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A SIMPLIFIED PROOF OF A THEOREM ON THE DIFFERENCE OF THE MOORE-PENROSE INVERSES OF TWO POSITIVE SEMI-DEFINITE MATRICES

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ABSTRACT

Simplified proofs are given of a standard result that establishes positive semi-definiteness of the difference of the inverses of two non-singular matrices, and of the extension of this result by Milliken and Akdeniz (1977) to the difference of the Moore-Penrose inverses of two singular matrices.

INTRODUCTION AND PROOFS

The following is a well known matrix theorem that is used extensively in the statistical literature to compare the covariance matrices of different statistical estimators: If A, B, and A-B are positive definite (pd), then $B^{-1} - A^{-1}$ is pd. Standard proofs of the result are given in texts such as Goldberger (1964, p. 38) and Graybill (1969, p. 330). Milliken and Akdeniz (1977) recently extended the theorem to the case where A and B are positive semi-definite (psd). They motivate this extension and provide examples of its applicability.

Here we provide substantially simplified proofs of both theorems.

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Theorem 1. Suppose A and B are pd and A-B is psd (pd), then $B^{-1} - A^{-1}$ is psd (pd).

Proof: The matrices A and B can be simultaneously diagonalized as $A = QQ'$ and $B = QDQ'$, where W and D are non-singular and D is diagonal (see Anderson (1971, Problem 30, p. 363)). Hence, A-B psd (pd) implies I-D psd (pd), and this, in turn, implies $D^{-1} - I$ psd (pd). Since $A^{-1} = (Q')^{-1}Q^{-1}$ and $B^{-1} = (Q')^{-1}D^{-1}Q^{-1}$, $D^{-1} - I$ psd (pd) implies $B^{-1} - A^{-1}$ psd (pd), as desired. □

Theorem 2 (Milliken-Akdeniz). Suppose A, B, and A-B are psd, then $B^{+} - A^{+}$ is psd if and only if $rk(A) = rk(B)$ (where A^{+} and $rk(A)$ denote the Moore-Penrose generalized inverse and the rank of A, respectively).

Proof: To show necessity, note that $B^{+} - A^{+}$ psd implies $rk(B) = rk(B^{+}) \geq rk(A^{+}) = rk(A)$, and A-B psd implies $rk(A) \geq rk(B)$.

To show sufficiency, write $A = FF'$ and $B = GG'$, where F and G are $n \times r$ full column rank matrices, and n is the dimension and r is the rank of A and B. Since A-B is psd, there is a non-singular matrix W such that $F = GW$. Now, $A-B = G(WW' - I)G'$ psd implies $WW' - I$ psd, and hence, $I - (WW')^{-1}$ psd, by Theorem 1. This implies $B^{+} - A^{+}$ is psd, since

$$\begin{aligned} B^{+} - A^{+} &= G(G'G)^{-2}G' - F(F'F)^{-2}F' \\ &= G(G'G)^{-1}[I - (WW')^{-1}](G'G)^{-1}G' \ . \ \square \end{aligned}$$

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