

A SHORTCUT TO LAD ESTIMATOR ASYMPTOTICS

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Using generalized functions of random variables and generalized Taylor series expansions, we provide quick demonstrations of the asymptotic theory for the LAD estimator in a regression model setting. The approach is justified by the smoothing that is delivered in the limit by the asymptotics, whereby the generalized functions are forced to appear as linear functionals wherein they become real valued. Models with fixed and random regressors, and autoregressions with infinite variance errors are studied. Some new analytic results are obtained including an asymptotic expansion of the distribution of the LAD estimator.

Shall I refuse my dinner because I do not fully understand the process of digestion?

O. HEAVISIDE

1. INTRODUCTION

Classical asymptotic methods such as those given in Cramér [7, Ch. 33] are usually thought to apply only in cases of regular estimation. Here, smoothness conditions facilitate the use of Taylor expansions of the objective function and the associated first-order conditions. When supplemented with preliminary consistency arguments, these expansions then yield the required asymptotic distribution theory in a few simple steps. With more tiresome algebra and higher-order smoothness conditions, they also supply the formulae for Edgeworth expansions to second and higher orders.

The absence of smoothness in the criterion function that occurs in what are called nonregular cases is generally thought to prevent the use of this classical approach. Early studies that dealt with nonregularity complications by direct methods were Daniels [8] and Huber [11]. More recently there has been a move toward the use of stochastic equicontinuity arguments and empirical process techniques to address the complications that are presented by the absence of smoothness and continuity in the criterion function. These techniques have been successful in accommodating a wide range of nonregular cases, including simulation-based optimization estimators such as those that are employed by McFadden [14] and Pakes and Pollard [16]. They have also

David Pollard's paper [23], as it was presented to a Yale econometrics seminar in October 1988, was the immediate stimulus for writing this note. My thanks go to two referees and David Pollard for comments on an earlier version, to Glenna Ames for wordprocessing, and to the NSF for research support under Grant No. SES 8821180.

found attractive applications in the development of asymptotics for semiparametric models [2]. The work of Pollard [20,21,22,23] has been especially influential in advancing the use of these techniques in econometric applications. However, as pointed out by Pollard [23], stochastic equicontinuity arguments are less accessible to many potential users even though they often capture the key technical difficulty in the asymptotics. Partly in response to this objection, Pollard presents an alternative approach in [23] for studying the asymptotic theory of the least absolute deviation (LAD) estimator in a simple regression context. Pollard's alternative approach builds on the convexity of the LAD criterion function to construct a quadratic approximation whose minimand is close enough to the LAD estimator for the latter to share the same asymptotic normal distribution.

The present note is related closely to Pollard's paper [23]. However, instead of putting forward an alternative approach, our objective is to show the serviceability of the classical approach in nonregular problems like that of the LAD estimator. The idea we put forward is very simple. If the criterion function has nonregularities like discontinuities in its derivatives, these may be accommodated directly by the use of generalized functions, provided the discontinuities are smoothed out asymptotically. First-order conditions and Taylor representations can be written down in the usual way but they take the form of generalized Taylor series. They may be formally interpreted as linear functionals in terms of the empirical distribution function. As the sample size $n \rightarrow \infty$, these linear functionals become well behaved provided some basic smoothness conditions are imposed on the underlying probability law of the data. In effect, with this generalization of the classical approach, the asymptotics provide the smoothness that is required to justify the Taylor development and thereby the resulting asymptotics.

Our approach is heuristic and we do not claim to deal rigorously with all of the mathematical issues that arise. However, it is hoped that our extended treatment of LAD asymptotics will serve to illustrate the utility of these ideas and to stimulate the interest of others in the use of these methods. To continue the theme put forward by Heaviside in the line that heads this article, we believe there is still good food to enjoy in classical dinners.

2. LAD ASYMPTOTICS: THE HEURISTICS

Suppose y_t is generated by the linear regression

$$y_t = x_t' \beta^0 + u_t \quad (t = 1, \dots, n), \quad (1)$$

where the parameter vector $\beta^0 \in \mathbb{R}^k$, the errors u_t satisfy (\mathcal{Q}_1) below and (x_t) constitutes a bounded, deterministic sequence for which

$$Q_n = n^{-1} \sum_1^n x_t x_t' \rightarrow Q, \quad (2)$$

a positive definite limit, as $n \rightarrow \infty$.

(\mathcal{Q}_1) The sequence (u_t) is i.i.d. with zero median and probability density $f(\cdot)$ that is positive and analytic at zero.

