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WEAK CONVERGENCE OF SAMPLE COVARIANCE MATRICES TO STOCHASTIC INTEGRALS VIA MARTINGALE APPROXIMATIONS

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Under general conditions the sample covariance matrix of a vector martingale and its differences converges weakly to the matrix stochastic integral $\int_0^1 BdB'$, where B is vector Brownian motion. For strictly stationary and ergodic sequences, rather than martingale differences, a similar result obtains. In this case, the limit is $\int_0^1 BdB' + \Lambda$ and involves a constant matrix Λ of bias terms whose magnitude depends on the serial correlation properties of the sequence. This note gives a simple proof of the result using martingale approximations.

1. INTRODUCTION

There has recently been a good deal of interest in time-series regressions that involve integrated processes. The theory makes extensive use of weak convergence methods in general, and multivariate invariance principles in particular. Some recent papers dealing with this topic are [2,4–18]. Much of the theory involves weak convergence of sample covariance matrices to matrix stochastic integrals of the form $\int_0^1 BdB' + \Lambda$, where B is vector Brownian motion and Λ is a constant matrix of bias terms. The result is of theoretical interest because it cannot be obtained from a routine application of the continuous mapping theorem and an invariance principle except in the scalar case [2,7,9]. It also has many useful applications in the theory of regression with integrated time series. The reader is referred to [13] for a recent review of the field.

To fix ideas, let $\{x_t\}_0^\infty$ be an n-vector time series generated by

$$x_t = x_{t-1} + u_t, \qquad t = 1,2,\ldots, \tag{1}$$

where x_0 is any random vector (including a constant) and $\{u_t\}_{-\infty}^{\infty}$ is a zero mean, strictly stationary, and ergodic sequence with continuous spectral den-

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sity $f_{uu}(\lambda)$. Define $X_T(r) = T^{-1/2}\Sigma_1^{[Tr]}u_j$. Then, as shown in [8,14], under quite general conditions as $T \to \infty$ we have

$$X_T(r) \Rightarrow B(r) \equiv BM(\Omega) \tag{2}$$

with

$$\Omega = 2\pi f_{uu}(0) = \Sigma + \Lambda + \Lambda',$$

$$\Sigma = E(u_0 u_0'), \qquad \Lambda = \sum_{k=1}^{\infty} E(u_0 u_k').$$

We also have

$$T^{-1}\Sigma_1^T x_{t-1} u_t' \Rightarrow \int_0^1 BdB' + \Lambda. \tag{3}$$

In (2) and (3), we use the symbol "$\Rightarrow$" to signify weak convergence as $T \to \infty$, "$\equiv$" to signify equality in distribution, and "$BM(\Omega)$" to denote Brownian motion with covariance matrix Ω.

Proofs of (3) that are presently available [9,17] are lengthy and difficult to follow. A more direct proof of the result seems desirable. When $\{u_t\}$ forms a square integrable martingale difference sequence with respect to the natural filtration of σ-fields $F_t = \sigma(u_t, u_{t-1}, \dots)$, then $\Lambda = 0$ and (3) has been proved recently by direct methods in [2]. In particular, we have the following lemma.

LEMMA (Chan and Wei). *If $\{u_t, F_t\}$ is a martingale difference sequence, if*

$$E(u_t' u_t | F_{t-1}) \le c \qquad \text{a.s.} \tag{4}$$

for some constant $c > 0$, and if (2) holds, then

$$T^{-1}\Sigma_1^T x_{t-1} u_t' \Rightarrow \int_0^1 BdB'.$$

The purpose of the present note is to show how (3) may be obtained quite simply when $\Lambda \neq 0$ by using this lemma and a martingale approximation to the process u_t. The approach we follow is inspired by the use of martingale approximations in central-limit theory for stationary processes. The reader is referred to [3, Chapter 5] for an excellent exposition of the approach.

2. MAIN RESULT AND PROOF

It will be convenient to let u_t in (1) be generated by the linear process

$$u_t = \sum_{j=-\infty}^{\infty} B_j e_{t-j}, \qquad \sum_{j=-\infty}^{\infty} \|B_j\| < \infty, \tag{5}$$

where the sequence of random vectors $\{e_t\}_{-\infty}^{\infty}$ is i.i.d.$(0,\Delta)$ with $\Delta > 0$ and where $\|B_j\| = \max_k\{\Sigma_l |b_{jkl}|\}$ with $B_j = (b_{jkl})$. This includes all stationary and invertible ARMA processes, for instance, and is therefore of wide applicability. The process u_t defined by (5) is strictly stationary and ergodic and has continuous spectral density given by

$$f_{uu}(\lambda) = (1/2\pi)(\Sigma_j B_j e^{ij\lambda})\Delta(\Sigma_j B_j e^{ij\lambda})^*.$$

For our theory, we need to strengthen the absolute summability requirement on $\{B_j\}$ in (5). We will use the following condition (based on (5.37) of Hall and Heyde [3])

$$\sum_{k=1}^{\infty}\left[\left\|\sum_{j=k}^{\infty} B_j\right\| + \left\|\sum_{j=k}^{\infty} B_{-j}\right\|\right] < \infty. \tag{6}$$

This condition holds for all sequences $\{B_j\}$ that are 1-summable in the sense of Brillinger [1, equation (2.7.14)]; see also Stock [17, fn. 7]. It is again satisfied by all stationary and invertible ARMA models.

