Cowles Foundation Paper #669

LARGE DEVIATION EXPANSIONS IN ECONOMETRICS

P. C. B. Phillips

ABSTRACT

The role in econometrics of the modern branch of probability theory known as the theory of large deviations is discussed. In this theory, the argument in the distribution function or probability density of a standardized statistic is allowed to vary and, in particular, is allowed to grow with the sample size. The theory therefore provides a convenient mechanism by which the limiting tail behavior of econometric statistics may be studied. This paper develops an associated asymptotic expansion, which we call the large deviation expansion. This expansion is developed for statistics which may be expressed as quite general functions of the sample moments of the data, and it is therefore of rather wide applicability. The new expansion is related to more conventional asymptotic expansions of the Edgeworth type. An application to tail probability expansions in the AR(1) is presented.

Advances in Econometrics, vol. 5, pages 199–226

ISBN: 0-89232-686-7

I. INTRODUCTION

The problem of approximating finite sample distributions by Edgeworth series expansions has attracted a good deal of attention over the past decade. Important advances have occurred in mathematical statistics and in econometrics. In particular, useful general results and algorithms have been established by Chambers (1967), Sargan (1976), Phillips (1977b) Bhattacharya and Ghosh (1978), and Sargan and Satchell (1986). Reviews of some of this research are given in Bhattacharya and Rao (1976), Phillips (1980, 1983), and Rothenberg (1984). In addition to providing more information than simple asymptotic theory about the finite sample behavior of various econometric estimators and facilitating comparisons between different estimators, Edgeworth expansions can be used to try to improve statistical testing in econometrics by providing second order size corrections to conventional asymptotic tests. Higher order analyses along these lines using Edgeworth expansions are given by Rothenberg (1982) and by Akahira and Takeuchi (1981).

Series expansions of the Edgeworth type can be viewed as extensions of the limit theorems which give us the asymptotic distribution of our estimators and test statistics. As such, they belong to the same branch in the theory of probability as the classical central limit theorem. Moreover, they share a common limitation with classical central limit theory: namely, that they are often not very informative about the tails or limiting tails of a statistic of interest. To clarify this remark it is helpful to refer to the simplest case of a standardized sum Z_T of T independent and identically distributed random variables $\{X_t: t = 1, \dots, T\}$ with a common distribution such that $E(X_t) = 0$ and $E(X_t^2) = \sigma^2$. Then classical theory tells us that

$$F_T(x) = P(Z_T \leq x) \to I(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\, dt, \qquad \text{as } T \uparrow \infty, \qquad (1)$$

which is of interest when $x = 0(1)$ as $T \to \infty$. But when the argument x is allowed to vary with T, the statement of the above theorem can appear trivial. For instance, if $x \to -\infty$ as $T \uparrow \infty$, then both sides of (1) tend to zero. In such cases, what we are often really interested in is the behavior of the limiting tails of $F_T(-x)$ and $1 - F_T(x)$. It is more useful in this case to consider the ratios of tail probabilities

$$\frac{F_T(-x)}{I(-x)} \quad \text{and} \quad \frac{1 - F_T(x)}{1 - I(x)}$$

under the assumption that $x \to \infty$ as $T \uparrow \infty$. If the limiting tails are normal, then these ratios will converge to unity as $T \uparrow \infty$. Clearly, the rate at which $x \to \infty$ with T determines how deep in the tails we are concentrating. When

$x = O(\sqrt{T})$, as $T \to \infty$, a number of important results have been obtained and these constitute the theory of large deviations. Possibly the best single reference work in the field is the treatise by Ibragimov and Linnik (1971), which contains an extensive survey of research on large deviations up to the late 1960's. Serious work in the field commenced with a paper by Richter (1957), but the ideas may be traced back to Cramér (1938). Chernoff (1956) also pointed out the relevance of this type of limit theory in applications where x may be quite large relative to T.

The large deviation theorem for tail probabilities of standardized sums corresponding to the classical result (1) tells us that if $x > 0$ and $x = O(\sqrt{T})$ as $T \to \infty$, then (Petrov, 1968):

$$\frac{P(Z_T > X)}{1 - I(x)} = \exp\left\{\frac{x^3}{\sqrt{T}}\Psi\left(\frac{x}{\sqrt{T}}\right)\right\}\left(1 + O\left(\frac{x}{\sqrt{T}}\right)\right) \tag{2}$$

and

$$\frac{P(Z_T \leq -x)}{I(-x)} = \exp\left\{-\frac{x^3}{\sqrt{T}}\Psi\left(\frac{-x}{\sqrt{T}}\right)\right\}\left(1 + O\left(\frac{x}{\sqrt{T}}\right)\right), \tag{3}$$

where

$$\Psi(z) = \psi_0 + \psi_1 z + \psi_2 z^2 + \cdots$$

is a power series whose coefficients $\Psi_i (i = 0, 1, \ldots)$ depend on the cumulants of X_t and which converges in a neighborhood of $z = 0$.

Although this theorem is clearly stronger than (1), it also depends on the stronger condition that

$$E\{\exp(a|X_t|)\} < \infty \tag{4}$$

for some $a > 0$ so that the moment generating function of the underlying variates exists and is analytic in a strip of the imaginary axis. The main import of (2) and (3) is that the limiting tails of Z_T are normal only if x does not tend to infinity with T too fast [to be precise $x = o(T^{1/6})$]. For if x tends to infinity as fast or faster than a constant multiple of $T^{1/6}$, then the limiting tails of Z_T are not normal, but will depend on the coefficients in the power series $\Psi(z)$. Thus if $x > 0$ and $x = O(T^{1/4})$ as $T \to \infty$, it is easy to see that

$$P(Z_T > x) = (1 - I(x))\exp\left\{\psi_0 \frac{x^3}{\sqrt{T}} + \psi_1 \frac{x^4}{T}\right\}\left(1 + O\left(\frac{x}{\sqrt{T}}\right)\right) \tag{5}$$

and, more generally, if $x > 0$ and $x = O(T^{k/2(k+2)})$ for some positive integer k, then

$$P(Z_T > x) = (1 - I(x))\left\{\exp \frac{x^3}{\sqrt{T}}\Psi^{[k]}\left(\frac{x}{\sqrt{T}}\right)\right\}\left(1 + O\left(\frac{x}{\sqrt{T}}\right)\right) \tag{6}$$

where $\Psi^{[k]}(z)$ represents the first k terms of the series $\Psi(z)$. Similar results hold for the negative tail.

In practice, we are frequently concerned with approximating the tails of the distribution of a test statistic whose exact distribution is unknown. In such cases, where x may be quite large relative to $\sqrt{T}$, it is known that the Edgeworth approximation can lead to unsatisfactory results, including negative probabilities. An alternative which should be available in many cases is to use the first few terms in a large deviation expansion such as (5) or (6). Note that these expansions have the advantage that they are positive for all x (although not necessarily less than unity) and might be expected to do well at least for a certain region in the tails.

One limitation, however, to the immediate application in econometrics of large deviation limit theory and its associated expansions is the fact that virtually all the results available so far seem to have been established for standardized sums of independent random variables (or vectors). One exception is the theorem in Phillips (1977a) which gives a general result for large deviations of multivariate statistics which are more general than standardized means but which depend on the sample size T in much the same way as $T \uparrow \infty$. The present study goes further and deals with rather general functions of such multivariate statistics. The results should be sufficiently general to be of wide applicability in regression and time series settings. In view of the generality of the problem our approach here will involve some sacrifice of mathematical rigor. In particular, we shall concentrate on obtaining the final large deviation formulas by formal methods of asymptotic expansion. We shall relate these formulas to known results on Edgeworth expansions in this general case. The formulas are then applied to the coefficient estimator in an AR(1).

II. THE LARGE DEVIATION EXPANSION

This section is concerned with deriving general formulas for large deviation expansions such as (5) and (6) when we are interested in approximating the tails of the sampling distribution of statistics which can be represented as quite general functions of the first and second sample moments of the data. Thus our starting point will be the same as that in earlier work by Chambers (1967), Sargan (1975, 1976), Phillips (1977b), and Sargan and Satchell (1986). The formulas we derive should apply in rather a wide range of different models, including those where lagged endogenous variables occur or where there are nonnormally distributed errors.

Our notation here is based on that of Phillips (1977b). In particular, we let q denote an m-vector of primitive variates which will usually comprise suitably standardized errors in the sample moments of the data. Our main focus of attention will be the function $e_T(q)$ of these primitive variates. Most commonly, $e_T(q)$ will represent the error of an estimator, say $\hat{\alpha} - \alpha$, where the estimator $\hat{\alpha} = \hat{\alpha}(q)$ is a function of the primitive variate q. We introduce the standardized statistic $t = \sqrt{T} e_T(q)$, which, under conventional conditions, will converge weakly to a normal variate. The statistic $t = t(q)$ may be a suitably standardized and centered estimator such as $t = \sqrt{T}(\hat{\alpha} - \alpha)$ or a t-ratio type test statistic. Our main requirement is the following:

ASSUMPTION 2.1: (a) *The error function $e_T(\ )$ satisfies $e_T(0) = 0$, is analytic in a neighborhood of the origin in q space, has derivatives which are uniformly bounded in this neighborhood as $T \uparrow \infty$, and if $e^0 = \partial e_T(0)/\partial q$ then the Euclidean length $\|e^0\|$ is bounded above zero as $T \uparrow \infty$.*

(b) *The mean vector of q is zero and all higher order cumulants of Tq exist and are of $O(T)$ as $T \uparrow \infty$. If $V_T(x)$ is the distribution function of Tq, then there exist positive numbers A, l_T, and L_T such that*

$$l_T \leq \left| \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{z'x}\, dV_T(x) \right| \leq L_T \tag{7}$$

for all z in the sphere $(z^*z)^{1/2} < A$, where z^* represents the complex conjugate transpose of z.

Condition (a) mirrors similar conditions in Sargan (1976), Phillips (1977b), and Bhattacharya and Ghosh (1978). But it goes further in requiring $e_T(\ )$ to be analytic rather than just continuously differentiable to a certain order. The first part of Condition (b) also corresponds with assumptions in Sargan (1976) and Phillips (1977b). The second part of (b) goes further and implies that the moment generating function of Tq is analytic in the sphere $(z^*z)^{1/2} < A$ (cf. Lukacs, 1970). In the simplest case, where Tq represents the sum $X_1 + \cdots + X_T$ of the independent and identically distributed random quantities X_t, we observe the close link between (7) and (4). In particular, we have from (4):

$$E\{\exp(a|X_t|)\} \leq L < \infty;$$

and thus

$$\left| \int_{-\infty}^{\infty} e^{z'x}\, dV(x) \right| = |[E\{\exp(zX_t)\}]^T| \leq L^T$$

when $|z| < a$. The lower bound follows in this case since $E\{\exp(zX_t)\}$ is continuous and equal to unity at the origin.