The LAD estimator $\hat{\beta}_n$ is chosen as a solution of the extremum problem

$$\hat{\beta}_n = \operatorname{argmin} \left[n^{-1} \sum_1^n |y_t - x_t' \beta| \right]. \quad (3)$$

This is the standard framework for deriving the consistency and asymptotic normality of the LAD estimator. The original work is due to Bassett and Koenker [3]. An extensive study with an analysis of autoregressions as well as the regression model (1) is provided by the monograph of Bloomfield and Steiger [6]. Some additional insights into LAD estimation and the history of its asymptotic theory are given in Bassett [4]. A recent and novel treatment of the subject that includes an historical overview of research and some additional references is Pollard [23]. Our condition (\mathcal{Q}_1) is stronger than Pollard's "error assumption" on u_t in that we require the density $f(\cdot)$ to be analytic rather than simply continuous at the origin. But there will be gains to making this stronger assumption. Not only does it help in developing generalized Taylor series but in so doing it facilitates the subsequent development of higher-order asymptotics.

In most cases of interest and certainly under the standard assumptions of the regression model given above, the consistency of $\hat{\beta}_n$ is easily established by conventional arguments that involve the limit of the objective function in (3). Amemiya [1, pp. 152–153] is a convenient source for the details of this approach. In what follows, we shall assume this argument has already been made and that $\hat{\beta}_n \xrightarrow{p} \beta^0$ as $n \rightarrow \infty$.

Our concern is with the asymptotic distribution of $\hat{\beta}_n$. Our approach is to proceed as if the problem were regular and the objective function were differentiable in β . Although the derivatives do not exist in the usual sense, they do have a meaning as generalized functions. Moreover, they are real valued and unique when they appear in an appropriate linear functional form. Since this is precisely how they do arise in the limit as $n \rightarrow \infty$, it turns out that we may proceed with the usual Taylor series expansion of the first-order conditions to extract the asymptotic theory.

We start with the first-order conditions for $\hat{\beta}_n$ from (3). These are

$$n^{-1} \sum_1^n \operatorname{sgn}(y_t - x_t' \hat{\beta}_n) x_t = 0, \quad (4)$$

where $\operatorname{sgn}(X) = -1$ for $X < 0$ and $= 1$ for $X \geq 0$. We now expand (4) in a Taylor series about its value at β^0 . Note that $d/dX(\operatorname{sgn}(X)) = 2\delta(X)$ where $\delta(X)$ is the delta (generalized) function (Gelfand and Shilov [9], hereafter simply GS, p. 4). We denote successive derivatives of the delta function by $\delta^{(k)}(X)$ for $k = 1, 2, \dots$. Proceeding in a purely formal way by treating $\operatorname{sgn}(\cdot)$ as analytic (it is already piecewise analytic) and its derivatives as ordinary functions, we would have the expansion

$$0 = n^{-1} \sum_1^n \operatorname{sgn}(u_t) x_t - 2n^{-1} \sum_1^n \delta(u_t) x_t x_t' (\hat{\beta}_n - \beta^0) + 2 \sum_{k=2}^{\infty} (-1)^k \left(\frac{1}{k!} \right) n^{-1} \sum_1^n \delta^{(k-1)}(u_t) x_t [x_t' (\hat{\beta}_n - \beta^0)]^k. \quad (5)$$

Let us now suppose that we can ignore all but the first two terms of (5) as $n \rightarrow \infty$. This is, of course, precisely what is done in the classical approach. Scaling (5) by $n^{1/2}$ and taking the error to be $o_p(1)$, we have

$$n^{-1/2} \sum_1^n \operatorname{sgn}(u_t) x_t = 2 \left[n^{-1} \sum_1^n \delta(u_t) x_t x_t' \right] n^{1/2} (\hat{\beta}_n - \beta^0) + o_p(1). \quad (6)$$

Since $(\operatorname{sgn}(u_t))$ is i.i.d.(0,1), the left side of (6) satisfies a multivariate extension of the Lindeberg-Lévy theorem, leading to

$$n^{-1/2} \sum_1^n \operatorname{sgn}(u_t) x_t \xrightarrow{d} N(0, Q). \quad (7)$$

On the right-hand side of (6) the matrix factor in square brackets satisfies a weak law of large numbers, viz.

$$n^{-1} \sum_1^n \delta(u_t) x_t x_t' \xrightarrow{p} \lim_{n \rightarrow \infty} n^{-1} \sum_1^n E(\delta(u_t)) x_t x_t' = f(0)Q. \quad (8)$$

Putting (6), (7), and (8) together we deduce directly the limit theory for the LAD estimator, that is,

$$n^{1/2} (\hat{\beta}_n - \beta^0) \xrightarrow{d} N(0, (1/2 f(0))^2 Q^{-1}). \quad (9)$$

3. AN ATTEMPT AT RIGOR

To attach some rigor to this heuristic skeleton, we need to justify three of the steps just taken, specifically, (i) the Taylor expansion (5); (ii) the $o_p(1)$ error in (6); and (iii) the weak law of large numbers (8).

