

# Best Median-Unbiased Estimation in Linear Regression With Bounded Asymmetric Loss Functions

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This article considers optimal median-unbiased estimation in a linear regression model with the distribution of the errors lying in a subclass of the elliptically symmetric distributions. The generalized least squares (GLS) estimator is shown to be best for any monotone loss function, that is, any loss function that is nondecreasing as the magnitude of underestimation or overestimation increases. This includes bounded asymmetric loss functions. For the same loss functions, a restricted GLS estimator is shown to be best when the estimand is known to lie in an interval. For the case of normal errors, a best median-unbiased estimator of the error variance  $\sigma^2$  is given for restricted and unrestricted parameter spaces. This estimator differs from the sample variance  $s^2$ . In comparison with best mean-unbiased estimators of regression and variance parameters, the best median-unbiased estimators considered here take advantage of restrictions on the parameter space and are optimal with respect to a much wider class of loss functions—in particular, both bounded and unbounded loss functions.

The choice of median-unbiasedness, as opposed to mean-unbiasedness, is not crucial when deriving an optimality result for the estimation of regression parameters when the model has elliptically symmetric errors, provided the parameter space is unrestricted or is restricted only by linear constraints. The reason is that many estimators considered in the literature have symmetric distributions about the estimand in this context and hence are both median- and mean-unbiased if their expectations exist. (Proper Bayes and shrinkage estimators are the two main classes of estimators that do not have symmetric distributions and are neither mean- nor median-unbiased.)

On the other hand, if the parameter space of the regression parameters is restricted by nonlinear constraints on the parameters, then the mean-unbiasedness condition becomes much more restrictive than median-unbiasedness. This occurs because estimators that take advantage of the restrictions on the parameters generally are mean-biased. Median-unbiased estimators, however, can be adjusted to take account of restrictions without losing their property of median-unbiasedness. Thus our use of the condition of median-unbiasedness, rather than mean-unbiasedness, is of little consequence when the parameter space is unrestricted and is a distinct advantage when the parameter space is restricted by nonlinear constraints on the parameters.

The class of error distributions that we consider consists of distributions that are consistent with elliptical symmetry for any sample size. Such distributions are rotated variance mixtures of multivariate normal distributions (and hence include multivariate normal distributions). Examples are given of cases in which such distributions may arise.

The contents of this article are organized as follows. Section 1 briefly reviews recent results by Kariya (1985) and Hwang (1985) that are related to the results given here. Section 2 shows that the GLS estimator is the best median-unbiased estimator of the regression parameters for quite general loss functions, when the parameter space is unrestricted. Of note is the fact that this result holds without moment restrictions. Thus the errors may have a multivariate Cauchy distribution. Section 3 shows that a restricted GLS estimator is best median-unbiased for a linear combination of the regression parameters, when that linear combination is restricted to lie in an interval. Certain other linear combinations of the

parameter vector may be subject to arbitrary additional restrictions. Section 4 presents best median-unbiased estimators of the error variance  $\sigma^2$ , as well as monotone functions of  $\sigma^2$ , when the errors are normally distributed. If  $\sigma^2$  is constrained to lie in a finite interval, the best estimator is a censored version of its unconstrained counterpart. When  $\sigma^2$  is constrained only to be positive, the best median-unbiased estimator is always larger than the best mean-unbiased estimator  $s^2$  and is approximately equal to  $s^2$  calculated with its degrees of freedom reduced by 66. The Appendix gives proofs of the results. These make use of results due to Lehmann (1959) and Pfanzagl (1979).

**KEY WORDS** Generalized least squares; Elliptically symmetric distribution; Restricted parameter space; Minimum risk; Variance estimation.

## 1. KARIYA'S AND HWANG'S OPTIMALITY RESULTS FOR GLS

The Gauss–Markov theorem states that for the linear regression model,

$$y = X\beta_0 + u_N, \quad E(u_N) = 0, \quad \text{and } \text{cov}(u_N) = \sigma^2\Sigma, \tag{1}$$

the generalized least squares (GLS) estimator,

$$\hat{\beta} = (X'\Sigma^{-1}X)^{-1}X'\Sigma^{-1}y, \tag{2}$$

is the best linear mean-unbiased estimator in the sense that  $c'\hat{\beta}$  minimizes the mean squared error for estimation of  $c'\beta_0$  for all fixed  $K$ -vectors  $c$ , provided that  $\Sigma$  is known. Here  $X$  is an  $N \times K$  fixed matrix of rank  $K$ ,  $\Sigma$  is a positive definite  $N \times N$  matrix, and  $\beta_0 \in R^K$ .

Two extensions of this result have appeared recently in the literature; see Kariya (1985) and Hwang (1985, cor. 3.2). In this section, we briefly review these extensions.

A common criticism of the Gauss–Markov theorem is that it only considers linear estimators. This has little or no justification. In contrast, Kariya's (1985) recent version of the Gauss–Markov theorem allows for nonlinear estimators. The class of estimators he considered is

$$e_1 = \{ \hat{\beta} \mid \hat{\beta}(y) = C(e)y, C(e) \text{ is a } K \times N \text{ matrix-valued measurable function of } e \text{ such that } C(e)X = I_K \text{ for all } e \text{ and } E\|\hat{\beta}\|^2 \text{ exists} \},$$

where  $e$  is the  $N$ -vector of ordinary least squares (OLS) residuals, that is,  $e = y - Xb$ , where  $b = (X'X)^{-1}X'y$  is the OLS estimator. For a more restricted class of error distributions than that considered in the Gauss–Markov

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framework, this class includes the class of linear mean-unbiased estimators. It contains both nonlinear and biased estimators. A typical example of an estimator in  $\mathcal{C}_1$  is the nonlinear feasible GLS estimator defined by  $C(e) = (X'\hat{\Sigma}^{-1}X)^{-1}X'\hat{\Sigma}^{-1}e$ , where the estimated covariance matrix  $\hat{\Sigma}$  depends only on the OLS residuals  $e$ .