We now let $F(\bar{q})$ be the distribution of $\bar{q} = \sqrt{T}q$. We have the following Taylor representation of $t(q)$ (we use the tensor summation convention and write derivatives of $e_T(\ )$ at the origin as, for instance, $e_{jk} = \partial^2 e_T(0)/\partial q_j\, \partial q_k$):

$$t = \sqrt{T}\left\{ e_k q_k + \sum_{j=2}^{\infty} \frac{1}{j!}\left(q_k \frac{\partial}{\partial q_k}\right)^j e_T(0)\right\}$$

$$= e_k \bar{q}_k + \sum_{j=2}^{\infty} \frac{1}{j!}\left(\frac{1}{\sqrt{T}}\right)^{j-1}\left(\bar{q}_k \frac{\partial}{\partial q_k}\right)^j e_T(0).$$

The characteristic function of t is given by

$$\psi(s) = \int e^{ist}\, dF(\bar{q})$$

$$= \int e^{ise_k\bar{q}_k} \exp\left\{ is \sum_{j=2}^{\infty} \frac{1}{j!}\left(\frac{1}{\sqrt{T}}\right)^{j-1}\left(\bar{q}_k \frac{\partial}{\partial q_k}\right)^j e_T(0)\right\} dF(\bar{q}), \tag{8}$$

where the integration is over the entire $\bar{q}$ space. The representation (8) is formal, but is sufficient for the asymptotic formulas that follow. In fact, under Assumption 2.1 t is not necessarily an entire function, and the power series representation used in (8) is valid only in a fixed neighborhood of the origin in q space. However, as $T \uparrow \infty$, the probability mass is confined to this neighborhood with a probability that approaches unity, and the error involved in the representation (8) may be neglected. A rigorous analysis may be conducted as in the author's related paper (1977a).

We now write the characteristic function of $\bar{q}$ as

$$\theta(z) = \int e^{iz'\bar{q}}\, dF(\bar{q}).$$

Taking the principal branch of the logarithm, we define the second characteristic (or cumulant generating function) of $\bar{q}$ as

$$\lambda(z) = \log[\theta(z)].$$

We introduce the notation

$$\bar{\psi}(is) = \psi(s), \qquad \bar{\theta}(iz) = \theta(z), \qquad \bar{\lambda}(iz) = \lambda(z),$$

and then from (8) we have

$$\bar{\psi}(w) = \int e^{we_k\bar{q}_k} \exp\left\{ w \sum_{j=2}^{\infty} \frac{1}{j!}\left(\frac{1}{\sqrt{T}}\right)^{j-1}\left(\bar{q}_k \frac{\partial}{\partial q_k}\right)^j e_T(0)\right\} dF(\bar{q}).$$

Noting that

$$\exp\left\{ w \sum_{j=2}^{\infty} \frac{1}{j!}\left(\frac{1}{\sqrt{T}}\right)^{j-1}\left(\bar{q}_k \frac{\partial}{\partial q_k}\right)^j e_T(0)\right\}$$

$$= \left[\exp\left\{ w \sum_{j=2}^{\infty} \frac{1}{j!}\left(\frac{1}{\sqrt{T}}\right)^{j-1}\left(\frac{\partial}{\partial \zeta_k}\frac{\partial}{\partial q_k}\right)^j e_T(0)\right\} e^{\zeta'\bar{q}}\right]_{\zeta=0},$$

we deduce the representation

$$\bar{\psi}(w) = \int \left[\exp\left\{ w \sum_{j=2}^{\infty} \frac{1}{j!}\left(\frac{1}{\sqrt{T}}\right)^{j-1} \left(\frac{\partial}{\partial \zeta_k} \frac{\partial}{\partial q_k}\right)^j e_T(0) \right\} e^{\zeta' \bar{q}} \right]_{\zeta=0} e^{w e_k \bar{q}_k} \, dF(\bar{q})$$

$$= \int \left[\exp\left\{ w \sum_{j=2}^{\infty} \frac{1}{j!}\left(\frac{1}{\sqrt{T}}\right)^{j-1} \left(\frac{\partial}{\partial \zeta_k} \frac{\partial}{\partial q_k}\right)^j e_T(0) \right\} e^{\zeta' \bar{q}} \right]_{\zeta=we^0} dF(\bar{q})$$

$$= \left[\exp\left\{ w \sum_{j=2}^{\infty} \frac{1}{j!}\left(\frac{1}{\sqrt{T}}\right)^{j-1} \left(\frac{\partial}{\partial \zeta_k} \frac{\partial}{\partial q_k}\right)^j e_T(0) \right\} \int e^{\zeta' \bar{q}} \, dF(\bar{q}) \right]_{\zeta=we^0}$$

$$= \left[\exp\left\{ w \sum_{j=2}^{\infty} \frac{1}{j!}\left(\frac{1}{\sqrt{T}}\right)^{j-1} \left(\frac{\partial}{\partial \zeta_k} \frac{\partial}{\partial q_k}\right)^j e_T(0) \exp[\bar{\lambda}(\zeta)] \right\} \right]_{\zeta=we^0}. \qquad (9)$$

Note that the argument leading to (9) involves the removal of both the operator

$$\exp\left\{ w \sum_{j=2}^{\infty} \frac{1}{j!}\left(\frac{1}{\sqrt{T}}\right)^{j-1} \left(\frac{\partial}{\partial \zeta_k} \frac{\partial}{\partial q_k}\right)^j e_T(0) \right\}$$

and the evaluation at $\zeta = we^0$ to the outside of the integral. These steps are justified by (7), which ensures the uniform convergence of the derived integrand in an open ball of the origin.

Equation (9) is a very general formula from which most existing results on asymptotic expansions may be derived. Since it involves the Taylor representation of the function $t = t(q)$, (9) is, in fact, an asymptotic formula. We shall use it to extract a large deviation expansion of the density and distribution function of t.

Assuming that the density of t exists and is given by $p_T(x)$, we have by inversion

$$p_T(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-isx} \psi(s) \, ds$$

$$= \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} e^{-wx} \bar{\psi}(w) \, dw,$$

and the path of integration in the last integral is along the imaginary axis. We now replace w in the above by $\sqrt{T}u$ so that

$$p_T(x) = \frac{\sqrt{T}}{2\pi i} \int_{-i\infty}^{i\infty} e^{-\sqrt{T}ux} \bar{\psi}(\sqrt{T}u) \, du. \qquad (10)$$

Returning to (9), it is now convenient to expand $\bar{\lambda}(\zeta)$ in Taylor series about the value $\zeta = \sqrt{T}ue^0$ as

$$\begin{aligned} \bar{\lambda}(\zeta) &= \bar{\lambda}(\sqrt{T}ue^0) + \bar{\lambda}_a(\sqrt{T}ue^0)(\zeta_a - \sqrt{T}ue_a) \\ &\quad + \tfrac{1}{2}\bar{\lambda}_{ab}(\sqrt{T}ue^0)(\zeta_a - \sqrt{T}ue_a)(\zeta_b - \sqrt{T}ue_b) + \cdots. \end{aligned} \qquad (11)$$

We note that under Assumption 2.1 $\bar{\lambda}(\zeta)$ has an analytic continuation to strips in the space of complex ζ for which $\|\mathrm{Re}(\zeta)\| = O(\sqrt{T})$. The expansion above then takes place within this region since, as we shall see later, u is selected so that $\mathrm{Re}(u) = o(1)$ as $T\uparrow\infty$. Later on, we also require the Taylor expansions of $\bar{\lambda}(\sqrt{T}ue^0)$, $\bar{\lambda}_a(\sqrt{T}ue^0)$, and $\bar{\lambda}_{ab}(\sqrt{T}ue^0)$ about the origin. To obtain these we note that (writing derivatives at the origin, as for instance $\lambda_{jk} = \partial^2\lambda(0)/\partial z_j\,\partial z_k$)

$$\lambda(z) = \tfrac{1}{2}\lambda_{jk}z_jz_k + \tfrac{1}{6}\lambda_{jkl}z_jz_kz_l + \tfrac{1}{24}\lambda_{jklm}z_jz_kz_lz_m + \cdots$$

so that

$$\bar{\lambda}(\zeta) = -\frac{1}{2}\lambda_{jk}\zeta_j\zeta_k + \frac{i}{6}\lambda_{jkl}\zeta_j\zeta_k\zeta_l + \frac{1}{24}\lambda_{jklm}\zeta_j\zeta_k\zeta_l\zeta_m + \cdots$$

and hence, from the order of magnitude of the cumulants of $\bar{q}$,

$$\bar{\lambda}(\sqrt{T}ue^0) = T\left\{-\frac{u^2}{2}\lambda_{jk}e_je_k + \frac{u^3}{6}iT^{1/2}\lambda_{jkl}e_je_ke_l + \frac{u^4}{24}T\lambda_{jklm}e_je_ke_le_m + O(u^5)\right\}$$

$$\bar{\lambda}_a(\sqrt{T}ue^0) = \sqrt{T}\left\{-u\lambda_{aj}e_j + \frac{u^2}{2}iT^{1/2}\lambda_{ajk}e_je_k + O(u^3)\right\}$$

$$\bar{\lambda}_{ab}(\sqrt{T}ue^0) = -\lambda_{ab} + O(u).$$

We now have

$$\begin{aligned}\bar{\psi}(\sqrt{T}u) = \Bigg[&\exp\left\{\sqrt{T}u\sum_{j=2}^{\infty}\frac{1}{j!}\left(\frac{1}{\sqrt{T}}\right)^{j-1}\left(\frac{\partial}{\partial\zeta_k}\frac{\partial}{\partial q_k}\right)^j e_T(0)\right\}\\ &\times\exp\Big\{\bar{\lambda}(\sqrt{T}ue^0) + \bar{\lambda}_a(\sqrt{T}ue^0)(\zeta_a - \sqrt{T}ue_a)\\ &+\frac{1}{2}\bar{\lambda}_{ab}(\sqrt{T}ue^0)(\zeta_a - \sqrt{T}ue_a)(\zeta_b - \sqrt{T}ue_b) + \cdots\Big\}\Bigg]_{\zeta=\sqrt{T}ue^0}\\ &\times\exp\{\bar{\lambda}(\sqrt{T}ue^0)\}\Bigg[\exp\left\{\sqrt{T}u\sum_{j=2}^{\infty}\frac{1}{j!}\left(\frac{1}{\sqrt{T}}\right)^{j-1}\left(\frac{\partial}{\partial\zeta_k}\frac{\partial}{\partial q_k}\right)^j e_T(0)\right\}\\ &\times\exp\Big\{\bar{\lambda}_a(\sqrt{T}ue^0)(\zeta_a - \sqrt{T}ue_a)\\ &+\frac{1}{2}\bar{\lambda}_{ab}(\sqrt{T}ue^0)(\zeta_a - \sqrt{T}ue_a)(\zeta_b - \sqrt{T}ue_b) + \cdots\Big\}\Bigg]_{\zeta=\sqrt{T}ue^0}\end{aligned}$$

$$= \exp\{\bar{\lambda}(\sqrt{T}ue^0)\}\Bigg[\exp\Bigg\{\frac{u}{2!}\Bigg(e_{kl}\frac{\partial^2}{\partial\zeta_k\,\partial\zeta_l}\Bigg)$$

$$+\frac{u}{3!}\Bigg(\frac{1}{\sqrt{T}}\Bigg)\Bigg(e_{klm}\frac{\partial^3}{\partial\zeta_k\,\partial\zeta_l\,\partial\zeta_m}\Bigg)+O\Bigg(\frac{u}{T}\Bigg)\Bigg\}$$

$$\times\exp\Bigg\{\bar{\lambda}_a(\sqrt{T}ue^0)(\zeta_a-\sqrt{T}ue_a)$$

$$+\frac{1}{2}\bar{\lambda}_{ab}(\sqrt{T}ue^0)(\zeta_a-\sqrt{T}ue_a)(\zeta_b-\sqrt{T}ue_b)+\cdots\Bigg\}\Bigg]_{\zeta=\sqrt{T}u}. \quad (12)$$