Our main result is as follows:

THEOREM. *If $\{x_t\}$ is generated by (1) and $\{u_t\}$ satisfies (5) and (6), then (3) holds.*

Proof: Under the stated conditions, we note first that the multivariate invariance principle (2) applies. When $n = 1$, this follows directly from Theorem 5.5 of Hall and Heyde [3, p. 141 and p. 146]. For $n > 1$, the result may again be deduced from this theorem by applying the argument of Theorem 2.1 of [8].

The remainder of the theorem is based on a martingale approximation of u_t. The construction is achieved in Theorems 5.4 and 5.5 of Hall and Heyde [3]. We let $M_k = \sigma\{e_j, j \le k\}$ and define

$$Y_0 = \sum_{l=-\infty}^{\infty} [E(u_l|M_0) - E(u_l|M_{-1})] = \left(\sum_{l=-\infty}^{\infty} B_l\right)e_0,$$

$$Z_0 = \sum_{k=0}^{\infty} E(u_k|M_{-1}) - \sum_{k=-\infty}^{-1} \{u_k - E(u_k|M_{-1})\}.$$

Setting $Y_k = U^k Y_0$ and $Z_k = U^k Z_0$, where U is the temporal displacement operator, we observe that $\{Y_k, M_k\}$ is a martingale difference sequence whose differences Y_k are strictly stationary, ergodic, and square integrable with covariance matrix $\Omega = (\Sigma_j B_j)\Delta(\Sigma_j B_j') = 2\pi f_{uu}(0)$. The process $\{Z_k\}$ is also strictly stationary, ergodic, and square integrable. With this construction, we have $u_0 = Y_0 + Z_0 - Z_1$ and thus

$$u_t = Y_t + Z_t - Z_{t+1}.$$

Note that $x_t = \Sigma_1^t u_j + x_0$, and writing $P_k = \Sigma_1^k Y_j$, we obtain

$$\begin{aligned} T^{-1}\Sigma_1^T x_{k-1} u_k' &= T^{-1}\Sigma_1^T (P_{k-1} + Z_1 - Z_k + x_0)(Y_k + Z_k - Z_{k+1})', \\ &= T^{-1}\Sigma_1^T P_{k-1} Y_k' + T^{-1}\Sigma_1^T (Z_1 - Z_k) Y_k' \\ &\quad + T^{-1}\Sigma_1^T P_{k-1}(Z_k - Z_{k+1})' \\ &\quad + T^{-1}\Sigma_1^T (Z_1 - Z_k)(Z_k - Z_{k+1})' + o_p(1). \end{aligned} \tag{7}$$

Now by ergodicity, we have

$$T^{-1}\Sigma_1^T Z_1 Y_k' \to 0 \qquad \text{a.s.}$$

$$T^{-1}\Sigma_1^T Z_k Y_k' \to E(Z_0 Y_0') \qquad \text{a.s.}$$

$$T^{-1}\Sigma_1^T (Z_1 - Z_k)(Z_k - Z_{k+1})' \to -E(Z_0(Z_0 - Z_1)') \qquad \text{a.s.}$$

and

$$\begin{aligned} T^{-1}\Sigma_1^T P_{k-1}(Z_k - Z_{k+1})' &= T^{-1}\Sigma_1^T P_{k-1} Z_k' \\ &\quad - [T^{-1}\Sigma_1^T P_k Z_{k+1}' - T^{-1}\Sigma_1^T Y_k Z_{k+1}'], \\ &= T^{-1} P_0 Z_1 - T^{-1} P_T Z_{T+1} \\ &\quad + T^{-1}\Sigma_1^T Y_k Z_{k+1}' \to E(Y_0 Z_1') \qquad \text{a.s.} \end{aligned}$$

Moreover, by the lemma

$$T^{-1}\Sigma_1^T P_{k-1} Y_k' \Rightarrow \int_0^1 B dB',$$

where $B(r) \equiv BM(\Omega)$. We deduce that

$$T^{-1}\Sigma_1^T x_{t-1} u_t' \Rightarrow \int_0^1 B dB' + K,$$

where

$$\begin{aligned} K &= E(Y_0 Z_1') - E(Z_0(Z_0 - Z_1)') - E(Z_0 Y_0'), \\ &= E(Y_0 Z_1') - E(Z_0 u_0'). \end{aligned}$$