Kariya's optimality result for the class  $\mathcal{C}_1$  is possible, because he considers a smaller class of error distributions than is considered in the Gauss–Markov theorem. Let  $\mathcal{F}_N^2$  be the class of distributions satisfying (1) such that when  $u_N$  is transformed into  $\tilde{u}_N = \Sigma^{-1/2}u_N$ , the distribution of  $\tilde{u}_N$  is orthogonally invariant; that is,  $\mathcal{L}(\Gamma\tilde{u}_N) = \mathcal{L}(\tilde{u}_N)$ , where  $\Gamma$  is any  $N \times N$  orthogonal matrix and  $\mathcal{L}(\cdot)$  denotes the distribution of  $\cdot$ .  $\mathcal{F}_N^2$  is the class of *elliptically symmetric*  $N$ -variate distributions with two moments finite (see Kelker 1970; King 1980). It contains the  $N$ -variate normal distribution and  $N$ -variate exponential and  $t$ -distributions with 3 or more degrees of freedom (see Bennett 1961; Dunnett and Sobel 1955; Lord 1954).

Kariya showed that if  $\mathcal{L}(u_N) \in \mathcal{F}_N^2$ , then the GLS estimator is best in the class  $\mathcal{C}_1$  in the sense of mean squared error. That is, for any  $\hat{\beta} \in \mathcal{C}_1$  and all  $c \in R^K$ ,

$$E(c'\hat{\beta} - c'\beta_0)^2 \geq E(c'\tilde{\beta} - c'\beta_0)^2 = \sigma^2 c'(X'\Sigma^{-1}X)^{-1}c. \quad (3)$$

As Kariya pointed out, this result has relevance whether or not  $\Sigma$  is known: If  $\Sigma$  is known it yields a best estimator; if  $\Sigma$  is unknown, it yields a sharp lower bound for the mean squared error of estimators in  $\mathcal{C}_1$ .

Although Kariya's class  $\mathcal{C}_1$  is more general than the class of linear mean-unbiased estimators, it is still quite restrictive. It excludes a wide variety of estimators in the literature that are mean- and median-unbiased in the present context. Such estimators include maximum likelihood for unknown  $\Sigma$ , robust  $M$ -,  $L$ -,  $R$ -, minimum distance, spectral, and adaptive estimators [see Andrews (1986) and the discussion in Sec. 2 here]. In addition, Kariya's result only establishes optimality with respect to the squared error loss function.

Hwang (1985, cor. 3.2) extended the Gauss–Markov theorem in a direction different from that of Kariya. He generalized the criterion of optimality considerably from mean squared error to risk under arbitrary symmetric monotone loss function (defined subsequently). The squared error loss function is of very special form and exhibits the general qualitative features of unboundedness and symmetry. In many circumstances, this loss function is not very appropriate. Hence it is important to see if the optimality of the GLS estimator is sensitive to this particular choice of loss function.

When the errors have multivariate normal distribution, it is known that the GLS estimator  $c'\tilde{\beta}$  is the best mean-unbiased estimator of  $c'\beta_0$  for any convex loss function (see Lehmann 1983, th. 3.4.3, p. 189). This is a generalization of the standard uniformly minimum variance unbiased (UMVU) result. It is quite useful, because it allows for asymmetric loss functions of fairly flexible shape and does not impose linearity of the estimators. Unfortunately,

the convexity condition implies unboundedness of the loss function, which may be inappropriate in many circumstances.

Hwang's result, on the other hand, imposes symmetry of the loss functions, but otherwise allows for quite general shape, including boundedness. For estimation of  $c'\beta_0$ , he considered nonnegative loss functions  $L(c'\hat{\beta} - c'\beta_0)$  that are symmetric about 0 and nondecreasing in  $|c'\hat{\beta} - c'\beta_0|$ . [In fact, Hwang's (1985) results carry through unchanged for loss functions  $L(\beta_0, c'\hat{\beta} - c'\beta_0)$  that are symmetric about 0 in their second argument and nondecreasing in  $|c'\hat{\beta} - c'\beta_0|$ , for each value of their first argument  $\beta_0 \in R^K$ . This extension can be important, because the magnitude of the loss attributable to overestimation or underestimation by a fixed amount often depends on the true value of the parameter.]

Hwang (1985, th. 2.3) showed that his class of loss functions is sufficiently general that given two estimators  $c'\hat{\beta}_1$  and  $c'\hat{\beta}_2$ , the risk of  $c'\hat{\beta}_1$  is less than or equal to that of  $c'\hat{\beta}_2$  for all symmetric monotone loss functions iff

$$|c'\hat{\beta}_1 - c'\beta_0| \stackrel{\text{ST}}{\leq} |c'\hat{\beta}_2 - c'\beta_0|, \quad (4)$$

where  $\stackrel{\text{ST}}{\leq}$  denotes "stochastically less than or equal to." Thus optimality under Hwang's class of loss functions is a strong result. The only clear deficiency is the restriction to *symmetric* loss function.

To show the optimality of GLS under symmetric monotone loss functions, Hwang assumed that the errors have elliptically symmetric distributions and maintained the restriction to linear estimators that is used in the Gauss–Markov theorem. In addition, he assumed that the estimators are either mean- or median-unbiased. In the latter case, the error distributions are not subject to any moment restrictions; that is,  $\mathcal{L}(u_N) \in \mathcal{F}_N^0$ . [The superscript 0 denotes the assumed number of well-defined moments. For distributions in  $\mathcal{F}_N^0$  that have infinite variances,  $\Sigma$  does not satisfy (1), since no covariance matrix exists. In this case,  $\Sigma$  is just the characteristic matrix that achieves spherical symmetry in the transformed coordinates.] In the former case, the error distributions are assumed to have one moment well defined; that is,  $\mathcal{L}(u_N) \in \mathcal{F}_N^1$ . Thus Hwang's error assumptions are stronger than those of the Gauss–Markov theorem with respect to the range of distributions with finite variances but are more general in terms of moment restrictions.

Under these assumptions, Hwang showed that the GLS estimator  $c'\tilde{\beta}$  of  $c'\beta_0$  is best in the class of linear mean- or median-unbiased estimators for all symmetric monotone loss functions. Thus

$$|c'\tilde{\beta} - c'\beta_0| \stackrel{\text{ST}}{\leq} |c'\hat{\beta} - c'\beta_0| \quad (5)$$

for all linear unbiased estimators  $c'\hat{\beta}$ .

