'(u_{0t}) - E\{\psi'(u_{0t})\}] (T^{-1/2} x_t) (T^{-1/2} x_t') \right] g \\
= O_p(T^{-1/2}),
\end{aligned} \tag{P.12}$$

uniformly in g because the expression in large brackets converges to a stochastic integral with random drift, just as in (20). Finally,

$$T^{-2} \sum_1^T E\{\psi'(u_{0t})\} g' x_t x_t' g \rightarrow_d E\{\psi'(u_{0t})\} g' \int_0^1 B_x B_x' g \tag{P.13}$$

and

$$T^{-1} \sum_1^T \psi(u_{0t}) x_t' g \rightarrow_d g' \left(\int_0^1 B_x dB_\psi + \Delta_{x\psi} \right), \tag{P.14}$$

because $\psi(u_{0t})$ satisfies the functional law

$$T^{-1/2} \sum_1^T \psi(u_{0t}) \rightarrow B_\psi \equiv BM(\Omega_{\psi\psi})$$

and the conditions for the convergence to the stochastic integral with drift in (P.12) in view of Assumption ML. Combining (P.13) and (P.14) with (P.11) and (P.12) provides the following limit for $Z_T(g)$:

$$Z_T(g) \rightarrow_d -g' \left(\int_0^1 B_x dB_\psi + \Delta_{x\psi} \right) + \left(\frac{1}{2} \right) E\{\psi'(u_{0t})\} g' \int_0^1 B_x B_x' g = Z(g).$$

We deduce that

$$\hat{g}_T \rightarrow_d \hat{g} = \left[E\{\psi'(u_{0t})\} \int_0^1 B_x B_x' \right]^{-1} \left[\int_0^1 B_x dB_\psi + \Delta_{x\psi} \right],$$

giving the required result.

In case (b), where $T^{1/2}(\beta_M - \beta) = o_p(1)$ and β_M satisfies (41), we expand the first-order conditions, giving

$$0 = T^{-1} \sum_1^T x_t \psi(u_{0t}) - T^{-2} \sum_1^T \psi'(u_{0t}) x_t x_t' T(\beta_M - \beta) + T^{-1} R_T,$$

where

$$T^{-1}R_T = T^{-1} \sum_1^T \{\psi'(u_{0t}) - \psi'(u_{0t}^*)\} x_t x_t' (\beta_M - \beta),$$

and u_{0t}^* lies between u_{0t} and $u_{0t} + x_t'(\beta - \beta_M)$. Now,

$$\begin{aligned} |T^{-1}R_T| &\leq KT^{-1} \sum_1^T \|x_t\|^3 |\beta_M - \beta|^2 \\ &= KT^{-1} \sum_1^T \|T^{-1/2}x_t\|^3 |T(\beta_M - \beta)| |T^{1/2}(\beta_M - \beta)| \\ &= |T(\beta_M - \beta)| o_p(1). \end{aligned}$$

Hence,

$$\begin{aligned} T(\beta_M - \beta) &= \left[T^{-2} \sum_1^T \psi'(u_{0t}) x_t x_t' + o_p(1) \right]^{-1} \left[T^{-1} \sum_1^T x_t \psi(u_{0t}) \right] \\ &\rightarrow_d \left[E\{\psi'(u_{0t})\} \int_0^1 B_x B_x' \right]^{-1} \left[\int_0^1 B_x dB_\psi + \Delta_{x\psi} \right], \end{aligned}$$

just as in the case of convex ρ . ■

9.1. Addendum to Theorem 5.1: $\psi(\cdot)$ Nonsmooth

We will consider here the case where $\psi(\cdot)$ is differentiable except for a countable number of points of \mathbb{R} . We will retain the other conditions of Assumption ML. The arguments follow the same general lines as those given in the proof of Theorem 4.1 for the LAD estimator.