To simplify the above, at least to obtain the first two terms in the large deviation expansion, we can use the following rules for manipulating differential operators. If $D = d/dx$, $F(D)$ denotes a polynomial in D, and $W(x)$ is an analytic function of x, we have

(i) $\quad F(D)\{e^{ax}W(x)\} = e^{ax}F(a+D)W(x)$

and

(ii) $$\exp(bD)\{e^{ax^2}W(x)\} = e^{a(x+b)^2}\exp(bD)\{W(x)\} = e^{a(x+b)^2}W(x+b).$$

Rule (i) is well known [see, for example, Piaggio (1962)], and rule (ii) is a direct consequence of the fact that $\exp(bD)$ is the Taylor series expansion operator.

From (12) we obtain by using rule (i) and noting the order of magnitude of the derivatives $\bar{\lambda}_a(\sqrt{T}ue^0)$:

$$\bar{\psi}(\sqrt{T}u) = \exp\{\bar{\lambda}(\sqrt{T}ue^0)\}\Bigg[\exp\Bigg\{\frac{u}{2}e_{kl}\Bigg(\bar{\lambda}_k(\sqrt{T}ue^0)+\frac{\partial}{\partial\zeta_k}\Bigg)$$

$$\times\Bigg(\bar{\lambda}_l(\sqrt{T}ue^0)+\frac{\partial}{\partial\zeta_l}\Bigg)$$

$$+\frac{u}{6}\Bigg(\frac{1}{\sqrt{T}}\Bigg)e_{klm}\Bigg(\bar{\lambda}_k(\sqrt{T}ue^0)+\frac{\partial}{\partial\zeta_k}\Bigg)\Bigg(\lambda_l(\sqrt{T}ue^0)+\frac{\partial}{\partial\zeta_l}\Bigg)$$

$$\times\Bigg(\lambda_m(\sqrt{T}ue^0)+\frac{\partial}{\partial\zeta_m}\Bigg)+O(u^5T)\Bigg\}$$

$$\times\exp\Bigg\{\frac{1}{2}\bar{\lambda}_{ab}(\sqrt{T}ue^0)(\zeta_a-\sqrt{T}ue_a)(\zeta_b-\sqrt{Tu}e_b)+\cdots\Bigg\}\Bigg]_{\zeta=\sqrt{T}ue^0}$$

$$= \exp\left\{\bar{\lambda}(\sqrt{T}ue^0) + \frac{u}{2}e_{kl}\bar{\lambda}_k(\sqrt{T}ue^0)\bar{\lambda}_l(\sqrt{T}ue^0)\right.$$

$$\left. + \frac{u}{6}\left(\frac{1}{\sqrt{T}}\right)e_{klm}\bar{\lambda}_k(\sqrt{T}ue^0)\bar{\lambda}_l(\sqrt{T}ue^0)\bar{\lambda}_m(\sqrt{T}ue^0) + O(u^5T)\right\}$$

$$\times \left[\exp\left\{ue_{kl}\bar{\lambda}_k(\sqrt{T}ue^0)\frac{\partial}{\partial\zeta_l} + \frac{u}{2}e_{kl}\frac{\partial^2}{\partial\zeta_k\,\partial\zeta_l} + O(u^3T^{1/2})\right\}\right.$$

$$\left.\times \exp\left\{\frac{1}{2}\bar{\lambda}_{ab}(\sqrt{T}ue^0)(\zeta_a - \sqrt{T}ue_a)(\zeta_b - \sqrt{T}ue_b) + \cdots\right\}\right]_{\zeta=\sqrt{T}ue^0}.$$

Using rule (ii) and introducing the vector $\nu = \zeta - \sqrt{T}ue^0$, we find that

$$\bar{\lambda}(\sqrt{T}u) = \exp\left\{\bar{\lambda}(\sqrt{T}ue^0) + \frac{u}{2}e_{kl}\bar{\lambda}_k(\sqrt{T}ue^0)\bar{\lambda}_l(\sqrt{T}ue^0)\right.$$

$$\left. + \frac{u}{6}\left(\frac{1}{\sqrt{T}}\right)e_{klm}\bar{\lambda}_k(\sqrt{T}ue^0)\bar{\lambda}_l(\sqrt{T}ue^0)\bar{\lambda}_m(\sqrt{T}ue^0) + O(u^5T)\right\}$$

$$\times \left[\exp\left\{\frac{u}{2}e_{kl}\frac{\partial^2}{\partial\nu_k\,\partial\nu_l} + O(u^3T^{1/2})\right\}\right.$$

$$\times \exp\left\{\frac{1}{2}\bar{\lambda}_{ab}(\sqrt{T}ue^0)\nu_a\nu_b + ue_{kl}\bar{\lambda}_k(\sqrt{T}ue^0)\bar{\lambda}_{la}(\sqrt{T}ue^0)\right.$$

$$+ \frac{1}{2}u^2e_{kl}e_{mn}\bar{\lambda}_k(\sqrt{T}ue^0)\bar{\lambda}_m(\sqrt{T}ue^0)\bar{\lambda}_m(\sqrt{T}ue^0)\bar{\lambda}_{ln}(\sqrt{T}ue^0)$$

$$\left.\left. + ue_{kl}\bar{\lambda}_k(\sqrt{T}ue^0)\frac{\partial}{\partial\nu_l}\right\} \times \exp\left\{\frac{1}{6}\bar{\lambda}_{abc}(\sqrt{T}ue^0)\nu_a\nu_b\nu_c + \cdots\right\}\right]_{\nu=0}$$

$$= \exp\left\{\bar{\lambda}(\sqrt{T}ue^0) + \frac{u}{2}e_{kl}\bar{\lambda}_k(\sqrt{T}ue^0)\bar{\lambda}_l(\sqrt{T}ue^0)\right.$$

$$+ \frac{u}{6}\left(\frac{1}{\sqrt{T}}\right)e_{klm}\bar{\lambda}_k(\sqrt{T}ue^0)\bar{\lambda}_l(\sqrt{T}ue^0)\bar{\lambda}_m(\sqrt{T}ue^0)$$

$$\left. + \frac{u^2}{2}e_{kl}e_{mn}\bar{\lambda}_k(\sqrt{T}ue^0)\bar{\lambda}_m(\sqrt{T}ue^0)\bar{\lambda}_{ln}(\sqrt{T}ue^0) + O(u^5T)\right\}$$

$$\times \left[\exp\left\{\frac{u}{2}e_{kl}\frac{\partial^2}{\partial\nu_k\,\partial\nu_l} + O(u^3T^{1/2})\right\}\right.$$

$$\times \exp\left\{\frac{1}{2}\bar{\lambda}_{ab}(\sqrt{T}ue^0)\nu_a\nu_b + ue_{kl}\bar{\lambda}_k(\sqrt{T}ue^0)\bar{\lambda}_{la}(\sqrt{T}ue^0)\nu_a\right\}$$

$$\times \exp\left\{ u e_{kl} \bar{\lambda}_k(\sqrt{T}ue^0) \frac{\partial}{\partial v_l} \right\}$$

$$\times \exp\left\{ \frac{1}{6} \bar{\lambda}_{abc}(\sqrt{T}ue^0) v_a v_b v_c + \cdots \right\} \Bigg]_{v=0}.$$

Expanding the exponential operators in the square brackets above and differentiating term by term, we obtain:

$$\bar{\psi}(\sqrt{T}u) = \exp\{G_T(u)\} J_T(u), \tag{13}$$

where

$$G_T(u) = \bar{\lambda}(\sqrt{T}ue^0) + \frac{u}{2} e_{ab} \bar{\lambda}_a(\sqrt{T}ue^0) \bar{\lambda}_b(\sqrt{T}ue^0)$$
$$+ \frac{u}{6}\left(\frac{1}{\sqrt{T}}\right) e_{abc} \bar{\lambda}_a(\sqrt{T}ue^0) \bar{\lambda}_b(\sqrt{T}ue^0) \bar{\lambda}_c(\sqrt{T}ue^0)$$
$$+ \frac{u^2}{2} e_{ab} e_{cd} \bar{\lambda}_a(\sqrt{T}ue^0) \bar{\lambda}_b(\sqrt{T}ue^0) \bar{\lambda}_{bd}(\sqrt{T}ue^0) + O(u^5 T)$$

and

$$J_T(u) = 1 + \frac{u}{2} e_{ab} \bar{\lambda}_{ab}(\sqrt{T}ue^0) + O(u^2) + O(u^5 T).$$