This is an interesting result, but it suffers greatly from the arbitrary restriction to linear estimators. In addition, the restriction to symmetric loss functions may be objectionable.

## 2. A STRONG OPTIMALITY RESULT FOR GLS

Each of the optimality results discussed previously is less general than desirable because of the class of loss functions considered and/or the class of estimators considered. For example, none of these results allows for bounded asymmetric loss functions. Further, the results of Kariya and Hwang arbitrarily restrict the class of estimators beyond the restriction due to mean- or median-unbiasedness (which itself may be subject to criticism). The result we present here removes these restrictions.

For estimation of  $c'\beta_0$ , given  $c \in R^K$ , we consider loss functions  $L(\beta_0, c'\hat{\beta} - c'\beta_0)$  that are subject only to the condition that loss is nondecreasing in  $c'\hat{\beta} - c'\beta_0$  for  $c'\hat{\beta} - c'\beta_0 > 0$  and nonincreasing in  $c'\hat{\beta} - c'\beta_0$  for  $c'\hat{\beta} - c'\beta_0 < 0$ , for each value of its first argument  $\beta_0 \in R^K$ . Such loss functions are called *monotone*. They were considered by Lehmann (1959, p. 83) and Pfanzagl (1979).

Given any  $d_1, d_2 > 0$ , there exists a monotone loss function such that the risk of an estimator  $c'\hat{\beta}$  is  $\Pr(-d_1 \leq c'\hat{\beta} - c'\beta_0 \leq d_2)$ . Thus an estimator that is optimal with respect to the class of monotone loss functions has a distribution more concentrated about the estimand than any other estimator considered. This is a strong optimality property.

The argument of Hwang (1985, th. 2.3) can be used to show that for two estimators  $c'\hat{\beta}_1$  and  $c'\hat{\beta}_2$ , the risk of  $c'\hat{\beta}_1$  is less than or equal to that of  $c'\hat{\beta}_2$  for all monotone loss functions iff

$$(c'\hat{\beta}_1 - c'\beta_0)_+ \stackrel{ST}{\leq} (c'\hat{\beta}_2 - c'\beta_0)_+ \quad \text{and} \quad (c'\hat{\beta}_1 - c'\beta_0)_- \stackrel{ST}{\leq} (c'\hat{\beta}_2 - c'\beta_0)_-, \quad (6)$$

where  $(\cdot)_+$  and  $(\cdot)_-$  denote the positive and negative parts of  $\cdot$ , that is, for  $\lambda \in R$ ,  $(\lambda)_+ = \max\{\lambda, 0\}$  and  $(\lambda)_- = \max\{-\lambda, 0\}$ . If an estimator  $\hat{\beta}_1$  satisfies (6) for all  $\hat{\beta}_2$  in a designated class, we say that  $\hat{\beta}_1$  is *stochastically best* in this class of estimators. This is a stronger property than optimality with respect to Hwang's stochastic condition (4).

A particular monotone loss function that may be of interest is the function  $L(\beta_0, s) = s^2/(1 + \lambda s^2)$  for  $\lambda > 0$ , where  $s = c'\hat{\beta} - c'\beta_0$ . This loss function is bounded, yet for small  $\lambda$  it is close to the common squared error loss function except when  $s^2$  is large. Of course, the (unbounded) squared error loss function is also a monotone loss function.

The class of error distributions that we consider is slightly less general than the class  $\mathcal{F}_N^0$  of elliptically symmetric  $N$ -variate distributions centered at the origin. In most applications of the linear regression model, the properties of the errors are not specific to the sample size under consideration. In particular, if an assumption such as elliptical symmetry of the errors is reasonable for sample size  $n$  equal to some  $N$ , then it is necessarily reasonable for sample size  $n$  equal to  $N - 1, N - 2, \dots, 1$ , and usually also is reasonable for sample sizes  $N + 1, N + 2, \dots$ . This being the case, it is not unduly restrictive to consider the subclass of error distributions of  $\mathcal{F}_N^0$  given by

$$\mathcal{G}_N = \{\mathcal{L}(u_N) \in \mathcal{F}_N^0 : \mathcal{L}(u_n) \in \mathcal{F}_n^0 \text{ for } n = 1, 2, \dots\},$$

where  $u_n$  denotes the vector of errors  $(u_{(1)}, u_{(2)}, \dots, u_{(n)})'$  when the sample size is  $n$ . That is,  $\mathcal{G}_N$  contains all distributions of the first  $N$  errors that can be generated by errors  $(u_{(1)}, u_{(2)}, \dots, u_{(n)})'$  that have elliptically symmetric distributions for any sample size  $n = 1, 2, \dots$ .

Distributions in  $\mathcal{G}_N$  are called *consistent elliptically symmetric* (CES)  $N$ -variate distributions, where the adjective "consistent" refers to the fact that the distributions are consistent with elliptical symmetry for any sample size  $n$ . Since  $\mathcal{G}_N$  is not restricted by moment conditions, it contains distributions with infinite variances and nonexistent means. In particular,  $\mathcal{G}_N$  contains the  $N$ -variate normal, exponential, and  $t$ -distributions, including the  $N$ -variate Cauchy distribution.

By theorem 10 of Kelker (1970),  $\mathcal{L}(u_N) \in \mathcal{G}_N$  iff the distribution of  $\Sigma^{-1/2}u_N$  is a variance mixture of  $N$  iid mean 0 normal random variables (with nonnegative mixing density). Thus CES distributions can be constructed and characterized quite simply.

In comparison with the error distributions considered in the Gauss–Markov theorem, the class of CES distributions restricts the range of distributions with finite variances considerably. On the other hand, this restriction weakens the conditions of mean- and median-unbiasedness substantially, as we now shall see.

The class of estimators that we consider consists of all *median-unbiased* estimators. By definition, an estimator  $c'\hat{\beta}$  of  $c'\beta_0$  is median-unbiased if

$$\Pr(c'\hat{\beta} \geq c'\beta_0) \geq \frac{1}{2} \quad \text{and} \quad \Pr(c'\hat{\beta} \leq c'\beta_0) \geq \frac{1}{2}, \quad (7)$$

for each error distribution  $\mathcal{L}(u_N)$  in  $\mathcal{G}_N$ . If  $\Pr(c'\hat{\beta} = c'\beta_0) = 0$ , as is usually the case, then this condition simplifies to  $\Pr(c'\hat{\beta} > c'\beta_0) = \Pr(c'\hat{\beta} < c'\beta_0) = \frac{1}{2}$ .