Take case (a), where ρ is convex. As in the LAD proof, we need to show that $Z_T(g) = \sum_1^T \{\rho(u_{0t} - T^{-1}x_t'g) - \rho(u_{0t})\}$ has a suitable quadratic approximation as $T \rightarrow \infty$. Because $\psi(\cdot)$ is not everywhere differentiable, we cannot use (P.10). Instead, we proceed by treating the ordinary function $\rho(\cdot)$ in $Z_T(g)$ as a generalized function by means of the corresponding regular sequence $\rho_m(\cdot)$ given by

$$\rho_m(u) = \int_{-\infty}^{\infty} \rho(v) S(m(v - u)) m e^{-v^2/m^2} dv.$$

(The existence of this integral poses no practical constraints on $\rho(v)$, which will, for robust estimation purposes, generally be bounded by a function that is at most $O(v^2)$ as $|v| \rightarrow \infty$.) Then, $Z_T(g)$ is defined by the regular sequence of processes

$$Z_{Tm}(g) = \sum_1^T \{\rho_m(u_{0t} - T^{-1}x_t'g) - \rho_m(u_{0t})\}.$$

Expanding Z_{Tm} in a Taylor series about $g = 0$ gives

$$Z_{Tm}(g) = T^{-1} \sum_1^T \psi_m(u_{0t}) x'_t g + \left(\frac{1}{2}\right) T^{-2} \sum_1^T \psi'_m(u_{0t}^*) g' x'_t x'_t g, \quad (\text{P.15})$$

where $\psi_m(\cdot) = \rho'_m(\cdot)$ is a regular sequence for $\psi(\cdot) = \rho'(\cdot)$ and $\psi'_m(\cdot)$ is a regular sequence for $\psi'(\cdot)$, where both ψ and ψ' are treated as generalized functions.

We examine the limit behavior of the two components of $Z_{Tm}(g)$ separately. First, as in Example 3.5, we get

$$T^{-1} \sum_1^T \psi_m(u_{0t}) x'_t g \rightarrow_d \left(\int_0^1 dB_{\psi_m} B'_x + \Delta'_{x\psi_m} \right) g, \quad (\text{P.16})$$

where $B_{\psi_m} = BM(\Omega_{\psi_m\psi_m})$, $\Omega_{\psi_m\psi_m} = \sum_{j=-\infty}^{\infty} E\{\psi_m(u_{0t})\psi_m(u_{0t+j})\}$, and $\Delta_{x\psi_m} = \sum_{j=0}^{\infty} E\{u_{xt}\psi_m(u_{0t+j})\}$. The limit process in (P.16) as $m \rightarrow \infty$ is

$$\lim_{m \rightarrow \infty} \left(\int_0^1 dB_{\psi_m} B'_x + \Delta'_{x\psi_m} \right) g \equiv \left(\int_0^1 dB_{\psi} B'_x + \Delta'_{x\psi} \right) g. \quad (\text{P.17})$$

In the second term of (P.15), $\psi'_m(\cdot)$ is a regular sequence and therefore is differentiable with a bounded derivative for each m . Thus,

$$|\psi'_m(u_{0t}^*) - \psi'_m(u_{0t})| \leq K_m T^{-1} x'_t g, \quad \forall m$$

for some $K_m > 0$ and

$$T^{-2} \sum_1^T \{\psi'_m(u_{0t}) - \psi'_m(u_{0t}^*)\} g' x'_t x'_t g \leq K_m T^{-3} \sum_1^T \|x_t\|^3 \|g\|^3 \rightarrow_p 0, \quad \forall m$$

uniformly over g in compact sets. Just as in Example 3.3, we now obtain the limit

$$T^{-2} \sum_1^T \psi'_m(u_{0t}) g' x'_t x'_t g \rightarrow_d E\{\psi'_m(u_{0t})\} \int_0^1 (g' B_x)^2. \quad (\text{P.18})$$