Let us start with (i). Obviously, (5) has no meaning as an ordinary equation or as an ordinary Taylor series expansion. But it can be interpreted in terms of generalized functions and as a generalized Taylor series. Thus, suppose φ is a suitable test function for linear functionals of a generalized function g . Suppose, for instance, that φ belongs to the space S of entire functions which, together with their derivatives, approach zero more rapidly than any power of $1/|u|$ as $|u| \rightarrow \infty$ (e.g., e^{-u^2}) and that the linear functional

$$(g(u+h), \varphi) = \int_{\mathbb{R}} g(u+h) \varphi(u) du = \int_{\mathbb{R}} g(s) \varphi(s-h) ds \quad (10)$$

is an ordinary analytic function of h in some neighborhood of $h = 0$ for all φ . Then, $g_h(u) = g(u+h)$ is a generalized analytic function of h (GS,

pp. 149–150). Indeed, by expanding $\varphi(s - h)$ about its value at $h = 0$ in (10) and noting that $(g^{(j)}, \varphi) = (g, (-1)^j \varphi^{(j)})$, we obtain

$$(g(u + h), \varphi) = \sum_{j=0}^{\infty} \left(\frac{1}{j!}\right) (-h)^j (g, \varphi^{(j)}) = \sum_{j=0}^{\infty} \left(\frac{1}{j!}\right) h^j (g^{(j)}, \varphi)$$

which we can write in formal terms as

$$g_h = \sum_{j=0}^{\infty} \left(\frac{1}{j!}\right) g_h^{(j)} h^j.$$

Setting $g_h(u) = \text{sgn}(u + h)$, $h = x'_t(\hat{\beta}_n - \beta^0)$, and noting that $\int_{-\infty}^{\infty} \text{sgn}(u + h)\varphi(u) du$ is analytic in h for all $\varphi \in S$, we deduce the expansion given in (5) above.

Next consider (iii). The limit given on the right of (8) is well defined because the generalized function $\delta(\cdot)$ arises only through the linear functional $E(\delta(u)) = \int \delta(u)f(u) du = f(0)$. Note that this last expression remains true even though the density $f(\cdot)$ may exist only in a neighborhood of zero. This is because generalized functions like $\delta(\cdot)$ may be defined locally in terms of their operation on test functions with support in arbitrarily small given neighborhoods of every point (see GS, p. 140).

To be more complete in deducing (8) we may replace $\delta(\cdot)$ with its inverse Fourier transform representation, which we signify by “ $F^{-1}(\cdot)$,” that is,

$$\delta(u_t) = F^{-1}(1) = (2\pi)^{-1} \int_{-\infty}^{\infty} e^{-iu_t\sigma} d\sigma.$$

Again, the integral is formal and the correspondence produces the Fourier transform pair of generalized functions $(\delta(\cdot), 1)$ (GS, p. 168). In place of (8), we may now show that

$$n^{-1} \sum_1^n e^{-iu_t\sigma} x_t x'_t \xrightarrow{p} \lim_{n \rightarrow \infty} n^{-1} \sum_1^n E(e^{-iu_t\sigma}) x_t x'_t = cf_u(-\sigma)Q. \tag{11}$$

But this follows immediately because $(e^{-iu_t\sigma} - E(e^{-iu_t\sigma}))$ is i.i.d. with zero mean and finite variance. Upon inversion of (11) we get (8), since

$$f(0) = \left(\frac{1}{2\pi}\right) \int_{-\infty}^{\infty} cf_u(\sigma) d\sigma.$$

In fact, (11) may be regarded as the appropriate way to interpret (8) as a weak law for generalized random variables.