In the present context, the class of median-unbiased estimators is very large—much larger than the class of mean- or median-unbiased estimators in the Gauss–Markov setup. The reason is that  $u_N$  is symmetrically distributed about the 0 vector [i.e.,  $\mathcal{L}(u_N) = \mathcal{L}(-u_N)$ ] when it has an elliptically symmetric distribution. Thus all estimators  $\hat{\beta}$  that are odd functions of the errors have distributions symmetric about  $\beta_0$  and yield median-unbiased estimators  $c'\hat{\beta}$  of  $c'\beta_0$ , for all  $c \in R^K$ . As shown in Andrews (1986), this result applies to the majority of non-Bayesian, nonshrinkage estimators considered in the literature. It holds for a wide class of nonlinear estimators that are defined as solutions to maximization problems or systems of equations, where initial estimators may be employed. This includes iterated estimators. In particular, the following estimators are covered: feasible GLS, quasi-maximum likelihood, Huber  $M$ -, bounded-influence  $M$ -,  $L$ -,  $R$ -, minimum distance, spectral, band spectral, generalized efficient  $M$ - (see Andrews 1983), adaptive, one-step asymptotically efficient, and instrumental variable. Note that these estimators also are mean-unbiased provided that their expectation exists.

Our main result is the following theorem. Its proof makes use of a result of Lehmann (1959, pp. 80–83) for best median-unbiased estimation in monotone likelihood ratio families of distributions that are indexed by a scalar pa-

parameter. A different proof of our result can be obtained by applying an extension of Lehmann's result due to Pfanzagl (1979).

The term "unique" is used in the theorem to mean unique almost everywhere with respect to Lebesgue measure.

*Theorem 1.* Consider the model  $y = X\beta_0 + u_N$ , where  $\beta_0 \in R^k$ ,  $X$  is full rank, and  $\mathcal{L}(u_N) \in \mathcal{G}_N$ . (a) The GLS estimator  $c'\hat{\beta}$  is the unique best median-unbiased estimator of  $c'\beta_0$  for any given  $c \in R^k$  in the sense of uniformly minimum risk for any monotone loss function. (b) Equivalently, the GLS estimator  $c'\hat{\beta}$  is the unique stochastically best median-unbiased estimator of  $c'\beta_0$  for any given  $c \in R^k$ .

The proof is given in the Appendix.

*Comment 1.* The theorem also holds if we restrict attention to errors with multivariate normal distributions. The requirement of median-unbiasedness under the larger class of CES distributions is not driving the optimality result by eliminating estimators from consideration.

*Comment 2.* The GLS estimator has infinite risk for some loss functions and some error distributions in  $\mathcal{G}_N$ . The theorem still has import in these circumstances, however, because it implies that every other median-unbiased estimator also has infinite risk.

*Comment 3.* In some cases, the ultimate object of interest is not  $c'\beta_0$  but a nonlinear function of  $c'\beta_0$ , say  $h(c'\beta_0)$ , because it has a particular interpretation or meaning in an underlying theoretical model. For example, we may want to estimate the logarithm of a regression parameter. If  $h(\cdot)$  is a monotone function, then given Theorem 1, it is not hard to see that not only is  $h(c'\hat{\beta})$  median-unbiased, but it is the best median-unbiased estimator for any monotone loss function (under the assumptions of the theorem). This is a very convenient result, especially in light of the difficulties in obtaining best mean-unbiased estimators of nonlinear functions of  $c'\beta_0$ . Such estimators do not equal  $h(c'\hat{\beta})$ , in general, and may not even exist.

*Comment 4.* For bounded loss functions, the risk of the GLS estimator is finite even when the errors have undefined means or infinite variances, for example, as in the  $N$ -variate Cauchy case. Thus we get the interesting result that situations exist in which the least squares estimator is strictly preferred over a wide variety of robust procedures, even though the errors may have no moments finite. This result is possible, because the errors are not independent, even if  $\Sigma = I_N$ , unless  $u_N$  has normal distribution. The optimality result depends heavily on the elliptically symmetric form of the underlying error distribution, as comparisons with results in the robustness literature clearly attest [compare Huber (1981)].

*Comment 5.* The class of estimators considered in Theorem 1 is much more general with respect to nonlinearity than is Kariya's (1985) class  $\mathcal{C}_1$ . It does not contain  $\mathcal{C}_1$ , however, because  $\mathcal{C}_1$  includes some median-biased estimators. On the other hand, if the function  $C(e)$  that defines Kariya's estimators is an even function of the

OLS residuals  $e$ , then  $\hat{\beta} = C(e)y$  is median-unbiased for  $\mathcal{L}(u_N) \in \mathcal{F}_N^0$ , since  $\hat{\beta} - \beta_0$  is an odd function of the errors. [Remember,  $C(e)X = I_k$ , by definition of  $C(e)$ .] Given the assumed symmetry of  $u_N$  about the 0 vector, the evenness of  $C(e)$  arises quite naturally and most estimators in  $\mathcal{C}_1$  that have been considered in the literature satisfy this property.

Nevertheless, for an optimality result it is desirable to avoid any restriction on the class of estimators, if possible. If one wishes to include the median-biased estimators of Kariya in an optimality result, one can proceed as follows: Consider the estimators of theorem 1 of Andrews (1986) where the assumption A1 is relaxed by requiring the defining function  $r$  of the estimators to be even in only its first argument rather than its first three arguments. Call the collection of such estimators  $\mathcal{C}_2$ . The class  $\mathcal{C}_2$  contains  $\mathcal{C}_1$ . One can show that for  $\beta_0 \in R^k$  and  $\mathcal{L}(u_N) \in \mathcal{G}_N$  the GLS estimator  $\hat{\beta}$  is the best estimator of  $\beta_0$  in the class  $\mathcal{C}_2$  in the sense of uniformly minimum risk for any symmetric convex loss function (see Andrews and Phillips 1985). This result generalizes Kariya's, because it considers much wider classes of loss functions and estimators (although it imposes slightly different error assumptions).