By definition of the regular sequence $\psi'_m(\cdot)$, we have the limit

$$\lim_{m \rightarrow \infty} \int_{-\infty}^{\infty} \psi'_m(u) h(u) du = \int_{-\infty}^{\infty} \psi'(u) h(u) du = - \int_{-\infty}^{\infty} \psi(u) h'(u) du,$$

which exist as ordinary Riemann integrals; that is, we have

$$E\{\psi'_m(u_{0t})\} \rightarrow E\{\psi'(u)\}. \quad (\text{P.19})$$

Combining (P.16) and (P.18) gives us

$$Z_{Tm}(g) \rightarrow_d - \left(\int_0^1 dB_{\psi_m} B'_x + \Delta'_{x\psi_m} \right) g \\ + \left(\frac{1}{2} \right) E\{\psi'_m(u_{0t})\} g' \int_0^1 B_x B'_x g = Z_m(g), \quad \text{say,}$$

and in view of (P.17) and (P.19)

$$\lim_{m \rightarrow \infty} Z_m(g) \equiv - \left(\int_0^1 dB_{\psi} B'_x + \Delta'_{x\psi} \right) g \\ + \left(\frac{1}{2} \right) E\{\psi'(u_{0t})\} g' \int_0^1 B_x B'_x g = Z(g).$$

This establishes the weak convergence of $Z_T(g) \rightarrow_d Z(g)$ as generalized processes uniformly in g over compact sets. The argument then follows as in the proof of Theorem 4.1, and we get

$$\hat{g}_T = \operatorname{argmin} Z_T(g) \rightarrow_d \hat{g} = \operatorname{argmin} Z(g),$$

and thus the conclusion of Theorem 5.1 continues to apply in this case where $\psi(\cdot)$ is not everywhere differentiable.

An analogous proof when ρ is not necessarily convex (i.e., case (b) of Theorem 5.1) is constructed by following the lines of the second part of the proof of Theorem 5.1 and using generalized functions of random variables in the same way as the earlier part of this addendum.

Proof of Theorem 5.2. The error of estimation is

$$T(\beta_M^+ - \beta) = T(\beta_M - \beta) \\ - \left\{ 1/T^{-1} \sum_1^T \psi'(\hat{u}_{0t}) \right\} (T^{-2} X' X)^{-1} [T^{-1} X' \Delta X \hat{\Omega}_{xx}^{-1} \hat{\Omega}_{x\psi} + \hat{\Delta}_{x\psi}^+].$$

Note that

$$\left| T^{-1} \sum_1^T \{\psi'(\hat{u}_{0t}) - \psi'(u_{0t})\} \right| \leq K T^{-1} \sum_1^T \|x_t\| \|\beta_M - \beta\| \rightarrow_p 0$$

and

$$T^{-1} \sum_1^T \psi'(u_{0t}) \rightarrow_{\text{a.s.}} E\{\psi'(u_{0t})\}$$

so that

$$T^{-1} \sum_1^T \psi'(\hat{u}_{0t}) \rightarrow_p E\{\psi'(u_{0t})\}.$$

In a similar way, we can replace $\psi(u_{0t})$ by $\hat{\psi} = \psi(\hat{u}_{0t})$ in the sample covariances that enter into the formulae for $\hat{\Omega}_{x\psi}$ and $\hat{\Delta}_{x\psi}$ and retain the consistency of these estimators. Then, using Theorem 5.1, we get