This leaves us with (ii). Working from (5) we have

$$n^{-1/2} \sum_1^n \text{sgn}(u_t)x_t = 2A_n n^{1/2}(\hat{\beta}_n - \beta^0), \tag{12}$$

where

$$A_n = n^{-1} \sum_1^n \delta(u_t) x_t x_t' + \sum_{j=1}^{\infty} (-1)^{j+1} ((j+1)!)^{-1} n^{-1} \sum_1^n \delta^{(j)}(u_t) x_t x_t' [x_t'(\hat{\beta}_n - \beta^0)]^j$$

and we have to show that the second term of A_n is $o_p(1)$ as $n \rightarrow \infty$.

As with (8) we have

$$n^{-1} \sum_1^n \delta^{(k)}(u_t) x_t x_t' \xrightarrow{p} \lim_{n \rightarrow \infty} \left[n^{-1} \sum_1^n E(\delta^{(k)}(u_t)) x_t x_t' \right] = (-1)^k f^{(k)}(0) Q,$$

where we use the fact that

$$(\delta^{(k)}, f) = \int_{-\infty}^{\infty} \delta^k(u) f(u) du = (\delta, (-1)^k f^{(k)}) = (-1)^k f^{(k)}(0),$$

(GS, p. 26). Moreover, since $f(\cdot)$ is analytic at zero, power series such as

$$\sum_{k=2}^{\infty} \left(\frac{1}{k!} \right) f^{(k)}(0) \epsilon^{k-1} = \frac{(f(\epsilon) - f(0) - f'(0)\epsilon)}{\epsilon}$$

are convergent and of order $O(\epsilon)$ for all ϵ in the vicinity of zero. However, $\hat{\beta}_n \xrightarrow{p} \beta^0$ and $x_t'(\hat{\beta}_n - \beta^0) \xrightarrow{p} 0$ uniformly in t . It follows that the second term of A_n converges in probability to zero as required.

4. AN ASYMPTOTIC EXPANSION

One advantage of the above approach is that it lends itself to the development of higher-order asymptotic expansions. To see how to proceed we set $q_n = n^{1/2}(\hat{\beta}_n - \beta^0)$, $\ell_1 = n^{-1/2} \sum_1^n \text{sgn}(u_t) x_t$, scale the expansion (5) by $n^{1/2}$, and write it in the form

$$0 = \ell_{1i} + \ell_{2ij} q_{nj} + n^{-1/2} \ell_{3ijk} q_{nj} q_{nk} + n^{-1} \ell_{4ijkl} q_{nj} q_{nk} q_{nm} + o_p(n^{-1}), \quad (13)$$

where we use the summation convention. Inverting (13), we have up to $O_p(n^{-1/2})$

$$q_{ni} = \ell_2^{ij} \ell_{1j} + n^{-1/2} \ell_2^{ij} \ell_{3jke} (\ell_2^{km} \ell_{1m}) (\ell_2^{ln} \ell_{1n}) + O_p(n^{-1}). \quad (14)$$

Next observe that the distribution of $\text{sgn}(u_t)$ is symmetric, that the distribution of ℓ_1 will admit a valid Edgeworth expansion, and that, because of the symmetry of the distribution of $\text{sgn}(u_t)$, there will be no skewness term in this expansion. Thus, we may write symbolically

$$\ell_1 \stackrel{d}{=} N(0, Q) + O_p(n^{-1}) = \ell_1 + O_p(n^{-1}), \text{ say.}$$

The second term of (14) is $O_p(n^{-1/2})$ and

$$\ell_{3jk\ell} = n^{-1} \sum_1^n \delta'(u_t) x_{ij} x_{tk} x_{t\ell} \xrightarrow{p} (-1) f'(0) r_{jk\ell} = \ell_{3jk\ell}, \text{ say,}$$

where

$$r_{jk\ell} = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_1^n x_{ij} x_{tk} x_{t\ell}.$$

Thus (14) would *appear* to yield a conventional Edgeworth expansion for the distribution of q_n . However, there is an additional complication that arises from the components ℓ_{2ij} and the elements ℓ_2^{ij} of its inverse. We have

$$\ell_{2ij} = 2n^{-1} \sum_1^n \delta(u_t) x_{ti} x_{tj} \xrightarrow{p} 2f(0) q_{ij} = \ell_{2ij}, \text{ say.}$$

Similarly, define $\ell_2^{ij} = (1/2f(0))q^{ij}$. Then (14) may be written as

$$q_{ni} = \ell_2^{ij} \ell_{1j} + (\ell_2^{ij} - \ell_2^{ij}) \ell_{1j} + n^{-1/2} \ell_2^{ij} \ell_{3jk\ell} (\ell_2^{km} \ell_{1m}) (\ell_2^{lm} \ell_{1m}) + o_p(n^{-1/2}). \quad (15)$$

The order of magnitude of the second term of (15) depends on that of

$$\ell_{2ij} - \ell_{2ij} = 2n^{-1} \sum_1^n (\delta(u_t) - f(0)) x_{ti} x_{tj} + o(1).$$

This term is much more difficult to analyze and is larger than $O_p(n^{-1/2})$.