From (10) and (13) we now have

$$p_T(x) = \frac{\sqrt{T}}{2\pi i} \int_{-i\infty}^{i\infty} \exp\left[T\left\{ -u\tau + \frac{1}{T} G_T(u) \right\} \right] J_T(u)\, du, \tag{14}$$

where $\tau = x/\sqrt{T}$. The next step is to deform the path of integration in the integral (14) so that it is the line of steepest descent [see, for example, Copson (1965)] through the saddlepoint u^0. The saddlepoint u^0 is the solution of the equation

$$\tau = \frac{1}{T} \frac{\partial G_T(u)}{\partial u}$$

or

$$\tau = \frac{1}{T} \frac{\partial \bar{\lambda}(\sqrt{T}ue^0)}{\partial u} + \frac{e_{ab}}{2} \left(\frac{\bar{\lambda}_a(\sqrt{T}ue^0)}{\sqrt{T}} \right) \left(\frac{\bar{\lambda}_b(\sqrt{T}ue^0)}{\sqrt{T}} \right)$$
$$+ \frac{e_{abc}}{6} \left(\frac{\bar{\lambda}_a(\sqrt{T}ue^0)}{\sqrt{T}} \right) \left(\frac{\bar{\lambda}_b(\sqrt{T}ue^0)}{\sqrt{T}} \right) \left(\frac{\bar{\lambda}_c(\sqrt{T}ue^0)}{\sqrt{T}} \right)$$
$$+ u e_{ab} e_{cd} \left(\frac{\bar{\lambda}_a(\sqrt{T}ue^0)}{\sqrt{T}} \right) \left(\frac{\bar{\lambda}_c(\sqrt{T}ue^0)}{\sqrt{T}} \right) \bar{\lambda}_{bd}(\sqrt{T}ue^0)$$

$$+ ue_{ab}\left(\frac{\bar{\lambda}_a(\sqrt{T}ue^0)}{\sqrt{T}}\right)\left(\frac{1}{\sqrt{T}}\frac{\partial\bar{\lambda}_b(\sqrt{T}ue^0)}{\partial u}\right)$$

$$+\frac{u}{2}e_{abc}\left(\frac{\bar{\lambda}_a(\sqrt{T}ue^0)}{\sqrt{T}}\right)\left(\frac{\bar{\lambda}_b(\sqrt{T}ue^0)}{\sqrt{T}}\right)\left(\frac{1}{\sqrt{T}}\frac{\partial\bar{\lambda}_c(\sqrt{T}ue^0)}{\partial u}\right)$$

$$+ u^2 e_{ab}e_{cd}\left(\frac{\bar{\lambda}_a(\sqrt{T}ue^0)}{\sqrt{T}}\right)\left(\frac{1}{\sqrt{T}}\frac{\partial\bar{\lambda}_c(\sqrt{T}ue^0)}{\partial u}\right)\bar{\lambda}_{bd}(\sqrt{T}ue^0) + O(u^4). \tag{15}$$

But

$$\frac{1}{T}\frac{\partial\bar{\lambda}(\sqrt{T}ue^0)}{\partial u} = -u\lambda_{jk}e_je_k + \frac{u^2}{2}iT^{1/2}\lambda_{jkl}e_je_ke_l + \frac{u^3}{6}T\lambda_{jklm}e_je_ke_le_m + O(u^4),$$

and

$$\frac{1}{T}\frac{\partial\bar{\lambda}_a(\sqrt{T}ue^0)}{\partial u} = -\lambda_{aj}e_j + uiT^{1/2}\lambda_{ajk}e_je_k + \frac{u^2}{2}T\lambda_{ajkl}e_je_ke_l + O(u^3),$$

so that (15) becomes, after collecting terms,

$$\tau = \gamma_1 u + \gamma_2 u^2 + \gamma_3 u^3 + O(u^4), \tag{16}$$

where

$$\gamma_1 = -\lambda_{jk}e_je_k,$$

$$\gamma_2 = \frac{iT^{1/2}}{2}\lambda_{jkl}e_je_ke_l + \frac{3}{2}(e_j\lambda_{ja})e_{ab}(\lambda_{bk}e_k),$$

and

$$\gamma_3 = \frac{T}{6}\lambda_{jklm}e_je_ke_le_m - 2e_{ab}(iT^{1/2}\lambda_{ajk}e_je_k)(\lambda_{bl}e_l)$$

$$-\frac{2}{3}e_{abc}(\lambda_{aj}e_j)(\lambda_{bk}e_k)(\lambda_{cl}e_l)$$

$$-2e_{ab}e_{cd}(\lambda_{aj}e_j)(\lambda_{ck}e_k)\lambda_{bd}.$$

When $x = o(\sqrt{T})$ as $T \to \infty$, $\tau = o(1)$ and will be small for large T so that (16) can be inverted [see, for example, Knopp (1956)] to give the position of the saddlepoint u^0 as a power series in τ; namely,

$$u^0 = \frac{\tau}{\gamma_1} - \frac{\gamma_2}{\gamma_1^3}\tau^2 + \left(\frac{2\gamma_2^2 - \gamma_1\gamma_3}{\gamma_1^5}\right)\tau^3 + O(\tau^4). \tag{17}$$

Returning to (14), we now write the exponent of the integrand as

$$H_T(u) = T\left\{-u\tau + \frac{1}{T}G_T(u)\right\},$$

and from the expansions of $\bar{\lambda}(\ )$ and its derivatives we have

$$H_T(u) = T\Big[-u\tau - \frac{1}{2}u^2\lambda_{jk}e_je_k + u^3\Big\{\frac{iT^{1/2}}{6}\lambda_{jkl}e_je_ke_l + \frac{e_{ab}}{2}(e_j\lambda_{ja})(\lambda_{bk}e_k)\Big\}$$

$$+ u^4\Big\{\frac{T}{24}\lambda_{jklm}e_je_ke_le_m - \frac{e_{ab}}{2}(iT^{1/2}\lambda_{ajk}e_je_k)(\lambda_{bl}e_l) \tag{18}$$

$$- \frac{e_{abc}}{6}(\lambda_{aj}e_j)(\lambda_{bk}e_k)(\lambda_{cl}e_l) - \frac{e_{ab}e_{cd}}{2}(\lambda_{aj}e_j)(\lambda_{ck}e_k)\lambda_{bd}\Big\} + O(u^5)\Big]$$

$$= T[-u\tau + \tfrac{1}{2}u^2\gamma_1 + u^3\eta_3 + u^4\eta_4 + O(u^5)], \text{ say.}$$

In place of (14) we now employ the equivalent inversion formula (Widder, 1946):

$$p_T(x) = \frac{\sqrt{T}}{2\pi i}\int_{u^0-i\infty}^{u^0+i\infty} \exp\{H_T(u)\}J_T(u)\,du.$$

We set $u = u^0 + iy$ and then, on the contour near u^0, we have the expansions:

$$H_T(u) = H_T(u^0) - \tfrac{1}{2}H_T^{(2)}(u^0)y^2 - \tfrac{1}{6}H_T^{(3)}(u^0)iy^3 + \tfrac{1}{24}H_T^{(4)}(u^0)y^4 + \cdots$$

and

$$J_T(u) = J_T(u^0) + J_T^{(1)}(u^0)iy - \tfrac{1}{2}J_T^{(2)}(u^0)y^2 - \tfrac{1}{6}J_T^{(3)}(u^0)iy^3 + \tfrac{1}{24}J_T^{(4)}(u^0)y^4 + \cdots.$$

Hence we have

$$p_T(x) = \frac{\sqrt{T}}{2\pi}\int_{-\infty}^{\infty} \exp\{H_T(u^0 + iy)\}J_T(u^0 + iy)\,dy$$

$$= \frac{\sqrt{T}}{2\pi}\exp\{H_T(u^0)\}\int_{-\infty}^{\infty}\exp\Big\{-\frac{1}{2}H_T^{(2)}(u^0)y^2$$

$$-\frac{1}{6}H_T^{(3)}(u^0)iy^3 + \frac{1}{24}H_T^{(4)}(u^0)y^4 + \cdots\Big\}$$

$$\times\Big[J_T(u^0) + J_T^{(1)}(u^0)iy - \frac{1}{2}J_T^{(2)}(u^0)y^2 + \cdots\Big]\,dy,$$

and, setting $v = \{H_T^{(2)}(u^0)\}^{1/2}y$, this can be written

$$p_T(x) = \frac{1}{2\pi}\Big\{\frac{T}{H_T^{(2)}(u^0)}\Big\}^{1/2}\exp\{H_T(u^0)\}$$

$$\times\int_{-\infty}^{\infty}\exp\Big\{\frac{-v^2}{2} - \frac{1}{6}L_3(u^0)iv^3 + \frac{1}{24}L_4(u^0)v^4 + \cdots\Big\}$$

$$\times\Big[J_T(u^0) + M_1(u^0)iv + \frac{1}{2}M_2(u^0)v^2 + \cdots\Big]\,dv, \tag{19}$$

where

$$L_j(u^0) = H_T^{(j)}(u^0)/\{H_T^{(2)}(u^0)\}^{j/2}, \qquad j \geq 3,$$
$$M_j(u^0) = J_T^{(j)}(u^0)/\{H_T^{(2)}(u^0)\}^{j/2}, \qquad j \geq 1.$$

From (18) we note that

$$H_T^{(2)}(u^0) = T\{\gamma_1 + 6u^0\eta_3 + 12(u^0)^2\eta_4 + O[(u^0)^3]\},$$
$$H_T^{(3)}(u^0) = T\{6\eta_3 + 24u^0\eta_4 + O[(u^0)^2]\},$$
$$H_T^{(4)}(u^0) = T\{24\eta_4 + O(u^0)\},$$

so that, since γ_1, η_3, and η_4 are all of $O(1)$ as $T\uparrow\infty$,

$$L_j(u^0) = O(T^{-j/2+1}).$$

Expanding part of the integrand in (19) we get

$$\begin{aligned}
p_T(x) &= \frac{1}{2\pi}\left\{\frac{T}{H_T^{(2)}(u^0)}\right\}^{1/2} \exp\{H_T(u^0)\} \int_{-\infty}^{\infty} e^{-v^2/2} \\
&\quad \times \left\{1 - \frac{1}{6}L_3(u^0)iv^3 + \left[\frac{1}{24}L_4(u^0)v^4 - \frac{1}{72}(L_3(u^0))^2v^6\right] + \cdots\right\} \\
&\quad \times \left\{J_T(u^0) + M_1(u^0)iv + \frac{1}{2}M_2(u^0)v^2 + \cdots\right\} dv \\
&= \left\{\frac{T}{2\pi H_T^{(2)}(u^0)}\right\}^{1/2} \exp\{H_T(u^0)\} \\
&\quad \times \left[J_T(u^0) + J_T(u^0)\left\{\frac{1}{8}L_4(u^0) - \frac{5}{24}(L_3(u^0))^2\right\}\right. \\
&\quad \left. + \frac{1}{2}M_1(u^0)L_3(u^0) + \frac{1}{2}M_2(u^0) + o(T^{-1})\right].
\end{aligned}$$

Since $J_T(u^0) = O(1)$ and is nonzero as $T\uparrow\infty$, we can write the above as:

$$\begin{aligned}
p_T(x) &= \left\{\frac{T}{2\pi H_T^{(2)}(u^0)}\right\}^{1/2} \exp\{H_T(u^0)\}J_T(u^0) \\
&\quad \times \left[1 + \left\{\frac{1}{8}L_4(u^0) - \frac{5}{24}(L_3(u^0))^2\right\}\right. \\
&\quad \left. + \frac{1}{2}\frac{M_1(u^0)L_3(u^0)}{J_T(u^0)} + \frac{1}{2}\frac{M_2(u^0)}{J_T(u^0)} + o(T^{-1})\right].
\end{aligned} \tag{20}$$

The first factor in the above can be regarded as the saddlepoint approximation [see Daniels (1954, 1956)].