$$\begin{aligned}
 T(\beta_M^+ - \beta) &\rightarrow_d \left[E\{\psi'(u_{0t})\} \int_0^1 B_x B_x' \right]^{-1} \left[\int_0^1 B_x dB_\psi + \Delta_{x\psi} \right] \\
 &\quad - (1/E\{\psi'(u_{0t})\}) \left(\int_0^1 B_x B_x' \right)^{-1} \\
 &\quad \times \left[\left(\int_0^1 B_x dB_x' + \Delta_{xx} \right) \Omega_{xx}^{-1} \Omega_{x\psi} + \Delta_{x\psi}^+ \right] \\
 &= \left[E\{\psi'(u_{0t})\} \int_0^1 B_x B_x' \right]^{-1} \left[\int_0^1 B_x dB_\psi - \int_0^1 B_x dB_x' \Omega_{xx}^{-1} \Omega_{x\psi} \right] \\
 &= \left[E\{\psi'(u_{0t})\} \int_0^1 B_x B_x' \right]^{-1} \left[\int_0^1 B_x dB_{\psi \cdot x} \right] \\
 &\equiv MN \left(0, \omega_{\psi \cdot x} [E\{\psi'(u_{0t})\}]^{-2} \left[\int_0^1 B_x B_x' \right]^{-1} \right)
 \end{aligned}$$

as given in (46). ■

Proof of Theorem 6.1. We first consider β_{LAD} , and our line of approach is the same as in the proof of Theorem 4.1. However, instead of (P.1), we take

$$Z_T(g) = \sum_1^T \{|u_{0t} - T^{-a}x_t'g| - |u_{0t}|\} \quad \text{and} \quad \hat{g}_T = \text{argmin } Z_T(g)$$

with $a = \frac{1}{2} + 1/\alpha$. As before, we treat $Z_T(g)$ as a generalized process, defined in terms of the regular sequence

$$Z_{Tm}(g) = \sum_1^T \{f_m(u_{0t} - T^{-a}x_t'g) - f_m(u_{0t})\},$$

and use the Taylor expansion

$$Z_{Tm}(g) = -T^{-a} \sum_1^T f_m^{(1)}(u_{0t})x_t'g + \left(\frac{1}{2}\right) T^{-2a} \sum_1^T f_m^{(2)}(u_{0t}^*)g'x_tx_t'g, \quad (\text{P.20})$$

where u_{0t}^* lies between u_{0t} and $u_{0t} - T^{-a}x_t'g$. Here the sequence $f_m^{(1)}(\cdot)$ is equivalent to $\text{sgn}_m(\cdot)$ and $f_m^{(2)}(\cdot)$ to $\delta_m(\cdot)$, as defined earlier.

First, consider the second term of (P.20). We use the "BN" decomposition

$$u_t = D(L)\varepsilon_t = D(1)\varepsilon_t + \tilde{\varepsilon}_{t-1} - \tilde{\varepsilon}_t \quad (\text{P.21})$$

(see Phillips and Solo, 1992), where $\tilde{\varepsilon}_t = \tilde{D}(L)\varepsilon_t$ and $\tilde{D}(L) = \sum_0^\infty \tilde{D}_j L^j$ with $\tilde{D}_j = \sum_{k=j+1}^\infty D_k$. Now, in view of (53),

$$\sum_0^\infty \|\tilde{D}_j\|^\delta \leq \sum_0^\infty k \|D_k\|^\delta < \infty,$$

and thus $\tilde{\varepsilon}_t = \sum_0^\infty \tilde{D}_j \varepsilon_{t-j}$ converges almost surely and $\tilde{\varepsilon}_t \in \mathcal{D}(\alpha)$. Now, set $P_t = \sum_1^t \varepsilon_j$ and $W_t = \sum_1^t u_j$. We have $W_t = D(1)P_t + \tilde{\varepsilon}_0 - \tilde{\varepsilon}_t$ and then

$$\begin{aligned} T^{-2a} \sum_1^T W_t W_t' &= D(1) T^{-2a} \sum_1^T P_t P_t' D(1)' + D(1) T^{-2a} \sum_1^T P_t (\tilde{\varepsilon}_0 - \tilde{\varepsilon}_t) \\ &\quad + T^{-2a} \sum_1^T (\tilde{\varepsilon}_0 - \tilde{\varepsilon}_t) P_t' D(1)' + T^{-2a} \sum_1^T (\tilde{\varepsilon}_0 - \tilde{\varepsilon}_t) (\tilde{\varepsilon}_0 - \tilde{\varepsilon}_t)'. \end{aligned} \quad (\text{P.22})$$