To see what is involved, set $x_t = 1$ and consider

$$n^{-1} \sum_1^n (\delta(u_t) - f(0)) = \int_{\mathbb{R}} \delta(u) d(F_n(u) - F(u)), \quad (16)$$

where F_n is the empirical distribution function of u_t and F is the c.d.f. of u_t . In general, we have the weak convergence (Billingsley [5, p. 141])

$$\sqrt{n}(F_n(u) - F(u)) \xrightarrow{d} Y(u), \quad (17)$$

where $Y(\cdot)$ is a Gaussian process with covariance kernel $F(u)(1 - F(v))$, $u \leq v$. However, we cannot employ (17) in (16) because the implied limit variate, viz. $\int_{\mathbb{R}} \delta(u) dY(u)$, does not exist. For instance, if u_t were uniformly distributed then $Y(u)$ would be a Brownian bridge process and $\int_{\mathbb{R}} \delta(u) dY(u)$ would be its "derivative," which, like the derivative of Brownian motion, does not exist as an ordinary random variable.

There is another way to proceed. Note that (16) is the derivative at $s = 0$ of

$$\begin{aligned} n^{-1} \sum_1^n \left(\left(\frac{1}{2} \right) \text{sgn}(u_t + s) - F(s) \right) &- n^{-1} \sum_1^n \left(\left(\frac{1}{2} \right) \text{sgn}(u_t) - F(0) \right) \\ &= [F_n(0) - F_n(-s)] - \int_0^s dF. \end{aligned}$$

For $s = -tn^{-1/3}$ this expression is

$$\begin{aligned}
 & -\{F_n(tn^{-1/3}) - F_n(0) - f(0)tn^{-1/3} - (\frac{1}{2})f'(0)t^2n^{-2/3}\} + o(n^{-2/3}) \\
 & = -X_n(t) + o(n^{-2/3}), \text{ say.}
 \end{aligned}$$

Now Kim and Pollard [12, Theorem 4.7 and Example 6.5, p. 216] show that

$$n^{2/3}\{F_n(tn^{-1/3}) - F_n(0) - f(0)tn^{-1/3}\} \xrightarrow{d} (\frac{1}{2})t^2f'(0) + f(0)^{1/2}W(t) \quad (18)$$

with $W(t)$ a two-sided Brownian motion. Thus,

$$n^{2/3}X_n(t) \xrightarrow{d} f(0)^{1/2}W(t).$$

Define $\hat{f}_n(0)$ to be the left derivative of the concave majorant of F_n (i.e., the smallest concave function on $[0, \infty)$ that is everywhere greater than or equal to F_n). Kim and Pollard further show that $n^{1/3}(\hat{f}_n(0) - f(0))$ has a limit distribution given by that of the slope at the origin of the concave majorant of Brownian motion with quadratic drift, a result that is originally due to Prakasa Rao [24]. Treating $\hat{f}_n(0) = n^{-1} \sum_1^n \delta(u_i)$ as an estimate of $f(0)$, these results suggest that (16) and, hence, $\ell_{2ij} - \underline{\ell}_{2ij}$ are both $O_p(n^{-1/3})$. But the second term in the expansion (15) involves the product $(\ell_2^{ij} - \underline{\ell}_2^{ij})\ell_{1j}$, where $\ell_1 \stackrel{d}{=} N(0, Q)$. Because ℓ_1 has zero mean this term will contribute to the asymptotic expansion of the distribution of q_n only through the variance of ℓ_1 and, hence, will produce an adjustment of $O(n^{-2/3})$. It follows that only the first and third members on the right side of (15) contribute to the expansion up to $O(n^{-1/2})$. In particular, setting $\xi = \ell_1 \stackrel{d}{=} N(0, Q)$, we have

$$\begin{aligned}
 q_{ni} &= (\frac{1}{2}f(0))q^{ij}\xi_j - n^{-1/2}(\frac{1}{2}f(0))^3f'(0)q^{ij}r_{jk\ell}(q^{km}\xi_m)(q^{\ell m}\xi_m) \\
 &+ o_p(n^{-1/2}). \quad (19)
 \end{aligned}$$