Now

$$Z_1 = \sum_{k=0}^{\infty} E(u_{k+1} | M_0) - \sum_{k=-\infty}^{-1} (u_{k+1} - E(u_{k+1} | M_0))$$

and

$$\begin{aligned} E(Y_0 Z_1') &= -\sum_{k=-\infty}^{-1} E(Y_0 u_{k+1}') + \sum_{k=-\infty}^{\infty} E\{Y_0 E(u_{k+1} | M_0)\}, \\ &= \sum_{k=0}^{\infty} E(Y_0 u_{k+1}'), \end{aligned}$$

since Y_0 is M_0-measurable. Next

$$E(Z_0 u_0') = -\sum_{k=-\infty}^{-1} E(u_k u_0') + \sum_{k=-\infty}^{\infty} E\{E(u_k|M_{-1})u_0'\},$$

$$= -\sum_{k=1}^{\infty} E(u_0 u_k') + \sum_{k=-\infty}^{\infty} E\{E(u_k|M_{-1})u_0'\}.$$

Hence,

$$K = \Lambda + \sum_{k=0}^{\infty} E(Y_0 u_{k+1}') - \sum_{k=-\infty}^{\infty} E\{E(u_k|M_{-1})u_0'\},$$

$$= \Lambda + \sum_{j=1}^{\infty} E(Y_{-j} u_0') - \sum_{k=-\infty}^{\infty} E\{E(u_k|M_{-1})u_0'\}. \tag{8}$$

Now

$$Y_{-j} = \sum_{l=-\infty}^{\infty} \{E(u_l|M_{-j}) - E(u_l|M_{-j-1})\}$$

and

$$\sum_{j=1}^{\infty} E(Y_{-j} u_0') = \sum_{l=-\infty}^{\infty} \left\{ \sum_{j=1}^{\infty} [E\{E(u_l|M_{-j})u_0'\} - E\{E(u_l|M_{-j-1})u_0'\}]\right\},$$

$$= \sum_{l=-\infty}^{\infty} E\{E(u_l|M_{-1})u_0'\}, \tag{9}$$

since $E(u_l|M_{-\infty}) = 0$ a.s. We deduce from (8) and (9) that $K = \Lambda$ and (3) follows immediately.

3. SOME REMARKS ON APPLICATIONS

Limit theorems involving stochastic integrals such as (3) seem to be of widespread importance. They have many applications in econometrics and arise frequently in time-series regressions with integrated processes and autoregressions with unit roots. Many examples are provided in the papers [3–6,12–15]. In addition, as indicated in other recent work [11], it seems likely that a general asymptotic theory for optimization estimators can be developed that uses limit theorems such as (3). With some extensions, this theory can accommodate limits to stochastic integrals that are taken with respect to more general continuous parameter martingales. Some of the interesting possibilities for such extensions are explored in Section 4 of [11].

REFERENCES

1. Brillinger, D.R. *Time series: data analysis and theory*. San Francisco, California: Holden-Day, 1981.

2. Chan. N.H. & C.Z. Wei. Limiting distributions of least-squares estimates of unstable autoregressive processes. *Annals of Statistics* 16 (1988): 367-401.
3. Hall, P. & C.C. Heyde *Martingale limit theory and its application*. New York: Academic Press, 1980.
4. Park, J.Y. & P.C.B. Phillips. Statistical inference in regressions with integrated processes: Part 1, *Econometric Theory* 4 (1988): 468–497.
5. Park, J.Y. & P.C.B. Phillips. Statistical inference in regressions with integrated processes: Part 2. *Econometric Theory* 5 (1989): forthcoming.
6. Phillips, P.C.B. Understanding spurious regressions in econometrics. *Journal of Econometrics* 33 (1986): 311-340.
7. Phillips, P.C.B. Time-series regression with a unit root. *Econometrica* 55 (1987): 277-301.
8. Phillips, P.C.B. Asymptotic expansions in nonstationary vector autoregressions. *Econometric Theory* 3 (1987): 45-68.
9. Phillips, P.C.B. Weak convergence to the matrix stochastic integral $\int_0^1 BdB'$. *Journal of Multivariate Analysis* 24 (1988): 252–264.
10. Phillips, P.C.B. Towards a unified asymptotic theory of autoregression. *Biometrika* 74 (1987): 535–547
11. Phillips, P.C.B. Regression theory for near integrated time series. *Econometrica* (1988): forthcoming.
12. Phillips, P.C.B. Partially identified econometric models. Cowles Foundation Discussion Paper No. 845, Yale University, July 1987.
13. Phillips, P.C.B. Multiple regression with integrated time series. *AMS/IMS/SIAM Conference on Stochastic Processes* (1988): forthcoming.
14. Phillips, P.C.B. & S.N. Durlauf. Multiple time-series regression with integrated processes. *Review of Economic Studies* 53 (1986): 473–496.
15. Sims, C.A., J.H. Stock & M.W. Watson. Inference in linear time-series models with some unit roots. Mimeographed, Minnesota University, 1986.
16. Solo, V. The order of differencing in ARMA models. *Journal of the American Statistical Association* 79 (1984): 916–921.
17. Stock, J.H. Asymptotic properties of least-squares estimators of cointegrating vectors. *Econometrica* 55 (1987): 1035–1057.
18. Stock, J.H. & M.W. Watson. Testing for common trends. Hoover Institution Working Paper No. E-87-2, 1987.