Note that $\tilde{\varepsilon}_t \in \mathcal{D}(\alpha/2)$, so that $T^{-2/\alpha} \sum_1^T \tilde{\varepsilon}_t \tilde{\varepsilon}_t' = O_p(1)$, $T^{-1/\alpha} \sum_1^T \tilde{\varepsilon}_t = O_p(1)$ and therefore the final term of (P.22) is $o_p(1)$. Also, $T^{-1-1/\alpha} \sum_1^T P_t = O_p(1)$ and $T^{-2a} \sum_1^T P_t \tilde{\varepsilon}_t = o_p(1)$ (the latter can be shown by using a further "BN" decomposition for $\tilde{\varepsilon}_t$). Hence, (P.22) is dominated by the first term. However, $T^{-1/\alpha} P_{[Tr]} \rightarrow_d U_\alpha(r)$ by (54), and by virtue of the continuous mapping theorem we obtain

$$\begin{aligned} T^{-2a} \sum_1^T W_t W_t' &= D(1) T^{-2a} \sum_1^T P_t P_t' D(1)' + o_p(1) \\ &\rightarrow_d D(1) \left(\int_0^1 U_\alpha(r) U_\alpha(r)' dr \right) D(1)'. \end{aligned} \quad (\text{P.23})$$

We deduce that

$$T^{-2a} \sum_1^T x_t x_t' \rightarrow_d \int_0^1 U_{x\alpha}(r) U_{x\alpha}(r)' dr,$$

where $U_{x\alpha}(r) = D_x' U_\alpha(r)$ and $D' = [D_0 D_x]$ is partitioned conformably with $u_t = (u_{0t}, u_{xt}')'$. In the same way as in the proof of Theorem 4.1, we can now show that

$$T^{-2a} \sum_1^T \{\delta_m(u_{0t}) - \delta_m(u_{0t}^*)\} g' x_t x_t' g \rightarrow_p 0$$

uniformly over g in compact sets, and

$$T^{-2a} \sum_1^T \delta_m(u_{0t}) g' x_t x_t' g \rightarrow_d E\{\delta_m(u_{0t})\} \int_0^1 (g' U_{x\alpha})^2.$$

Next, consider the first term of (P.20). Noting that $f_m^{(1)}(u_{0t}) = \text{sgn}_m(u_{0t})$, which is a sequence of strictly stationary bounded functions of u_{0t} , we have the martingale difference decomposition (see Hall and Heyde, 1980)

$$\text{sgn}_m(u_{0t}) = Y_{mt} + Q_{mt} - Q_{mt-1}, \quad \forall m, \quad (\text{P.24})$$

where the Y_{mt} are stationary square integrable ergodic martingale differences (with respect to the filtration generated by $\{u_{0j} : j \leq t\}$) and the Q_{mt} are square integrable stationary processes $\forall m$. As in the proof of Lemma 2

of Knight (1991), we can use (P.24) and the BN decomposition for u_{xt} that follows from (P.21) to establish the weak convergence

$$T^{-a} \sum_1^T \text{sgn}_m(u_{0t}) x'_t = \sum_1^T (T^{-1/2} \text{sgn}_m(u_{0t})) (T^{-1/\alpha} x'_t) \rightarrow_d \int_0^1 dB_m U_{x\alpha}^{-'}, \quad (\text{P.25})$$

where $U_{x\alpha}^{-}$ signifies the left limit of $U_{x\alpha}(\cdot)$. In the limiting stochastic integral (P.25), the Brownian motion $B_m \equiv BM(\Omega_m)$ is stochastically independent of the stable process $U_{x\alpha}$. There is also no drift or bias term in the limit (P.25), unlike the finite variance case. The independence is a consequence of the different rates of convergence to B_m and $U_{x\alpha}$ and follows from a result originally shown by Resnick and Greenwood (1979).