*Comment 6.* The result of Theorem 1 can be extended to allow homogeneous or nonhomogeneous linear restrictions on  $\beta_0$  and to allow less than full rank  $X$  matrix (provided that identifying linear side conditions on  $\beta_0$  are specified). If  $\beta_0$  is subject to inequality constraints, however, then Theorem 1 no longer holds, but a restricted GLS estimator can be shown to possess similar strong optimality properties, as the next section illustrates.

*Comment 7.* As stated, Theorem 1 does not cover the standard multivariate regression model. It is not difficult, however, to use the proof of Theorem 1 to establish an analogous result for this model. Such a result is important, since the multivariate regression model is of considerable interest in econometrics because of its application to demand systems, among others.

The multivariate regression model consists of  $T$  observations on  $g$  equations and can be written as

$$\begin{matrix} Y & = & Z & A_0 & + & U, \\ (T \times g) & & (T \times m) & (m \times g) & & (T \times g) \end{matrix}$$

where  $Y$ ,  $Z$ ,  $A_0$ , and  $U$  are matrices of dependent variables, regressors, unknown parameters, and errors, respectively. The parameter matrix  $A_0$  may contain zeros and redundant elements and hence is assumed to satisfy  $\text{vec}(A_0) = S\beta_0$ , where  $S$  is a  $gm \times p$  known selection matrix (with  $p \leq gm$ ),  $\beta_0$  is the vector of basic unknown parameters,  $\beta_0 \in R^p$ , and  $\text{vec}(\cdot)$  denotes the row-by-row vectorization operator. Equivalently, this model can be written as  $y = X\beta_0 + u$ , where  $y = \text{vec}(Y)$ ,  $X = (I_g \otimes Z)S$ , and  $u = \text{vec}(U)$ . Write  $U = (u_1, \dots, u_T)'$ . Suppose that the error vectors  $u_1, \dots, u_T$  are independent across observations and each error vector  $u_i$  has some elliptically symmetric distribution with  $g \times g$  full rank characteristic matrix  $\Omega$ , and no probability mass at the origin. (The vectors  $u_1, \dots, u_T$  need not be identically distributed.) Let  $\hat{\beta}$  denote the GLS estimator of  $\beta_0$  given by Equation (2)

with  $\Sigma = \text{diag}(\Omega_1, \dots, \Omega_T)$ . The aforementioned class of distributions of  $u$  does not equal  $\mathcal{G}_{gT}$ , and hence Theorem 1 does not apply. Nevertheless, it is straightforward to alter the proof of Theorem 1 to show that the optimality results (a) and (b) of Theorem 1 hold for the GLS estimator  $\hat{\beta}$  in this multivariate regression model.

*Comment 8.* Two examples of situations in which non-normal elliptically symmetric error distributions may arise are the following: First, consider the classical regression model based on an agricultural experiment. Suppose that the dependent variable is crop yield, and the independent variables include fertilizer treatment. The error may be composed of several factors, including differential land quality. The seed for each plot is taken from the same stock. The quality of this stock may be viewed as the outcome of a random draw (with different points in time or different geographic origins of the stock yielding different draws). Conditional on the stock of seed used, it may be reasonable to assume that the errors have a normal distribution. Different stocks of seed may interact differently with the environment to yield different conditional variances of the errors. To make inferences that are valid for the population of seed stocks, then, one needs to treat the errors as a variance mixture of normal distributions.

Second, consider a regression model with economic variables where the observations correspond to different firms in an industry observed at the same point in time. Suppose that the errors are identically distributed across firms and the state of the macroeconomy affects the size of the error variance for each firm. It may be reasonable to assume that the errors have a normal distribution conditional on the state of the macroeconomy. If so, then one needs to treat the errors as a variance mixture of normals if one wishes to make inferences that are valid for different points in the business cycle.

These examples suggest that there are a number of situations in which it may be reasonable to assume that the errors have nonnormal elliptically symmetric distributions. Of course, there are many additional situations for which the assumption of normality is appropriate.

*Comment 9.* As a final remark, we mention that the restriction to unbiased estimators (whether mean- or median-unbiased) is more difficult to justify with asymmetric loss functions than with symmetric loss functions [e.g., see Jones and Rothenberg (1981) and Zellner (1986)].

### 3. OPTIMAL ESTIMATION WITH A RESTRICTED PARAMETER SPACE

In this section we discuss optimal estimation of  $c'\beta_0$  when  $\beta_0$  is subject to certain nonlinear restrictions. In particular, we consider the case in which  $c'\beta_0$  is known to lie in an interval (possibly infinite) that does not depend on  $\beta_0$ , and certain linear combinations of  $\beta_0$ , denoted by  $c'_2\beta_0, \dots, c'_K\beta_0$ , are restricted in any fashion not involving  $c'\beta_0$ . A simple example is when we wish to estimate some element of  $\beta_0$  subject to the sole constraint that this element is positive or lies in  $[0, 1]$ .

Suppose that the only restriction on  $\beta_0$  is that  $c'\beta_0$  lies in a nondegenerate interval strictly contained in  $R$ . The best estimator of  $c'\beta_0$  from a subclass of mean-unbiased estimators is the GLS estimator that ignores the constraints, according to the Gauss–Markov theorem, Kariya's (1985) results, or various generalized UMVU results. The reason is that any attempt to improve the GLS estimator to take account of the constraints results in a mean-biased estimator. In this context, the mean-unbiasedness condition is overly restrictive.

On the other hand, estimators  $\hat{\delta}$  of  $c'\beta_0$  that are median-unbiased when no constraints are present can be adjusted quite naturally to take advantage of the restriction that  $c'\beta_0$  lies in an interval, or any subset of  $R$ , without losing their property of median-unbiasedness. Whenever  $\hat{\delta}$  lies outside the parameter space of  $c'\beta_0$ , set the adjusted estimator  $(\hat{\delta})_R$  equal to the closest value in the closure of the parameter space; otherwise leave the estimator as is. The resultant estimator  $(\hat{\delta})_R$  is median-unbiased for the restricted parameter space and lies in its closure. Thus the condition of median-unbiasedness is a relatively attractive condition for restricting the class of estimators when the parameter space of  $c'\beta_0$  is restricted.