Observe that when $f'(0) = 0$, which will be the case for symmetric error distributions, the term of $O_p(n^{-1/2})$ itself drops out, leaving only the first-order asymptotic term. In the general case, we can derive an asymptotic expansion of the density of some linear combination such as $r = c'q_n = c_i q_{ni}$ of the standardized error vector q_n . The formulae for the expansion of the density up to $O(n^{-1/2})$ may then be deduced from those in the literature (e.g., Sargan [25] and Phillips [17,18], where the last reference puts them in a form that is especially simple to interpret). With a little algebra, we obtain the following explicit expansion to $O(n^{-1/2})$:

$$\text{p.d.f.}(r) = (1/\omega)\varphi(r/\omega) [1 + n^{-1/2}\{a_1(r/\omega) + a_3(r/\omega)^3\}] + o(n^{-1/2}),$$

where $\varphi(x) = (2\pi)^{-1/2} \exp(-(\frac{1}{2})x^2)$ and the parameters are:

$$\omega^2 = (1/2f(0))^2 c_i q^{ij} c_j,$$

$$a_1 = (1/2\omega)q_{st}f_{st} - 3a_3,$$

$$a_3 = (1/2\omega^3)(1/2f(0))^2 f_{st} c_s c_t,$$

$$f_{st} = -2(1/2f(0))^3 f'(0) c_i q^{ij} r_{jkl} q^{ks} q^{lt}.$$

5. ESTIMATING THE COVARIANCE MATRIX

In conventional Taylor expansions of the first-order conditions, the Hessian is often used to produce an estimate of the asymptotic covariance matrix of the estimator. The same idea may be applied here. Taking the dominant member of the ‘‘Hessian,’’ $2A_n$, in the expansion (12) to be $2n^{-1} \sum_1^n \delta(u_t) x_t x_t'$ we simply replace the delta function in this expression by a *delta sequence* $\delta_m(\cdot)$ for which

$$\lim_{m \rightarrow \infty} \int_{-\infty}^{\infty} \delta_m(x) \varphi(x) dx = \varphi(0) \tag{20}$$

for all $\varphi \in S$. For example, the delta sequence $\delta_m(x) = (m/\pi)^{1/2} e^{-mx^2}$ satisfies (20) and corresponds to a density estimate based on a normal kernel. The covariance matrix estimate arising from this delta sequence takes the form

$$2n^{-1} \sum_1^n \delta_m(u_t) x_t x_t'.$$

The errors, u_t , in this expression can be replaced by residuals to produce a feasible covariance matrix estimate. The parameter m is like a bandwidth (for the normal kernel given above we would have $h = 1/m^{1/2}$) and must be chosen so that $m \rightarrow \infty$, $nm^{-1/2} \rightarrow \infty$ as $n \rightarrow \infty$ (i.e., $h \rightarrow 0$, $nh \rightarrow \infty$) for consistency (e.g., Silverman [26, p. 71]).

Since delta sequences encompass most density estimates (e.g., Walter and Blum [28]), this approach to the estimation of the covariance matrix is really quite general.

6. MODEL EXTENSIONS

It would appear that the approach suggested here remains valid for a large class of models with weakly dependent, rather than i.i.d., errors. All that is required is that $\ell_1 = n^{-1/2} \sum_1^n \text{sgn}(u_t) x_t$ satisfy some central limit theorem for weakly dependent errors. For example, suppose that u_t is strong mixing (e.g., Hall and Heyde [10, p. 132]) and satisfies the following.

(Q₂) The sequence (u_t) is strictly stationary and strong mixing with mixing coefficients β_k that satisfy $\sum_1^\infty \beta_k < \infty$, and u_t has zero median and probability density $f(\cdot)$ that is positive and analytic at zero.