However, it is possible to find u^0, and hence $H_T(u^0)$ and $J_T(u^0)$, only in very special cases. For this reason it is not possible to find the saddlepoint approximation analytically in a general case such as this. An obvious alternative is to use the first few terms of the expansion (17) of u^0 in powers of $\tau = x/\sqrt{T}$ to develop a corresponding expansion of $H_T(u^0)$ and $J_T(u^0)$. As pointed out in the introduction, this expansion should accurately describe at least a region of the limiting tails of $p_T(x)$.

From (17) and (18) we obtain

$$H_T(u^0) = T\Bigg[-\left\{\frac{\tau^2}{\gamma_1} - \frac{\gamma_2}{\gamma_1^3}\tau^3 + \left(\frac{2\gamma_2^2 - \gamma_1\gamma_3}{\gamma_1^5}\right)\tau^4\right\} + \left\{\frac{\tau^2}{\gamma_1^2} - \frac{2\gamma_2}{\gamma_1^4}\tau^3 + \left(\frac{5\gamma_2^2 - 2\gamma_1\gamma_3}{\gamma_1^6}\right)\tau^4\right\}\frac{\gamma_1}{2} + \left\{\frac{\tau^3}{\gamma_1^3} - \frac{3\gamma_2}{\gamma_1^5}\tau^4\right\}\eta_3 + \frac{\tau^4}{\gamma_1^4}\eta_4 + O(\tau^5)\Bigg], \tag{21}$$

and noting that

$$\bar{\lambda}_{ab}(\sqrt{T}ue^0) = -\lambda_{ab} + uiT^{1/2}\lambda_{abj}e_j + O(u^2)$$

we deduce from (17) and the definition of $J_T(u)$ the expansion

$$J_T(u^0) = 1 - \frac{1}{2}\frac{\tau}{\gamma_1}e_{ab}\lambda_{ab} + O(\tau^2) + O(\tau^5 T).$$

We now use the following notation which corresponds to that in Phillips (1977c):

$$\alpha_1 = \lambda_{jkl}e_je_ke_l, \qquad \alpha_2 = \lambda_{jklm}e_je_ke_le_m,$$
$$\alpha_3 = \sigma_a e_{ab}\sigma_b, \qquad \alpha_4 = \lambda_{ab}e_{ab},$$
$$\alpha_5 = \delta_{ab}e_{ab}, \qquad \alpha_6 = e_{abc}\sigma_a\sigma_b\sigma_c,$$
$$\alpha_7 = e_{abc}\lambda_{ab}\sigma_c, \qquad \alpha_8 = e_{ab}e_{cd}\sigma_a\sigma_c\lambda_{bd},$$
$$\alpha_9 = e_{ab}\lambda_{bc}e_{cd}\lambda_{da}, \qquad \alpha_{10} = \sigma_a e_{ab}\beta_b,$$

where

$$\omega^2 = -\lambda_{jk}e_je_k, \qquad \sigma_a = \lambda_{ak}e_k,$$
$$\beta_a = \lambda_{ajk}e_je_k, \qquad \delta_{ab} = \lambda_{abk}e_k,$$

so that, in terms of the above, we have

$$\gamma_1 = \omega^2, \qquad \gamma_2 = \frac{iT^{1/2}\alpha_1 + 3\alpha_3}{2},$$

$$\eta_3 = \frac{iT^{1/2}\alpha_1 + 3\alpha_3}{6} \qquad \eta_4 = \frac{T}{24}\alpha_2 - \frac{iT^{1/2}}{2}\alpha_{10} - \frac{\alpha_6}{6} - \frac{\alpha_8}{2}.$$

Then, after collecting terms in (21), we obtain:

$$H_T(u^0) = -\frac{1}{2}\left(\frac{x}{\omega}\right)^2 + \frac{\psi_0}{\sqrt{T}}\left(\frac{x}{\omega}\right)^3 + \frac{\psi_1}{T}\left(\frac{x}{\omega}\right)^4 + O\left(\frac{x^5}{T^{3/2}}\right), \tag{22}$$

where

$$\psi_0 = \frac{iT^{1/2}\alpha_1 + 3\alpha_3}{6\omega^3}$$

and

$$\psi_1 = -\frac{1}{8}\left(\frac{iT^{1/2}\alpha_1 + 3\alpha_3}{\omega^3}\right)^2 + \frac{1}{\omega^4}\left(\frac{T\alpha_2}{24} - \frac{iT^{1/2}\alpha_{10}}{2} - \frac{\alpha_6}{6} - \frac{\alpha_8}{2}\right).$$

Similarly, we find

$$\begin{aligned} H_T^{(2)}(u^0) = T\omega^2\Bigg[1 &- \left(\frac{x/\omega}{\sqrt{T}}\right)\left(\frac{iT^{1/2}\alpha_1 + 3\alpha_3}{\omega^3}\right) \\ &+ \left(\frac{x/\omega}{\sqrt{T}}\right)^2\left\{\frac{1}{2\omega^4}(T\alpha_2 - 12iT^{1/2}\alpha_{10} - 4\alpha_6 - 12\alpha_8)\right. \\ &\left. - \frac{1}{2}\left(\frac{iT^{1/2}\alpha_1 + 3\alpha_3}{\omega^3}\right)^2\right\} + O\left(\left(\frac{x}{\sqrt{T}}\right)^3\right)\Bigg], \end{aligned} \tag{23}$$

$$\begin{aligned} J_T(u^0) = 1 &- \frac{1}{2}\left(\frac{x/\omega}{\sqrt{T}}\right)\frac{\alpha_4}{\omega} + \frac{1}{2}\left(\frac{x/\omega}{\sqrt{T}}\right)^2\left\{\frac{\alpha_4}{\omega}\left(\frac{iT^{1/2}\alpha_1 + 3\alpha_4}{2\omega^3}\right)\right. \\ &\left. + \frac{iT^{1/2}\alpha_5}{2\omega^2}\right\} + O\left(\left(\frac{x}{\sqrt{T}}\right)^3\right) + O\left(\frac{x^5}{T^{3/2}}\right), \end{aligned} \tag{24}$$

$$J_T^{(1)}(u^0) = -\frac{1}{2}\alpha_4 + \left(\frac{x/\omega}{\sqrt{T}}\right)\frac{iT^{1/2}\alpha_5}{\omega} + O\left(\left(\frac{x}{\sqrt{T}}\right)^2\right) + O\left(\frac{x^4}{T}\right), \tag{25}$$

and

$$J_T^{(2)}(u^0) = iT^{1/2}\alpha_5 + O\left(\frac{x}{\sqrt{T}}\right) + O\left(\frac{x^3}{\sqrt{T}}\right), \tag{26}$$

so that from (20), (22), (23), and (24)-(26) we deduce that

$$\begin{aligned} p_T(x) = &\frac{1}{\sqrt{2\pi}\omega\left\{1 + 6\psi_0\left(\frac{x/\omega}{\sqrt{T}}\right) + O\left(\left(\frac{x}{\sqrt{T}}\right)^2\right)\right\}^{1/2}} \\ &\times \exp\left[-\frac{1}{2}\left(\frac{x}{\omega}\right)^2\right]\exp\left[\frac{(x/\omega)^3}{\sqrt{T}}\left\{\psi_0 + \psi_1\left(\frac{x/\omega}{\sqrt{T}}\right) + O\left(\left(\frac{x}{\sqrt{T}}\right)^2\right)\right\}\right] \end{aligned}$$

$$\times\left\{1-\frac{1}{2}\left(\frac{x/\omega}{\sqrt{T}}\right)\frac{\alpha_4}{\omega}+O\left(\left(\frac{x}{\sqrt{T}}\right)^2\right)+O\left(\frac{x^5}{T^{3/2}}\right)\right\} \tag{27}$$
$$\times\left\{1+O(T^{-1})+O\left(\left(\frac{x}{\sqrt{T}}\right)^4\right)+O\left(\left(\frac{x}{\sqrt{T}}\right)^3\right)+o(T^{-1})\right\}.$$

When $x = O(T^{1/4})$ as $T\uparrow\infty$, it follows from (27) that

$$p_T(x)=\frac{1}{\sqrt{2\pi}\omega}\exp\left[-\frac{1}{2}\left(\frac{x}{\omega}\right)^2\right]\exp\left[\frac{(x/\omega)^3}{\sqrt{T}}\Psi^{[2]}\left(\frac{x}{\sqrt{T}}\right)\right]\left\{1+O\left(\frac{x}{\sqrt{T}}\right)\right\}, \tag{28}$$

where

$$\Psi^{[2]}\left(\frac{x}{\sqrt{T}}\right)=\psi_0+\psi_1\left(\frac{x/\omega}{\sqrt{T}}\right).$$

The first two factors on the right side of (28) then describe the limiting tails of $p_T(x)$ for large deviations up to (and including) $x = O(T^{1/2})$.

Expansions corresponding to (28) for tail probabilities rather than tail ordinates can be formally derived by integrating by parts and using Mills ratio.[1] We have for $x > 0$ and $x = O(T^{1/2})$

$$P(t\geq x)=\left\{1-I\left(\frac{x}{\omega}\right)\right\}\exp\left[\frac{(x/\omega)^3}{\sqrt{T}}\Psi^{[2]}\left(\frac{x}{\sqrt{T}}\right)\right]\left\{1+O\left(\frac{x}{\sqrt{T}}\right)\right\} \tag{29}$$

and

$$P(t\leq -x)=I(-x)\exp\left[\frac{-(x/\omega)^3}{\sqrt{T}}\Psi^{[2]}\left(\frac{-x}{\sqrt{T}}\right)\right]\left\{1+O\left(\frac{x}{\sqrt{T}}\right)\right\}. \tag{30}$$

III. RELATIONSHIP BETWEEN THE LARGE DEVIATION AND EDGEWORTH EXPANSIONS

The coefficients in the truncated power series $\psi^{[2]}(x/\sqrt{T})$ depend on the cumulants of the underlying variates q (up to the fourth order) and the derivatives of the error function $e_T(\ )$ (to the third order). Moreover, when we expand the exponential in the second factor on the right side of (28), we obtain the asymptotic normal density multiplied by a polynomial in x, just as in the Edgeworth expansion. In this way, we can establish a simple link between the two expansions.