We now define the linear combinations  $(c'_2\beta_0, \dots, c'_K\beta_0)$  of  $\beta_0$  that may be subject to additional restrictions beyond that on  $c'\beta_0$ . Let  $\tilde{X} = \Sigma^{-1/2}X$ . Since  $\tilde{X}$  is full rank  $K$ ,  $c'$  is proportional to some linear combination of the rows of  $\tilde{X}$ . Say,  $c' \propto d'_1\tilde{X}$ , where  $d_1$  is an orthonormal  $N$ -vector. Take any  $K - 1$  orthonormal  $N$ -vectors  $d_2, \dots, d_K$  that are orthogonal to  $d_1$  and are such that  $(d_1, \dots, d_K)$  span the column space of  $\tilde{X}$ . Then, the vectors  $c_j$  ( $j = 2, \dots, K$ ) are given by  $c_j = d'_j\tilde{X}$  for  $j = 2, \dots, K$ . As a simple example, suppose that  $\Sigma = I_N$ ,  $c' = (0, \dots, 0, 1)$ , so  $c'\beta_0 = \beta_{0K}$ , and the  $K$ th column of  $X$  is orthogonal to its other columns. In this case,  $(\beta_{01}, \dots, \beta_{0K-1})$  can be restricted in any way (not involving  $\beta_{0K}$ ) without affecting the optimality of the best median-unbiased estimator of  $\beta_{0K}$ .

The main result of this section gives a strong optimality property for the restricted GLS estimator  $(c'\hat{\beta})_R$ :

*Theorem 2.* Consider the model  $y = X\beta_0 + u_N$ , when  $\mathcal{L}(u_N) \in \mathcal{G}_N$ ,  $X$  has full rank,  $c'\beta_0$  lies in a known (possibly infinite) interval  $I_c$  that does not depend on  $\beta_0$ , and the linear combinations  $(c'_2\beta_0, \dots, c'_K\beta_0)$  of  $\beta_0$  are restricted in any fashion not involving  $c'\beta_0$ . Then, the restricted GLS estimator  $(c'\hat{\beta})_R$  is the unique best median-unbiased estimator of  $c'\beta_0$  in the sense of uniformly minimum risk for any monotone loss function. Equivalently, it is the unique stochastically best median-unbiased estimator of  $c'\beta_0$  for given  $c \in R^K$ .

The proof of this result makes use of the theorem of Pfanzagl (1979) (see the Appendix).

*Comment.* When the restrictions on  $\beta_0$  are such that the interval containing  $c'\beta_0$  depends on  $\beta_0$ , a uniformly best median-unbiased estimator of  $c'\beta_0$  does not exist. We still can obtain a lower bound on the risk of a median-unbiased

estimator of  $c'\beta_0$ , however, by using the method of the proof of Theorem 2. In particular, if we suppose that  $(c_2'\beta_0, \dots, c_K'\beta_0)$  are known, then the interval containing  $c'\beta_0$  is known [call it  $I_c(\beta_0)$ ], and the stochastically best median-unbiased estimator of  $c'\beta_0$  is the restricted GLS "estimator"  $(c'\hat{\beta})_R$ , restricted to the closure of  $I_c(\beta_0)$ . The risk of  $(c'\hat{\beta})_R$  as a function of  $\beta_0$  gives the desired lower bound.

#### 4. OPTIMAL ESTIMATION OF $\sigma^2$

In this section we specialize to the case of linear regression with iid normal errors with mean 0 and variance  $\sigma^2$ . We consider estimation of  $\sigma^2$  and various monotone transformations of  $\sigma^2$ , such as  $\sigma$  and  $d\sigma^2$  for some constant  $d \neq 0$ .

First, we discuss the optimality properties of the most commonly used estimators, namely,  $s^2$ ,  $s$ , and  $ds^2$  for  $\sigma^2$ ,  $\sigma$ , and  $d\sigma^2$ , respectively, where  $s^2 = (1/(N - K))(y - X\hat{\beta})'(y - X\hat{\beta}) \equiv (1/(N - K))SSR$  and  $\hat{\beta}$  is the least squares estimator. The use of  $s^2$  to estimate  $\sigma^2$  is justified in this context by the fact that it is the best mean-unbiased estimator in the sense of uniformly minimum risk for any convex loss function (see Lehmann 1983, th. 3.4.1, p. 185). This optimality property carries over to the estimation of  $d\sigma^2$  by  $ds^2$  but does not hold for the standard error of estimate  $s$  of  $\sigma$ , since  $s$  is biased. If  $\sigma^2$  is known to lie in a nondegenerate interval strictly contained in  $R^+$ , then  $s^2$  is still the best mean-unbiased estimator of  $\sigma^2$  for convex loss, even though it ignores the restrictions on  $\sigma^2$ .

For squared error loss, the risk of  $s^2$  is uniformly dominated by that of the mean-biased estimator  $\bar{s}^2 = (1/(N - K + 2))SSR$  (e.g., see Rao 1973, p. 316). This result is not of great concern, however, since the *symmetric* squared error loss function is usually quite inappropriate for estimation of  $\sigma^2$ . For example, it implies that the maximum loss from underestimation is bounded, whereas that from overestimation is unbounded. Furthermore, by appropriate choice of asymmetric squared error loss function,  $s^2$  dominates  $\bar{s}^2$  and any other scalar multiple of  $SSR$ .

We now consider an alternative to  $s^2$  and  $h(s^2)$  for estimating  $\sigma^2$  and  $h(\sigma^2)$ , where  $h(\cdot)$  is any monotone function. This alternative has several desirable properties. Suppose that  $\sigma^2$  is known to lie in an interval with endpoints  $a, b$  where  $0 \leq a < b \leq \infty$ . Define the estimator  $\tau^2$  by

$$\begin{aligned} \tau^2 &= b && \text{when } SSR/m_{N-K} \geq b \\ &= SSR/m_{N-K} && \text{when } SSR/m_{N-K} \in [a, b] \\ &= a && \text{when } SSR/m_{N-K} \leq a, \end{aligned} \quad (8)$$

where  $m_{N-K}$  is the median of a chi-squared random variable with  $N - K$  degrees of freedom.