Then $\text{sgn}(u_t)$ is also strong mixing with mixing coefficients that satisfy the same summability condition (e.g., White [27, Theorem 3.49, p. 47]) and we have the central limit theorem

$$n^{-1/2} \sum_1^n \text{sgn}(u_t)x_t \xrightarrow{d} N(0, V),$$

where

$$V = \sum_{k=-\infty}^{\infty} R_k E\{\text{sgn}(u_0)\text{sgn}(u_k)\}, \quad \text{with } R_k = \lim_{n \rightarrow \infty} n^{-1} \sum_1^n x_t x'_{t+k}.$$

When $x_t = 1$ we have

$$\begin{aligned} V &= E(\text{sgn}(u_0))^2 + 2 \sum_{k=1}^{\infty} E(\text{sgn}(u_0)\text{sgn}(u_k)) \\ &= 1 + 2 \sum_{k=1}^{\infty} [\{P(u_0 < 0, u_k < 0) + P(u_0 > 0, u_k > 0)\} \\ &\quad - \{P(u_0 > 0, u_k < 0) + P(u_0 < 0, u_k > 0)\}]. \end{aligned}$$

Using the same expansion (5) as before, we deduce the following limit theory for the LAD estimator

$$n^{1/2}(\hat{\beta}_n - \beta^0) \xrightarrow{d} N(0, (1/2 f(0))^2 Q^{-1} V Q^{-1}).$$

For inference V , as well as $f(0)$, now needs to be estimated. The situation is entirely analogous to the estimation of autocorrelation consistent covariance matrix estimation as it arises in conventional regression contexts (cf. White [27], Newey and West [15]).

Models in which the carrier variables x_t are random may also be accommodated. The details are close to those given in [23] so we shall only touch on them briefly here. Suppose the x_t are strictly stationary, ergodic, square integrable and \mathcal{F}_{t-1} measurable for some increasing sequence of σ -fields \mathcal{F}_t , suppose u_t is independent of \mathcal{F}_{t-1} and satisfies (\mathcal{G}_1) , and let $V_n = \sum_1^n E(z_t z'_t) = nE(x_t x'_t) = n\Omega$, where $z_t = \text{sgn}(u_t)x_t$. Then the following two conditions hold:

$$\begin{aligned} \max_{t \leq n} \|V_n^{-1/2} z_t\| &= \max_{t \leq n} \|\Omega^{-1/2}(n^{-1/2} z_t)\| \xrightarrow{p} 0, \\ V_n^{-1/2} \left(\sum_1^n z_t z'_t \right) V_n^{-1/2} &= \Omega^{-1/2} \left(n^{-1} \sum_1^n z_t z'_t \right) \Omega^{-1/2} \xrightarrow{p} I_k \end{aligned}$$

and we may use a martingale central limit theorem (e.g. [10, Theorem 3.2, p. 58]) to establish that

$$V_n^{-1/2} \sum_1^n \text{sgn}(u_t)x'_t \xrightarrow{d} N(0, I).$$

Moreover,

$$n^{-1} \sum_1^n e^{-iu_t \sigma} x_t x'_t \xrightarrow{p} E(e^{-iu_t \sigma} x_t x'_t) = cf_u(-\sigma)\Omega$$

and so

$$n^{-1} \sum_1^n \delta(u_t) x_t x_t' \xrightarrow{p} f(0)\Omega,$$

where $f(\cdot)$ is the probability density of u_t . These results enable us to deduce the limit law

$$n^{1/2}(\hat{\beta}_n - \beta^0) \xrightarrow{d} N(0, (1/2f(0))^2 \Omega^{-1})$$

in the same way as before. This covers the case of strictly exogenous stationary regressors x_t with $\Omega = E(x_t x_t')$. It also covers the case of stable autoregressions with $x_t' = (y_{t-1}, \dots, y_{t-k})$ with $(u_t) = \text{i.i.d.}(0, \sigma^2)$ (the finite variance σ^2 ensures that the elements of x_t are square integrable) and $\Omega = (\omega_{ij})$ with $\omega_{ij} = E(y_{t-i} y_{t-j})$.

7. AUTOREGRESSIONS WITH INFINITE VARIANCE ERRORS AND A UNIT ROOT

We shall consider the AR(1) model with a unit root, viz.

$$y_t = \beta^0 y_{t-1} + u_t, \quad \text{with } \beta^0 = 1 \quad (21)$$

and where the errors are i.i.d. and have infinite variance. This model has recently been studied by Knight [13].