To be precise, we recall the form of the Edgeworth expansion of $P(t \leq x)$ up to $O(T^{-1})$ given by Sargan (1976):

$$I\left(\frac{x}{\omega}\right)+i\left(\frac{x}{\omega}\right)\left\{a_0+a_1\left(\frac{x}{\omega}\right)+a_2\left(\frac{x}{\omega}\right)^2+a_3\left(\frac{x}{\omega}\right)^3+a_5\left(\frac{x}{\omega}\right)^5\right\}, \tag{31}$$

where

$$a_0 = \frac{1}{2\sqrt{T}\omega}\alpha_4 + \frac{1}{6\sqrt{T}\omega}(iT^{1/2}\alpha_1 + 3\alpha_3),$$

$$a_1 = \frac{1}{8\omega^2 T}(4iT^{1/2}\alpha_5 + 4\alpha_7 + \alpha_4 + 2\alpha_9)$$
$$+ \frac{1}{8\omega^4 T}(T\alpha_2 + 12iT^{1/2}\alpha_{10} - 2iT^{1/2}\alpha_1\alpha_4 - 4\alpha_6 - 6\alpha_3\alpha_4 - 12\alpha_8)$$
$$- \frac{5}{24T\omega^6}(iT^{1/2}\alpha_1 + 3\alpha_3)^2,$$

$$a_2 = -\frac{1}{6\sqrt{T}\omega^3}(iT^{1/2}\alpha_1 + 3\alpha_3),$$

$$a_3 = -\frac{1}{24\omega^4 T}(T\alpha_2 - 12iT^{1/2}\alpha_{10} - 2iT^{1/2}\alpha_1\alpha_4 - 4\alpha_6 - 6\alpha_3\alpha_4 - 12\alpha_8)$$
$$+ \frac{5}{36T\omega^6}(iT^{1/2}\alpha_1 + 3\alpha_3)^2,$$

$$a_5 = -\frac{1}{72T\omega^6}(iT^{1/2}\alpha_1 + 3\alpha_3)^2.$$

Equation (31) is in the same form and notation as equation (13) of Phillips (1977c). We sometimes call (31) the Edgeworth (A) approximation. An alternative representation of (31) which is accurate up to $O(T^{-1})$ is (Phillips, 1977c, p. 474)

$$I\left(\frac{x}{\omega} + b_0 + b_1\left(\frac{x}{\omega}\right) + b_2\left(\frac{x}{\omega}\right)^2 + b_3\left(\frac{x}{\omega}\right)^3\right), \tag{32}$$

where $b_0 = a_0$, $b_1 = a_1 + a_0^2/2$, $b_2 = a_2$, and $b_3 = a_3 + a_0 a_2$. Equation (32) is sometimes called the Edgeworth (B) approximation.

From (31) we have the corresponding density approximation

$$\frac{1}{\omega}i\left(\frac{x}{\omega}\right)\left\{1 + a_1 + (2a_2 - a_0)\left(\frac{x}{\omega}\right) + (3a_3 - a_1)\left(\frac{x}{\omega}\right)^2 - a_2\left(\frac{x}{\omega}\right)^3\right.$$
$$\left. + (5a_5 - a_3)\left(\frac{x}{\omega}\right)^4 - a_5\left(\frac{x}{\omega}\right)^6\right\}. \tag{33}$$

To reconcile (33) with the large deviation expansion we note the order of magnitude of the coefficients

$$a_0 = O(T^{-1/2}), \quad a_1 = O(T^{-1}), \quad a_2 = O(T^{-1/2}),$$
$$a_3 = O(T^{-1}), \quad a_5 = O(T^{-1}),$$

so that, in the tails, where x may be large relative to T the polynomial in braces in (33) is dominated by the last three terms which are respectively of order $O(x^3/\sqrt{T})$, $O(x^4/T)$, and $O(x^6/T)$. The other terms are of $O(x/\sqrt{T})$ or smaller.

It is now easy to see how the large deviation expansion selects those terms in (33) which dominate in determining the behavior of tails. Returning to (27), we have

$$
\begin{aligned}
p_T(x) = {} & \frac{1}{\sqrt{2\pi}\omega} \exp\left[-\frac{1}{2}\left(\frac{x}{\omega}\right)^2\right]\left\{1 + 6\psi_0\left(\frac{x/\omega}{\sqrt{T}}\right) + O\left(\left(\frac{x}{\sqrt{T}}\right)^2\right)\right\}^{-1/2} \\
& \times \exp\left[\psi_0 \frac{(x/\omega)^3}{\sqrt{T}} + \psi_1 \frac{(x/\omega)^4}{T} + O\left(\frac{x^5}{T^{3/2}}\right)\right] \\
& \times \left\{1 - \frac{1}{2}\left(\frac{x/\omega}{\sqrt{T}}\right)\frac{\alpha_4}{\omega} + O\left(\left(\frac{x}{\sqrt{T}}\right)^2\right) + O\left(\frac{x^5}{T^{3/2}}\right)\right\} \\
& \times \left\{1 + O(T^{-1}) + O\left(\left(\frac{x}{\sqrt{T}}\right)^3\right) + O(T^{-1})\right\} \\
= {} & \frac{1}{\sqrt{2\pi}\omega} \exp\left[-\frac{1}{2}\left(\frac{x}{\omega}\right)^2\right]\left\{1 - 3\psi_0\left(\frac{x/\omega}{\sqrt{T}}\right) + O\left(\left(\frac{x}{\sqrt{T}}\right)^2\right)\right\} \\
& \times \left\{1 + \psi_0 \frac{(x/\omega)^3}{\sqrt{T}} + \psi_1 \frac{(x/\omega)^4}{T} + \frac{1}{2}\psi_0^2 \frac{(x/\omega)^6}{T} \right. \\
& \left. + O\left(\left(\frac{x^3}{\sqrt{T}}\right)^3\right) + O\left(\frac{x^5}{T^{3/2}}\right)\right\} \\
& \times \left\{1 - \frac{1}{2}\left(\frac{x/\omega}{\sqrt{T}}\right)\frac{\alpha_4}{\omega} + O\left(\left(\frac{x}{\sqrt{T}}\right)^2\right) + O\left(\frac{x^5}{T^{3/2}}\right)\right\} \\
& \times \left\{1 + O(T^{-1}) + O\left(\left(\frac{x}{\sqrt{T}}\right)^3\right) + O(T^{-1})\right\} \\
= {} & \frac{1}{\sqrt{2\pi}\omega} \exp\left[-\frac{1}{2}\left(\frac{x}{\omega}\right)^2\right]\left\{1 - \frac{1}{\sqrt{T}}\left(\frac{1}{2}\frac{\alpha_4}{\omega} + 3\psi_0\right)\left(\frac{x}{\omega}\right) + \frac{\psi_0}{\sqrt{T}}\left(\frac{x}{\omega}\right)^3 \right. \\
& + \frac{1}{T}\left(\psi_1 - \psi_0\left(\frac{1}{2}\frac{\alpha_4}{\omega} + 3\psi_0\right)\right)\left(\frac{x}{\omega}\right)^4 + \frac{1}{2}\frac{\psi_0^2}{T}\left(\frac{x}{\omega}\right)^6 \\
& + \frac{1}{T}\left(\psi_1 - \psi_0\left(\frac{1}{2}\frac{\alpha_4}{\omega} + 3\psi_0\right)\right)\left(\frac{x}{\omega}\right)^4 + \frac{1}{2}\frac{\psi_0^2}{T}\left(\frac{x}{\omega}\right)^6 \\
& \left. + O\left(\left(\frac{x}{\sqrt{T}}\right)^2\right) + O(T^{-1})\right\}.
\end{aligned}
\tag{34}
$$

It now follows from the definition of ψ_0 and ψ_1 that

$$-\frac{1}{\sqrt{T}}\left(\frac{1}{2}\frac{\alpha_4}{\omega}+3\psi_0\right) = 2a_2 - a_0,$$

$$\frac{\psi_0}{\sqrt{T}} = -a_2,$$

$$\frac{1}{T}\left(\psi_1 - \psi_0\left(\frac{1}{2}\frac{\alpha_4}{\omega}+3\psi_0\right)\right) = 5a_5 - a_3,$$

and

$$\frac{1}{2}\frac{\psi_0^2}{T} = -a_5,$$

which reconciles (34) with the Edgeworth expansion (33) up to order $O((x/\sqrt{T})^2)$.

IV. TAIL PROBABILITY EXPANSIONS IN THE AR(1)

In this section we apply the large deviation expansion given above to the sampling distribution of the least squares estimator in the first order autoregression

$$y_t = \alpha y_{t-1} + u_t, \qquad t = \ldots, -1, 0, 1, \ldots, \quad |\alpha| < 1,$$

where the u_t are independent and identically distributed $N(0, \sigma^2)$. Writing the least squares estimator of α as $\hat{\alpha} = y'C_1y/y'C_2y$ as in Phillips (1977c), where $y' = (y_0, \ldots, y_T)$,

$$C_1 = \begin{bmatrix} 0 & \frac{1}{2} & \cdots & 0 & 0 \\ \frac{1}{2} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \frac{1}{2} & 0 \end{bmatrix}, \quad \text{and} \quad C_2 = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & & 1 & 0 \end{bmatrix},$$

we obtain

$$\hat{\alpha} - \alpha = e_T(q) = \frac{q_1 - \alpha q_2}{q_2 + \mu_2/T},$$

where $q_i = (y'C_iy - \mu_i)/T$, $\mu_i = E(y'C_iy)$, and $q' = (q_1, q_2)$.

As in Section II we introduce the vector $\bar{q} = \sqrt{T}q$, and from Phillips (1977c, p. 472) we know that the second characteristic of $\bar{q}$ is given by

$$\lambda(z) = -\frac{1}{2}\log\det\left\{I - \frac{2i}{\sqrt{T}}(z_1C_1 + z_2C_2)\Omega\right\} - \frac{i\mu_1 z_1}{\sqrt{T}} - \frac{i\mu_2 z_2}{\sqrt{T}},$$

where Ω is the matrix whose (i, j)th element is given by $\alpha^{|i-j|}\sigma^2/(1-\alpha^2)$. Successive derivatives of $\lambda(z)$ evaluated at the origin are

$$\lambda_a = 0,$$

$$\lambda_{ab} = -\frac{2}{T}\text{tr}\{(C_a\Omega)(C_b\Omega)\},$$

$$\lambda_{abc} = -\frac{4i}{T^{3/2}}\text{tr}\{(C_a\Omega)(C_b\Omega)(C_c\Omega)\} - \frac{4i}{T^{3/2}}\text{tr}\{(C_a\Omega)(C_c\Omega)(C_b\Omega)\},$$

$$\begin{aligned}\lambda_{abcd} = \frac{8}{T^2}[&\text{tr}\{(C_d\Omega)(C_c\Omega)(C_a\Omega)(C_b\Omega)\} + \text{tr}\{(C_d\Omega)(C_a\Omega)(C_c\Omega)(C_b\Omega)\}\\ &+ \text{tr}\{(C_d\Omega)(C_b\Omega)(C_c\Omega)(C_a\Omega)\} + \text{tr}\{(C_b\Omega)(C_a\Omega)(C_d\Omega)(C_c\Omega)\}\\ &+ \text{tr}\{(C_b\Omega)(C_a\Omega)(C_c\Omega)(C_d\Omega)\} + \text{tr}\{(C_a\Omega)(C_b\Omega)(C_c\Omega)(C_d\Omega)\}].\end{aligned}$$