The estimator  $\tau^2$  has the following properties: (i)  $\tau^2$  is the best median-unbiased estimator of  $\tau^2$  for any monotone loss function. Equivalently, it is the stochastically best median-unbiased estimator of  $\sigma^2$ . In contrast to the optimality results for  $s^2$ , this result includes bounded asymmetric loss functions. (ii) The optimality result of (i) holds even when

$\beta_0$  is subject to restrictions, provided that the parameter space of  $\beta_0$  has a nonempty interior. (iii)  $\tau^2$  takes advantage of the restrictions on  $\sigma^2$ . This is a distinct advantage of  $\tau^2$  over the best mean-unbiased estimator  $s^2$ . (iv) The estimator  $h(\tau^2)$  of  $h(\sigma^2)$  inherits the same optimality properties as  $\tau^2$ , provided that  $h(\cdot)$  is monotone on  $[a, b]$ . In particular,  $\tau$  and  $d\tau^2$  are best median-unbiased estimators of  $\sigma$  and  $d\sigma^2$ , respectively, for any monotone loss function. This result not only guarantees the existence of a best median-unbiased estimator for many estimands  $h(\sigma^2)$  of interest, it also provides very simple expressions for such estimators. Best mean-unbiased estimators of  $h(\sigma^2)$  do not exist for some functions  $h(\cdot)$ , and even when they do exist, they are more difficult to determine than the best median-unbiased estimator.

Results (i) and (ii) follow by showing that the present problem is covered by Pfanzagl's (1979) theorem and that the estimator  $\tau^2$  is the optimal estimator defined in his proof. Result (iv) follows from (i) and (ii) using the fact that both  $h(\cdot)$  and the loss functions under consideration are monotone.

Since it is natural to compare  $\tau^2$  and  $s^2$ , we might ask: In what ways, and to what extent, do  $\tau^2$  and  $s^2$  differ? To answer the first part of this question, we note that  $m_{N-K} < N - K$ , because  $m_{N-K}$  and  $N - K$  are the median and mean of a chi-squared random variable, respectively. Hence  $s^2 \leq b$  iff  $s^2 \leq \tau^2$ . That is,  $\tau^2$  is larger than  $s^2$  unless  $s^2$  takes a value larger than any value in the parameter space of  $\sigma^2$ .

The extent to which  $\tau^2$  and  $s^2$  differ depends on two separate factors: (i) whether  $a$  is positive and/or  $b$  is finite, and if so, on the proximity of the true parameter  $\sigma^2$  to one or other of the endpoints  $a$  or  $b$ , and (ii) the size of  $N - K$ . When  $a > 0$  and/or  $b < \infty$ ,  $\tau^2$  is a censored or doubly censored version of  $SSR/m_{N-K}$ . The closer is the true value  $\sigma^2$  to  $a$  or  $b$ , the greater is the extent of the censoring.

If  $a = 0$  and  $b = \infty$ , the only difference between  $\tau^2$  and  $s^2$  is in the multiplicative constants  $1/m_{N-K}$  and  $1/(N - K)$ . As  $N - K \rightarrow \infty$ ,  $(N - K)/m_{N-K} = \tau^2/s^2 \rightarrow 1$ , as expected. For degrees of freedom  $N - K$  equal to 10, 20, and 30,  $m_{N-K}$  equals 9.342, 19.34, and 29.34, and  $\tau^2$  exceeds  $s^2$  by 7.1%, 3.6%, and 2.4%, respectively. [See Thompson (1941) and Pearson and Hartley (1958, p. 130) for tables giving the medians of chi-squared random variables with degrees of freedom less than or equal to 100.]  $\tau^2$  is equal to  $s^2$  with its degrees of freedom reduced by .66 when  $N - K$  is in [8, 50) and by .67 when  $N - K$  is in [50, 100]. More sizable differences between  $\tau^2$  and  $s^2$  occur only if the parameter space is restricted.

#### APPENDIX: PROOFS OF THEOREMS 1 AND 2

*Proof of Theorem 1.* The distribution of  $\Sigma^{-1/2}u_N$  can be decomposed into a probability mass at the origin and an absolutely continuous component. Since the GLS estimator equals  $\beta_0$  if  $u_N = \mathbf{0}$ , we can assume that  $u_N$  has no mass at the origin, without loss of generality. Then,  $\Sigma^{-1/2}u_N$  has Lebesgue density  $\int (2\pi w)^{-N/2} \exp(-\|u_N\|^2/2w) dG(w)$ , where  $G(w)$  is a distribution on  $(0, \infty)$  and  $\|\cdot\|$  denotes the Euclidean norm on  $R^N$ . Let  $W$

denote the scalar mixing random variable with distribution  $G(\cdot)$ . Conditional on  $W = w$ , the distribution of  $u_N$  is multivariate normal with mean  $\mathbf{0}$  and covariance  $w\Sigma$ .

Condition on  $W = w$ . Let  $\tilde{X} = (1/\sqrt{w})\Sigma^{-1/2}X$ . We can construct an  $N \times N$  orthogonal matrix  $D$  such that the first  $K$  rows of  $D$  span the column space of  $\tilde{X}$  and the first row of  $D\tilde{X}$  is proportional to  $c'$ . That is,  $d_1'\tilde{X} = \gamma c'$  for the constant  $\gamma = \|d_1'\tilde{X}\tilde{X}'d_1\|/\|c\|$ , where  $d_1$  denotes the first row of  $D$  written as a column. Transform the model by premultiplication by  $(1/(\gamma\sqrt{w}))D\Sigma^{-1/2}$  to get  $y^* = X^*\beta_0 + u_N^*$ , where  $y^* = (1/(\gamma\sqrt{w}))D\Sigma^{-1/2}y$ ,  $X^* = (1/(\gamma\sqrt{w}))D\Sigma^{-1/2}X$ , and  $u_N^* = (1/(\gamma\sqrt{w}))D\Sigma^{-1/2}u_N \sim N(\mathbf{0}, (1/\gamma^2)I_N)$ . Define  $\eta = (\eta_1, \dots, \eta_N)'$  and  $\eta = X^*\beta_0$ . By the choice of  $D$ , we have  $(\eta_{K+1}, \dots, \eta_N) = (0, \dots, 0)$  and  $\eta_1 = e_1'D\tilde{X}\beta_0/\gamma = c'\beta_0$ , where  $e_1 = (1, 0, \dots, 0)'$ . Thus the estimand is  $\eta_1$ .