To develop a limit theory we shall assume that u_t is in the domain of attraction of a stable law with characteristic exponent α . Specifically we say that $u \in \mathfrak{D}(\alpha)$ if

$$P(u > x) = c_1 x^{-\alpha} L(x) (1 + \alpha_1(x)), \quad x > 0, \quad c_1 \geq 0 \quad (22)$$

and

$$P(u < -x) = c_2 x^{-\alpha} L(x) (1 + \alpha_2(x)), \quad x > 0, \quad c_2 \geq 0 \quad (23)$$

with $0 < \alpha < 2$, $L(x)$ a slowly varying function at ∞ , and $\alpha_i(x) \rightarrow 0$ as $|x| \rightarrow \infty$. If $L(x) = 1$ in (22) and (23), then u is in the normal domain of attraction of a stable law with parameter α and we write $u \in \mathfrak{ND}(\alpha)$. With this terminology in hand, we assume the following

(\mathfrak{R}_3) *The sequence (u_t) is i.i.d. with $u_t \in \mathfrak{D}(\alpha)$, where $0 < \alpha < 2$, with zero median and with probability density $f(\cdot)$ that is positive and analytic at zero. If $\alpha > 1$, $E(u_t) = 0$ and if $\alpha = 1$ then $u_t \stackrel{d}{=} -u_t$ (i.e., u_t is symmetrically distributed).*

Define the normalizing sequence

$$a_n = \inf\{x: P(|u| > x) \leq n^{-1}\}.$$

For $u \in \mathcal{D}(\alpha)$, we have $a_n = n^{1/\alpha}L(n)$ for some slowly varying function $L(n)$. For $u \in \mathcal{ND}(\alpha)$, we have $a_n = cn^{1/\alpha}$ for some constant $c > 0$.

Since $\beta^0 = 1$, the output y_t of (21) is an integrated process. In consequence, we have the weak convergence (e.g. [13], [19])

$$a_n^{-1}y_{[n \cdot]} = a_n^{-1} \sum_n^{[n \cdot]} u_t \xrightarrow{d} S_\alpha(\cdot),$$

where $S_\alpha(\cdot)$ is an α -stable process. Further, we have

$$n^{-1/2} \sum_1^{[n \cdot]} \text{sgn}(u_t) \xrightarrow{d} W(\cdot)$$

and

$$n^{-1/2}a_n^{-1} \sum_1^n \text{sgn}(u_t)y_{t-1} = n^{-1/2} \sum_1^n (a_n^{-1}y_{t-1})\text{sgn}(u_t) \xrightarrow{d} \int_0^1 S_\alpha^- dW, \quad (24)$$

where $W(\cdot)$ is standard Brownian motion and $S_\alpha^-(r)$ signifies the left limit of the process $S_\alpha(\cdot)$ at r . Next observe that

$$n^{-1}a_n^{-2} \sum_1^n y_{t-1}^2 = n^{-1} \sum_1^n (a_n^{-1}y_{t-1})^2 \xrightarrow{d} \int_0^1 S_\alpha^2$$

and

$$n^{-1} \sum_1^n (\delta(u_t) - f(0))(a_n^{-1}y_{t-1})^2 \xrightarrow{p} 0$$

so that

$$n^{-1} \sum_1^n \delta(u_t)(a_n^{-1}y_{t-1})^2 \xrightarrow{d} f(0) \int_0^1 S_\alpha^2. \quad (25)$$

Combining (24) and (25) we have

$$n^{1/2}a_n(\hat{\beta}_n - 1) \sim \frac{n^{-1/2}a_n^{-1} \sum_1^n \text{sgn}(u_t)y_{t-1}}{2n^{-1}a_n^{-2} \sum_1^n \delta(u_t)y_{t-1}^2} \xrightarrow{d} \frac{\int_0^1 S_\alpha^- dW}{2f(0) \int_0^1 S_\alpha^2},$$

as given in Theorem 3 of Knight [13]. Since $a_n = n^{1/\alpha}L(n)$ we have the remarkable result, due to Knight, that the LAD estimator converges at a faster rate in the unit root model for $0 < \alpha < 2$ than the OLS estimator. As remarked by Knight, robust estimators such as LAD retain the advantages of the strong signal from y_{t-1} (due to integration and thick tailed errors) but attenuate the effects of outliers in the error u_t to the extent that they occur in the sample covariance between $\text{sgn}(u_t)$ and y_{t-1} (in the LAD case).

8. CONCLUSION

The methods outlined here seem to offer some promise as tools for the analysis of regression asymptotics in nonregular cases like that of the LAD estimator. Our approach has not always been rigorous. Nonetheless, it is hoped that the results obtained point to the usefulness of the approach. One would like to hope that the approach can be made entirely rigorous by providing a tight probabilistic framework for the use of the generalized random variables that appear in our derivations. In the meantime it seems reasonable to conclude that the classical approach warrants more attention than it has yet received. It may indeed offer some advantages that empirical process methods do not seem to presently enjoy, viz. the capacity to develop higher-order asymptotics.

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