To derive the large deviation expansions (29) and (30) we need the coefficients ψ_0 and ψ_1 in the truncated power series $\Psi^{[2]}(x/\sqrt{T})$. These coefficients in turn depend on the cumulants of $\tilde{q}$ up to the fourth order, which we derive in the Appendix, and derivatives of $e_T(\ )$ to the third order at the origin. The latter are as follows:

$$e_1 = \frac{1-\alpha^2}{\sigma^2}, \qquad e_2 = \frac{-\alpha(1-\alpha^2)}{\sigma^2}$$

$$e_{11} = 0, \qquad e_{12} = \frac{-(1-\alpha^2)^2}{\sigma^4}, \qquad e_{22} = \frac{2\alpha(1-\alpha^2)^2}{\sigma^4},$$

$$e_{111} = 0, \qquad e_{112} = 0, \qquad e_{122} = \frac{\alpha(1-\alpha^2)^3}{\sigma^6}, \qquad e_{222} = \frac{-6\alpha(1-\alpha^2)^3}{\sigma^6}.$$

Using the results in the Appendix and omitting the details of the algebra, which is at times quite heavy, we now obtain

$$\begin{aligned}\omega^2 &= -\lambda_{jk}e_je_k = \left(\frac{1-\alpha^2}{\sigma^2}\right)^2\frac{2}{T}\text{tr}[(C_1 - C_2)\Omega]^2 = \left(\frac{1-\alpha^2}{\sigma^2}\right)^2\left(\frac{\sigma^4}{1-\alpha^2}\right)\\ &= 1-\alpha^2,\end{aligned}$$

$$\begin{aligned}iT^{1/2}\alpha_1 &= iT^{1/2}\lambda_{jkl}e_je_ke_l = \left(\frac{1-\alpha^2}{\sigma^2}\right)^3\left(\frac{8}{T}\text{tr}[(C_1 - \alpha C_2)\Omega]^3\right)\\ &= \left(\frac{1-\alpha^2}{\sigma^2}\right)^3\left(\frac{6\sigma^6\alpha}{(1-\alpha^2)^2} + O(T^{-1})\right)\\ &= 6\alpha(1-\alpha^2) + O(T^{-1}),\end{aligned}$$

$$\begin{aligned}
\alpha_3 &= (e_j\lambda_{ja})e_{ab}(\lambda_{bk}e_k)\\
&= 4\left(\frac{1-\alpha^2}{\sigma^2}\right)^2\left(\frac{1}{T}\operatorname{tr}\{[(C_1-C_2)\Omega]C_a\Omega\}\right)\\
&\quad\times e_{ab}\left(\frac{1}{T}\operatorname{tr}\{C_b\Omega[(C_1-\alpha C_2)\Omega]\}\right)\\
&= -4\alpha(1-\alpha^2)+O(T^{-1}),\\
\alpha_2 &= \lambda_{jklm}e_je_ke_le_m\\
&= \frac{48}{T^2}\operatorname{tr}(e_jC_j\Omega)^4\\
&= \left(\frac{1-\alpha^2}{\sigma^2}\right)^4\left(\frac{48}{T^2}\operatorname{tr}[(C_1-\alpha C_2)\Omega]^4\right)\\
&= \frac{1}{T}\left(\frac{1-\alpha^2}{\sigma^2}\right)^4\left(\frac{6(3+7\alpha^2)}{(1-\alpha^2)^3}\right)\\
&= \frac{6(3+7\alpha^2)(1-\alpha^2)}{T},\\
iT^{1/2}\alpha_{10} &= (iT^{1/2}\lambda_{ajk}e_je_k)e_{ab}(\lambda_{bl}e_l)\\
&= -\left\{\frac{8}{T}\operatorname{tr}[C_a\Omega(e_jC_j\Omega)^2]\right\}e_{ab}\left\{\frac{2}{T}\operatorname{tr}[C_b\Omega(e_kC_k\Omega)]\right\}\\
&= -\left(\frac{1-\alpha^2}{\sigma^2}\right)^3\left(\frac{8}{T}\operatorname{tr}[C_a\Omega(C_1-\alpha C_2)\Omega]\right)\\
&\quad\times e_{ab}\left(\frac{2}{T}\operatorname{tr}[C_b\Omega(C_1-\alpha C_2)\Omega)\right)\\
&= 4(1+5\alpha^2)(1-\alpha^2)+O(T^{-1}),\\
\alpha_6 &= e_{abc}(\lambda_{aj}e_j)(\lambda_{bk}e_k)(\lambda_{cl}e_l)\\
&= -8e_{abc}\left(\frac{1}{T}\operatorname{tr}[C_a\Omega(e_jC_j\Omega)]\right)\left(\frac{1}{T}\operatorname{tr}[C_b\Omega(e_kC_k\Omega)]\right)\\
&\quad\times\left(\frac{1}{T}\operatorname{tr}[C_c\Omega(e_lC_l\Omega)]\right)\\
&= -24\alpha^2(1-\alpha^2)+O(T^{-1}),
\end{aligned}$$

$$\begin{aligned}\alpha_8 &= e_{ab}e_{cd}(\lambda_{aj}e_j)(\lambda_{ck}e_k)\lambda_{bd}\\ &= -8e_{ab}e_{cd}\left(\frac{1}{T}\operatorname{tr}[C_a\Omega(e_jC_j\Omega)]\right)\left(\frac{1}{T}\operatorname{tr}[C_c\Omega(e_kC_kl)]\right)\\ &\quad\times\left(\frac{1}{T}\operatorname{tr}(C_b\Omega C_d\Omega)\right)\\ &= -2(1+7\alpha^2)(1-\alpha^2).\end{aligned}$$

It now follows that

$$\begin{aligned}\psi_0 &= \frac{1}{6(1-\alpha^2)^{3/2}}\{6\alpha(1-\alpha^2)-12\alpha(1-\alpha^2)\}+O(T^{-1})\\ &= \frac{-\alpha}{(1-\alpha^2)^{1/2}}+O(T^{-1})\end{aligned}$$

and

$$\begin{aligned}\psi_1 = &-\left\{\frac{1}{8}\left(\frac{6\alpha}{(1-\alpha^2)^{1/2}}\right)^2+\left(\frac{1}{1-\alpha^2}\right)^2\frac{1}{4}(3+7\alpha^2)(1-\alpha^2)\right.\\ &\left.-2(1+5\alpha^2)(1-\alpha^2)+4\alpha^2(1-\alpha^2)+(1+7\alpha^2)(1-\alpha^2)\right\}\\ &+O(T^{-1})\\ =&-\frac{1}{4}\left(\frac{1+7\alpha^2}{1-\alpha^2}\right)+O(T^{-1}).\end{aligned}$$

In the present case, therefore, we have for $x > 0$ and $x = O(T^{1/4})$ as $T\uparrow\infty$

$$P\left(\frac{\sqrt{T}(\hat{\alpha}-\alpha)}{(1-\alpha^2)^{1/2}} > x\right) = (1-I(x))\exp\left\{-\frac{\alpha}{(1-\alpha^2)^{1/2}}\frac{x^3}{\sqrt{T}}-\frac{1}{4}\left(\frac{1+7\alpha^2}{1-\alpha^2}\right)\frac{x^4}{T}\right\}\times\left[1+O\left(\frac{x}{\sqrt{T}}\right)\right] \tag{35}$$

and

$$P\left(\frac{\sqrt{T}(\hat{\alpha}-\alpha)}{(1-\alpha^2)^{1/2}} \leq -x\right) = I(-x)\exp\left\{\frac{\alpha}{(1-\alpha^2)^{1/2}}\frac{x^3}{\sqrt{T}}-\frac{1}{4}\left(\frac{1+7\alpha^2}{1-\alpha^2}\right)\frac{x^4}{T}\right\}\times\left[1+O\left(\frac{x}{\sqrt{T}}\right)\right]. \tag{36}$$

V. FINAL REMARKS

The theory developed in this paper is appropriate for large deviations of the form $x = O(T^{1/4})$ as $T \uparrow \infty$. In particular, the general formulas (29) and (30) provide the scale factors needed to modify the asymptotic normal tails when the region of the tail is restricted to a zone of $O(T^{1/4})$. These results may be extended to apply over zones of $O(T^{1/2-\varepsilon})$ for any $\varepsilon > 0$ as in formula (6) for the case of a standardized sum of iid variates.

This theory of large deviations is sometimes distinguished from the theory of very large deviations in which the argument x is not restricted to a zone of $o(T^{1/2})$. Ibragimov and Linnik (1971) report some results on very large deviations for standardized sums, showing how the limiting tails can be represented as the sum of a rational function in x and the normal integral. This type of expansion seems likely to be useful in practice only on the extreme tail of a distribution. However, in cases where the statistic under study has finite moments only up to a certain order such expansions are relevant, even though they may not yield very good approximations to tail probabilities in practice. A general theory of such expansions in terms of increasing powers of $1/x$ is developed by the author in a recent paper (1985) using the theory of Fourier transforms of generalized functions.