Consider estimation of  $\eta_1$  when the single observation  $y_1^* \sim N(\eta_1, 1/\gamma^2)$  is observed and  $1/\gamma^2$  is assumed known, where  $y_1^*$  is the first element of  $y^* = (y_1^*, \dots, y_N^*)'$ . The family of densities of  $y_1^*$  for  $\eta_1 \in R$  forms a monotone likelihood ratio family, and the likelihood ratios are a nondecreasing function of the continuous random variable  $y_1^*$ . Hence, by the confidence-bound results of Lehmann (1959, cor. 3, p. 80 and p. 83) for scalar parameters, the unique uniformly minimum risk median-unbiased estimator of  $\eta_1$  (based on observing  $y_1^*$  only) is  $y_1^*$ , for any monotone loss function, over the class of nonrandomized and randomized estimators. [See Lehmann (1959, p. 81) for construction of the randomized confidence bounds needed to compare the risk of  $y_1^*$  with the risks of randomized estimators.]

Now, any unconditionally median-unbiased estimator  $\hat{\delta}(y, X)$  of  $c'\beta_0$  also is median-unbiased conditional on  $W = w$ , because the conditional distribution of  $u_N$  is itself a CES distribution. We can write  $\hat{\delta}(y, X)$  as  $\hat{\delta}(y^*, X^*)$ . For purposes of comparing the risk of  $\hat{\delta}(y^*, X^*)$  with that of  $y_1^*$ , suppose that the vector  $(\eta_2, \dots, \eta_N)$  is known. The independence of  $y_1^*$  and  $(y_2^*, \dots, y_N^*)$ , plus the knowledge of  $X^*$  and the distribution of  $(y_2^*, \dots, y_N^*)$ , implies that  $\hat{\delta}(y^*, X^*)$  has the same conditional distribution as some randomized estimator of  $\eta_1$  based on the single observation  $y_1^*$ . Lehmann's result then implies that conditional on  $W = w$ , the risk of  $y_1^*$  is less than or equal to that of  $\hat{\delta}(y^*, X^*)$ . Since the optimal estimator  $y_1^*$  does not depend on  $\gamma$ , the assumption of known  $\gamma$  is innocuous. The optimality of  $y_1^*$  holds for all  $w$ , so integrating out  $w$  yields the unconditional optimality of  $y_1^*$ . This gives the desired result, because  $y_1^*$  is the GLS estimator of  $c'\beta_0$ . Let  $\tilde{y} = (1/\sqrt{w})\Sigma^{-1/2}y$ , then  $y_1^* = (1/\gamma)d_1'\tilde{y} = (1/\gamma)d_1'\tilde{X}(\tilde{X}'\tilde{X})^{-1}\tilde{X}'\tilde{y} = c'(\tilde{X}'\tilde{X})^{-1}\tilde{X}'\tilde{y} = c'\beta$ .

This proof could be shortened somewhat by applying Pfanzagl's (1979) theorem, instead of Lehmann's result. This is not done, however, because the proof given here is needed in the proof of Theorem 2 to attain the stated generality of Theorem 2. In addition, the proof of Pfanzagl's result is more complicated than that of Lehmann, because Pfanzagl considered cases in which the best estimator is randomized. Thus the reference to the simpler result of Lehmann may be helpful to the reader.

The extension of Theorem 1 to include non-CES distributions [as considered by Kariya (1985) and Hwang (1985)] is problematic using the method of proof given here. Almost all elliptically symmetric distributions can be written as variance mixtures of multivariate normal distributions (see Chu 1973). For non-CES distributions, however, the mixing "densities" are somewhere negative. The risk inequalities that hold for given variance values are reversed for negative values of the mixing density and hence cannot be integrated up over the range of values of the mixing density. Fortunately, as the discussion of Section 2 indicates, the

restriction to CES distributions is not serious. The elliptically symmetric distributions of greatest relevance are CES distributions.

*Proof of Theorem 2.* Proceed as in the proof of Theorem 1 to transform the model such that  $\eta_1$  is the estimand. The restricted estimators  $(y_1^*)_R$  is median-unbiased and equals  $(c'\hat{\beta})_R$  by arguments given previously.

Condition on  $W = w$ . The linear combinations  $(c_2'\beta_0, \dots, c_K'\beta_0)$  equal  $\gamma\sqrt{w}(\eta_2, \dots, \eta_K)$ . Thus the restrictions on  $(c_2'\beta_0, \dots, c_K'\beta_0)$  do not affect the conditional distribution of  $y_1^*$  or the parameter space  $I_c$  of  $\eta_1$ . Hence we can mimic the proof of Theorem 1 and assume that  $w$  and  $(\eta_2, \dots, \eta_K)$  are known for the purposes of comparing the risk of an arbitrary (conditionally and unconditionally) median-unbiased estimator  $\hat{\delta}(y^*, X^*)$  with that of  $(y_1^*)_R$ .  $\hat{\delta}(y^*, X^*)$  has distribution equal to that of some randomized estimator of  $\eta_1$  for the case in which only  $y_1^*$  is observed. Thus it suffices to show that conditional on  $W = w$ ,  $(y_1^*)_R$  is the unique best median-unbiased estimator of  $c'\beta_0$  based on the single observation  $y_1^* \sim N(\eta_1, 1/\gamma^2)$ . This follows by Pfanzagl's (1979) theorem and the proof is complete.

Pfanzagl's (1979) theorem allows for nuisance parameters and hence could be applied in the proof of Theorem 2 by treating  $(\eta_2, \dots, \eta_K)$  as nuisance parameters. This approach limits the restrictions on  $(\eta_2, \dots, \eta_K)$ , however, because it requires the assumption that the restricted parameter space of  $(\eta_2, \dots, \eta_K)$  contains a nonempty interior.

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