Combining these results produces

$$\begin{aligned} Z_{Tm}(g) &\rightarrow_d - \left(\int_0^1 dB_m U_{x\alpha}^{-'} \right) g + E\{\delta_m(u_{0t})\} g' \left(\int_0^1 U_{x\alpha} U_{x\alpha}' \right) g \\ &= Z_m(g), \quad \text{say } \forall m, \end{aligned}$$

and, as in the proof of Theorem 4.1, the convergence holds uniformly over g in compact sets. Again, because $\lim_{m \rightarrow \infty} B_m(r) = B_v \equiv BM(\Omega_{vv})$ and $\lim_{m \rightarrow \infty} E\{\delta_m(u_{0t})\} = E\{\delta(u_{0t})\} = \text{pdf}(0)$, we have

$$\lim_{m \rightarrow \infty} Z_m(g) \equiv - \left(\int_0^1 dB_v U_{x\alpha}^{-'} \right) g + \text{pdf}(0) g' \left(\int_0^1 U_{x\alpha} U_{x\alpha}' \right) g = Z(g), \quad \text{say},$$

which is an ordinary random variable. The remainder of the argument now follows exactly as in Theorem 4.1, and the result for $T^a(\beta_{\text{LAD}} - \beta)$ is established.

Next, consider the estimator β_{LAD}^+ . We have

$$\begin{aligned} T^a(\beta_{\text{LAD}}^+ - \beta) &= T^a(\beta_{\text{LAD}} - \beta) \\ &\quad - \left[2\hat{h}(0) T^{-2a} \sum_1^T x_t x'_t \right]^{-1} \\ &\quad \times \left[T^{-2a} \left(\sum_1^T x_t u_{xt}' \right) \hat{\Omega}_{xx}^{-1} \hat{\Omega}_{xv} + T^{1-2a} \hat{\Delta}_{xv}^+ \right]. \quad (\text{P.26}) \end{aligned}$$

We need to show that the second term on the right of (P.26) is $o_p(1)$. Because β_{LAD} is consistent (from the first part of the proof) and LAD residuals are used in the construction of $\hat{h}(0)$ and the long-run variance matrix estimates that appear in (P.26), we may proceed as if these estimates were constructed using the true errors u_{0t} . Then, $\hat{h}(0) \rightarrow_p h(0)$, and following the same line of argument as that given in Section 2.3 of Phillips (1991b), we find that $T^{1-2a} \hat{\Omega}_{xx} = O_p(1)$, $T^{1-2a} \hat{\Delta}_{xx} = O_p(1)$, $T^{1-2a} \hat{\Omega}_{xv} = o_p(1)$, and $T^{1-2a} \hat{\Delta}_{xv} = o_p(1)$. Then,

$$\begin{aligned}
& T^{-2a} \left(\sum_1^T x_t u'_{xt} \right) \hat{\Omega}_{xx}^{-1} \hat{\Omega}_{xv} + T^{1-2a} \Delta_{xv}^+ \\
&= \left(T^{1-2a} \sum_1^T x_t u'_{xt} \right) (T^{1-2a} \hat{\Omega}_{xx})^{-1} (T^{1-2a} \hat{\Omega}_{xv}) \\
&\quad + T^{-2/\alpha} \Delta_{xv} - (T^{1-2a} \hat{\Delta}_{xx}) (T^{1-2a} \hat{\Omega}_{xx})^{-1} (T^{-2/\alpha} \hat{\Omega}_{xv}) \\
&= (O_p(1))(O_p(1))^{-1} o_p(1) + o_p(1) - O_p(1)(O_p(1))^{-1} o_p(1) = o_p(1).
\end{aligned}$$

We deduce from (P.26) that $T^a(\beta_{\text{LAD}}^+ - \beta) = T^a(\beta_{\text{LAD}} - \beta) + o_p(1)$ and the stated result follows.

A similar argument gives the limit distribution of $T^a(\beta_M - \beta)$ and shows the asymptotic equivalence of β_M and β_M^+ . ■

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