APPENDIX

In this Appendix we detail derivations of the cumulants of q that are needed in order to obtain explicit representations of the expansions in Section IV of the paper. Cumulants up to order 3 were obtained in Phillips (1977c), and we now extend the analysis there to higher order cumulants. It is convenient to work with the cumulants of the quadratic form

$$Q(r) = y'(C_1 - rC_2)y$$

as a function of r. From Phillips (1977c, p. 465) the sth cumulant of $Q(r)$ is given by

$$k_s(r) = (s-1)!2^{s-1}\operatorname{tr}[(C_1 - rC_2)\Omega]^s \tag{A.1}$$

$$= (s-1)!\,2^{s-1}\left(\frac{T}{2\pi}\right)\sum_{j=0}^{s}\binom{s}{j}(-1)^{s-j}r^{s-j} \times \int_{-\pi}^{\pi}[2\pi f(\lambda)]^s(\cos\lambda)^j\,d\lambda + O(1), \tag{A.2}$$

where Ω is the covariance matrix of y and $f(\lambda)$ is the spectral density of

y_t. The integrals

$$\int_{-\pi}^{\pi} [2\pi f(\lambda)]^s (\cos \lambda)^j \, d\lambda$$

can then be written as linear combinations of integrals such as

$$\int_{-\pi}^{\pi} [2\pi f(\lambda)]^s (\cos k\lambda) \, d\lambda \tag{A.3}$$

for k integer and $k \le j$. Using the fact that

$$[2\pi f(\lambda)]^s = \frac{\sigma^{2s}}{|(1 - \alpha\, e^{i\lambda})^s|^2}$$

(A.3) can now be evaluated by residue theory as in the Appendix of Phillips (1977c) using the general formula

$$\int_{-\pi}^{\pi} [2\pi f(\lambda)]^s \cos(k\lambda) \, d\lambda = \sigma^{2s} 2\pi \lim_{z \to \alpha} \left\{ \frac{1}{(s-1)!} \frac{d^{s-1}}{dz^{s-1}} \frac{z^{k+s-1}}{(1 - \alpha z)^s} \right\}. \tag{A.4}$$

In this way, we can find expressions for the cumulants $k_s(r)$ which are correct up to $O(1)$. For $s = 2$ and $s = 3$, the results are given in Phillips (1977c). In particular, we have

$$k_2(r) = \frac{T\sigma^4}{(1-\alpha^2)^3}(1 + 4\alpha^2 - \alpha^4) - 8\alpha r + 2(1 + \alpha^2) r^2 + O(1) \tag{A.5}$$

and

$$k_3(r) = \frac{8T\sigma^6}{(1-\alpha^2)^5} \frac{\alpha}{4}(9 + 19\alpha^2 - 5\alpha^4 + \alpha^6) - \tfrac{3}{2}(1 + 10\alpha^2 + \alpha^4) r$$
$$+ 9\alpha(1 + \alpha^2) r^2 - (1 + 4\alpha^2 + \alpha^4) r^3 + O(1). \tag{A.6}$$

Hence

$$k_2(\alpha) = \frac{T\sigma^4}{1 - \alpha^2}, \qquad k_3(\alpha) = \frac{6T\sigma^6\alpha}{(1-\alpha^2)^2} + O(1),$$

and equating coefficients of like powers of r in (A.1) and (A.2) we have from (A.5) and (A.6)

$$\operatorname{tr}(C_1\Omega)^2 = \frac{T\sigma^4}{2(1-\alpha^2)^3}(1 + 4\alpha^2 - \alpha^4) + O(1),$$

$$\operatorname{tr}(C_1\Omega C_2\Omega) = \frac{2T\alpha\sigma^4}{(1-\alpha^2)^3} + O(1),$$

$$\operatorname{tr}(C_2\Omega)^2 = \frac{T\sigma^4(1 + \alpha^2)}{(1-\alpha^2)^3} + O(1),$$

$$\mathrm{tr}(C_1\Omega)^3 = \frac{T\sigma^6\alpha}{4(1-\alpha^2)^5}(9+19\alpha^2-5\alpha^4+\alpha^6)+O(1),$$

$$\mathrm{tr}[(C_1\Omega)^2C_2] = \frac{T\sigma^6\alpha}{2(1-\alpha^2)^5}(1+10\alpha^2+\alpha^4)+O(1),$$

$$\mathrm{tr}[C_1\Omega(C_2\Omega)^2] = \frac{3T\sigma^6}{(1-\alpha^2)^5}(1+\alpha^2)+O(1),$$

$$\mathrm{tr}(C_2\Omega)^3 = \frac{T\sigma^6}{(1-\alpha^2)^5}(1+4\alpha^2+\alpha^4)+O(1).$$

When $s = 4$, we have

$$\begin{aligned} k_4(r) = \frac{48T}{2\pi}\Big\{(-r)^4\int_{-\pi}^{\pi}[2\pi f(\lambda)]^4\,d\lambda + 4(-r)^3\int_{-\pi}^{\pi}[2\pi f(\lambda)]^4\cos\lambda\,d\lambda \\ + 6(-r)^2\int_{-\pi}^{\pi}[2\pi f(\lambda)]^4\cos^2\lambda\,d\lambda \\ +4(-r)\int_{-\pi}^{\pi}[2\pi f(\lambda)]^4\cos^3\lambda\,d\lambda + \int_{-\pi}^{\pi}[2\pi f(\lambda)]^4\cos^4\lambda\,d\lambda\Big\} + O(1), \end{aligned} \tag{A.7}$$

and setting

$$R_j = \frac{\int_{-\pi}^{\pi}[2\pi f(\lambda)]^4\cos(j\lambda)\,d\lambda}{2\pi\sigma^{2s}},$$

we obtain from (A.4) after some lengthy but routine algebra

$$\begin{aligned} R_j = {} & (1-\alpha^2)^{-4}(j+1)(j+2)(j+3)\alpha^j \\ & +12(1-\alpha^2)^{-5}(j+2)(j+3)\alpha^{j+2}+60(1-\alpha^2)^{-6}(j+3)\alpha^{j+4} \\ & +120(1-\alpha^2)^{-7}\alpha^{j+6}. \end{aligned} \tag{A.8}$$

Then, by elementary trigonometry, (A.7) becomes

$$\begin{aligned} k_4(r) &= 48\sigma^8T\{(-r)^4R_0+4(-r)^3R_1 \\ &\quad + 6(-r)^2\tfrac{1}{2}(R_0+R_2)+4(-r)\tfrac{1}{4}(3R_1+R_3) \\ &\quad + \tfrac{1}{8}(3R_0+4R_2+R_4)\}+O(1) \\ &= 6\sigma^8R\{8r^4R_0-32r^3R_1+24r^2(R_0+R_2) \\ &\quad - 8r(3R_1+R_3)+(3R_0+4R_2+R_4)\}+O(1). \end{aligned} \tag{A.9}$$

We can now substitute (A.8) directly into (A.9). In the paper we need only $k_4(\alpha)$, and this is obtained by setting $r = \alpha$ after substitution of (A.8) into

(A.9). We find

$$
\begin{aligned}
k_4(\alpha) &= 6\sigma^8 T(1-\alpha^2)^{-7}(7\alpha^{10} - 25\alpha^8 + 30\alpha^6 - 10\alpha^4 - 5\alpha^2 + 3) + O(1) \\
&= 6\sigma^8 T(1-\alpha^2)^{-7}(1-\alpha^2)^4(3+7\alpha^2) + O(1) \\
&= \frac{6\sigma^8 T(3+7\alpha^2)}{(1-\alpha^2)^3} + O(1).
\end{aligned}
$$

ACKNOWLEDGMENTS

My thanks as always, go to Glena Ames for her skill and effort in typing the manuscript of this paper. The research reported here was supported in part by the SSRC under Grant No. HR3432 and in part by the NSF under Grant No. SES 8218792.

NOTE

1. A rigorous development will be more difficult Integral large deviation limit theorems are generally proved by using auxiliary variates, and direct proofs by this method encounter greater difficulties than the corresponding local theorems for densities See, for example, Richter (1964).

REFERENCES

Akahira, M. and K Takeuchi (1981). *Asymptotic Efficiency of Statistical Estimators: Concepts and Higher Order Asymptotic Efficiency*, Berlin Springer-Verlag.

Bhattacharya, R. N and J K. Ghosh (1978) On the Validity of the Formal Edgeworth Expansion, *Annals of Statistics* 6, 434–451

Bhattacharya, R. N. and R. R. Rao (1976) *Normal Approximations and Asymptotic Expansions*, New York: Wiley.

Chambers, J M. (1967). On Methods of Asymptotic Approximation for Multivariate Distributions, *Biometrika* 54, 367–383

Chernoff, H (1956) Large Sample Theory: Parametric Case, *Annals of Mathematical Statistics* 27, 1–22.

Copson, E. T. (1965). *Asymptotic Expansions*, Cambridge, England: Cambridge University Press.

Cramér, H. (1938). Sur un Nouveau Théorème-Limite de la Théorie des Probabilités, *Actualités Scientifiqués et Industrielles* 736, 5–23.

Daniels, H. E. (1954). Saddlepoint Approximations in Statistics, Annals of Mathematical Statistics 25, 631–650.

Daniels, H E. (1956). The Approximate Distribution of Serial Correlation Coefficients, *Biometrika* 43, 169–185.

Ibragimov, I. A. and Y. V. Linnik (1971). *Independent and Stationary Sequences of Random Variables*, Groningen Wolters-Noordhoff

Knopp, K. (1956). *Infinite Sequences and Series*, New York: Dover.

Lukacs, E. (1970) *Characteristic Functions*, 2nd ed., London: Griffin.

Petrov, V. V (1968). Asymptotic Behaviour of Probabilities of Large Deviations, *Theory of Probability and Its Applications* 13, 408–420.

Phillips, P. C. B. (1977a). A Large Deviation Limit Theorem for Multivariate Distributions, *Journal of Multivariate Analysis* 7, 50–62.

Phillips, P. C. B. (1977b). A General Theorem in the Theory of Asymptotic Expansions as Approximations to the Finite Sample Distributions of Econometric Estimators, *Econometrica* 45, 1517–1534.

Phillips, P. C. B. (1977c). Approximations to Some Finite Sample Distributions Associated with a First Order Stochastic Difference Equation, *Econometrica* 45, 463–486.

Phillips, P. C. B. (1980). Finite Sample Theory and the Distributions of Alternative Estimators of the Marginal Propensity to Consume, *Review of Economic Studies* 47, 183–224.

Phillips, P. C. B (1983). Exact Small Sample Theory in the Simultaneous Equations Model In Intriligator, M. D. and Z. Griliches (eds.), *Handbook of Econometrics*. Amsterdam: North Holland.

Phillips, P. C. B. (1985) A Theorem on the Tail Behaviour of Probability Distributions with an Application to the Stable Family, *Canadian Journal of Economics* 18, 58–65.

Piaggio, H. T. H. (1962). *Differential Equations*, London: Bell and Sons.

Rothenberg, T. J. (1982). Comparing Alternative Asymptotically Equivalent Tests. In Hildenbrand, W. (ed.), *Advances in Econometrics*. Cambridge, England: Cambridge University Press.

Rothenberg, T. J. (1984). Approximating the Distributions of Econometric Estimators and Test Statistics. In Intriligator, M. D. and Z. Griliches (eds.), *Handbook of Econometrics*. Amsterdam: North Holland.

Richter, W. (1957). Local Limit Theorems for Large Deviations, *Theory of Probability and Its Applications* 2, 206–219.

Richter, W. (1964). Multi-Dimensional Limit Theorems for Large 'Deviations and Their Application to the χ^2 Distribution, *Theory of Probability and Its Applications* 9, 28–37.

Sargan, J. D. (1975). Gram-Charlier Approximations Applied to t-Ratios of k-Class Estimators, *Econometrica* 43, 327–346.

Sargan, J. D. (1976). Econometric Estimators and the Edgeworth Approximation, Walras lecture at the Econometric Society Meeting, San Francisco, December 1974 and *Econometrica* 44, 421–448.

Sargan, J. D. and S. E. Satchell (1986). A Theorem of Validity for Edgeworth Expansions, *Econometrica* 54, 189–214.

Widder, D. V. (1946). *The Laplace Transform*, Princeton, NJ: Princeton University Press.