

SPHERICAL MATRIX DISTRIBUTIONS AND CAUCHY QUOTIENTS

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Abstract: It is shown that matrix quotients of submatrices of a spherical matrix are distributed as matrix Cauchy. This generalizes known results for scalar ratios of independent normal variates. The derivations are simple and make use of the theory of invariant measures on manifolds.

Keywords: Cauchy quotients, invariant measures, matrix variates, spherical distributions

1. Spherical matrix distributions

Let X be a random real matrix of dimension $n \times q$ ($n \geq q$). If the distribution of X is invariant under left and right orthogonal transformations we say that X is spherical (or spherically symmetric). Such distributions have been studied extensively in the recent literature. Dawid (1977, 1981), in particular, provides many useful properties and gives several applications to problems of inference in the multivariate linear model.

The most prominent member of the spherical family is the matrix normal $N_{n,q}(0, I_{nq})$ with density

$$\text{pdf}(X) = (2\pi)^{-nq/2} e^{-\frac{1}{2}XX'}. \tag{1}$$

Clearly, (1) is invariant under the transformations

$$X \rightarrow H_1 X H_2, \quad H_1 \in O(n), \quad H_2 \in O(q), \tag{2}$$

where $O(m) = \{H(m \times m): H'H = I_m\}$ is the orthogonal group of order m . Moreover, invariance under (2) together with the requirement that the components of X be independent actually implies (1) (see Bartlett (1934), Kac (1939), Hartman and Wintner (1940) and, for a recent proof, Muirhead (1982)).

Other important members of the spherical family are: (i) the matrix t with density

$$\text{pdf}(X) = [\pi^{nq/2} \Gamma_n((n+k-1)/2)]^{-1} \Gamma_n((n+k+q-1)/2) [\det(I+XX')]^{-(n+k+q-1)/2} \tag{3}$$

where $\Gamma_n(\cdot)$ denotes the multivariate gamma function and $k \geq 1$ is the degrees of freedom parameter; and (ii) the uniform distribution on the Stiefel manifold $V_{q,n} = \{H(n \times q): H'H = I_q\}$ with density (with respect to Haar measure):

$$\text{pdf}(X) = [\text{Vol}(V_{q,n})]^{-1} = \Gamma_q(n/2) / 2^q \pi^{qn/2}. \tag{4}$$

The uniform distribution on $V_{q,n}$ is, in fact, uniquely determined by its invariance under the orthogonal transformations (2) (see James (1954)).

Some specializations of (1), (3) and (4) are worth mentioning. When $q = 1$, (1) is the multivariate $N(0, I_n)$, (3) is (proportional to) the multivariate t with degrees of freedom k and (4) is the uniform distribution on the unit sphere $S_n = \{h(n \times 1) \mid h'h = 1\}$. When $k = 1$, (3) is the matrix Cauchy distribution. Moreover, as shown by Dawid (1981), all of these distributions are maintained under marginalization. In particular, all submatrices of a matrix Cauchy variate are themselves matrix Cauchy.

2. Quotients of spherical distributions

Let X be spherical and partition X as follows:

$$X = \begin{bmatrix} X_1 \\ \vdots \\ X_2 \end{bmatrix} \begin{matrix} q \\ p \\ q \end{matrix}$$

Define the matrix quotient $R = X_1 X_2^{-1}$ and let

$$K = X(X'X)^{-1/2} = \begin{bmatrix} X_1(X'X)^{-1/2} \\ X_2(X'X)^{-1/2} \end{bmatrix} = \begin{bmatrix} K_1 \\ K_2 \end{bmatrix}$$

Since X is spherical we have

$$X \equiv H_1 X H_2, \quad H_1 \in O(n), \quad H_2 \in O(q), \tag{5}$$

where the symbol “ \equiv ” signifies equality in distribution. We find that

$$\begin{aligned} H_1 K H_2 &= H_1 X H_2 H_2' (X' X)^{-1/2} H_2 \\ &= H_1 X H_2 (H_2' X' H_1' H_1 X H_2)^{-1/2} \\ &\equiv X (X' X)^{-1/2} = K \end{aligned}$$

in view of (5), so that K is also spherical. But, since $K \in V_{q,n}$ and there is a unique invariant measure (given by the uniform distribution) on the manifold $V_{q,n}$, we deduce that K has the uniform distribution on $V_{q,n}$. This is true for any choice of the original spherical variate X .

We may write the matrix quotient R in terms of K as

$$R = K_1 K_2^{-1}.$$

But the distribution of K (and, hence, R) is invariant to the choice of original spherical distribution for X . We may therefore choose X to be matrix normal as in (1). Since the elements of X are independent we deduce quite simply that

$$R = K_1 K_2^{-1} \equiv [N_{p,q}(0, I)] [N_{q,q}(0, I_{q^2})]^{-1} \equiv \text{matrix Cauchy.} \tag{6}$$

The final “ \equiv ” here generalizes to matrix quotients the well known result for scalar random variables that a ratio of independent standard normals has a Cauchy distribution.

To prove (6) we proceed as follows. Define $S = X_2 X_2'$, $H = X_2' (X_2 X_2')^{-1/2}$ and $R = X_1 X_2^{-1}$. We transform

$$X \rightarrow (R, S, H)$$

and note that the measure transforms according to

$$dX = 2^{-q}(\det S)^{(p-1)/2} dR(dS)(dH) \quad (7)$$

where (dH) denotes the unnormalized invariant measure on $O(q)$ and (dS) denotes Lebesgue measure on the space of positive definite matrices. From (1) and (7) we obtain by a simple application of the multivariate gamma integral

$$\begin{aligned} \text{pdf}(R) &= (2\pi)^{-nq/2} 2^{-q} \int_{s>0} e^{\text{tr}\{-(1/2)(I+R'R)S\}} (\det S)^{(p-1)/2} (dS) \int_{O(q)} (dH) \\ &= \left\{ (2\pi)^{-nq/2} 2^{-q} \right\} \left\{ \Gamma_q((p+q)/2) \det(I+R'R)^{-(p+q)/2} 2^{(p+q)q/2} \right\} \\ &\quad \times \left\{ 2^q \pi^{q^2/2} / \Gamma_q(q/2) \right\} \\ &= \left[\pi^{pq/2} \Gamma_q(q/2) \right]^{-1} \Gamma_q((p+q)/2) [\det(I+R'R)]^{-(p+q)/2}. \end{aligned}$$

This is seen to be matrix $(p \times q)$ Cauchy (as in (3) with $k = 1$ and $n = p$) by noting that

$$\frac{\Gamma_q(q/2)}{\Gamma_q((p+q)/2)} = \frac{\Gamma_p(p/2)}{\Gamma_p((p+q)/2)}.$$

As observed above, R is matrix Cauchy for any choice of underlying spherical matrix X . In particular, X itself may be $(n \times q)$ matrix Cauchy. In this case, the submatrices X_1 and X_2 of X are themselves matrix Cauchy and they are statistically dependent. The quotient $R = X_1 X_2^{-1}$ then has the same matrix Cauchy distribution as X_1 itself.

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