Cowles Foundation Paper #625

ECONOMETRICA

VOLUME 53 JULY, 1985 NUMBER 4

THE EXACT DISTRIBUTION OF THE SUR ESTIMATOR

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This paper derives the exact finite sample distribution of the two-stage generalized least squares (GLS) estimator in a multivariate linear model with general linear parameter restrictions. This includes the seemingly unrelated regression (SUR) model as a special case and generalizes presently known exact results for the latter system. The usual classical assumptions are made concerning nonrandom exogenous variables and normally distributed errors. The theoretical results of this paper are made possible by the author's development of a matrix fractional calculus. This operator calculus is the main theoretical tool of the paper and may be used to solve a wide range of other unsolved problems in econometric distribution theory.

1. INTRODUCTION

IN THE EARLY 1960's Zellner [10] developed a two-stage GLS estimator for the coefficients in a linear multivariate system that is now popularly known as the SUR model. This two-stage procedure has since been used in many empirical applications. GLS also forms the basis of other commonly used estimators both in linear models with heteroscedastic or autocorrelated errors and in simultaneous equation systems where it leads to three stage least squares (3SLS). In spite of extensive research and perhaps surprisingly in view of the popularity of GLS methods in empirical work, the exact finite sample distribution of the SUR estimator is known only in highly specialized cases. These cases effectively restrict attention to two equation systems and models with orthogonal regressors [2]. Existing distribution theory is even more limited in the case of other commonly used GLS estimators, such as the two-stage estimator in linear models with heteroscedastic errors. Here, only low order moment formulae are known and then only in the simplest two sample setting.

The research underlying the present paper is motivated by the deficiencies outlined above. Our initial object of study was the exact distribution of the SUR estimator in the general case. But the methods we have developed open the way to an exact distribution theory for econometric estimators in a much wider setting than the SUR model. The present paper will derive the exact finite sample distribution of the two-stage GLS estimator in the multivariate linear model subject to general linear parameter restrictions. This generalizes all presently known distribution theory for the SUR model itself. Two important specializations of our results will be illustrated in detail: the unrestricted multivariate linear model; and the Zellner model with pairwise orthogonal regressors.

The analytical results reported here are made possible by the introduction of a fractional matrix calculus. This calculus is developed in terms of the action of

[1] I am grateful to the referees for comments on the earlier version of the paper My thanks go also to Glena Ames for her skill and effort in typing the manuscript of this paper and to the NSF for research support under Grant Number SES 8218792

a linear pseudodifferential operator on a complex analytic function defined on a certain matrix space. The development has a simple intuitive interpretation that follows from the definition of the multivariate gamma integral; and it can be regarded as a generalization of the scalar Weyl calculus. The mathematical theory of these matrix operational methods goes well beyond the application that we report to SUR-like systems.

In Section 2 of this paper the mathematical theory is developed in a form most suitable for immediate application in problems of distribution theory; and rules are given there for the fractional differentiation of the most frequently encountered elementary functions of matrix argument. The power and elegance of these matrix operational methods is then well demonstrated in Section 4 by the application to the SUR estimator. The tensor algebraic representation of this estimator severely inhibits the use of conventional multivariate methods but readily permits the application of the new operational methods to extract the form of the exact density of this estimator in a general setting.

The primary intent of the present paper is to introduce the fractional matrix calculus as a new tool of distribution theory and illustrate its use to SUR-like systems. The paper is therefore largely theoretical. More work needs to be done on the analysis and application of the results presented here for the SUR system. One immediate application, for example, which we only briefly mention in Section 7 is the simple derivation of asymptotic formulae from the finite sample results given below. Another is the derivation of moment formulae and the analysis of tail behavior for the SUR estimator. These and other applications will be left for subsequent work.

The methods given here open up an exact distribution theory for a wide class of econometric estimators and test statistics. In [**5**] the author has used similar techniques in deriving the distribution of the Stein-rule estimator in linear regression; in [**7**] the author uses the methods of Section 2 to find the distribution of the Wald statistic for testing general linear restrictions in the multivariate linear model; and in [**6**] Cramér's formula for the density of a ratio is generalized to matrix quotients by employing related operational techniques. Other potential applications of these methods include GLS estimators and test statistics for the heteroscedastic linear model and systems estimators in simultaneous equations models. The author has some work on these other problems underway.

2 FRACTIONAL MATRIX CALCULUS

This section extends the theory of fractional operators in differential calculus to matrix spaces. Those readers who are unfamiliar with scalar fractional operators as a generalization of traditional differential and integral calculus may wish to refer to the reviews in [**8** and **9**]. Additionally, [**8**] provides an historical survey which traces the theory of fractional differentiation to the work of Leibnitz, Liouville, and Euler. Interestingly, l'Hospital seems to have raised the possibility of a fractional derivative as early as 1695 in correspondence with Leibnitz.

Let $\mathscr{A}$ denote the space of $n \times n$ symmetric matrices, $O(n)$ the group of $n \times n$ orthogonal matrices, and $\mathscr{C}$ the class of symmetric functions on $\mathscr{A}$. $\mathscr{C}$ is defined as the set of all complex analytic functions on $\mathscr{A}$ for which $f(X)=f(HXH')$ for all $H\in O(n)$ and where $X\in\mathscr{A}$. Since $f\in\mathscr{C}$ is a complex analytic function of the n elementary symmetric functions of X (viz. $\sigma_1=\operatorname{tr}(X), \sigma_2=\cdots, \sigma_n=\det X$), the domain of definition of f may be extended to all complex $n\times n$ matrices X for which $f(X)$ continues to be defined [1]. In what follows we let X be an arbitrary complex $n\times n$ matrix and we use the notation ∂X to denote the matrix operator $\partial/\partial X$.

DEFINITION. If f is a complex analytic function of X and α is a complex number for which $\operatorname{Re}(\alpha)>(n-1)/2$ we define the fractional matrix operator $(\det\partial X)^{-\alpha}$ by the integral

$$(1)\qquad (\det\partial X)^{-\alpha}f(X)=\frac{1}{\Gamma_n(\alpha)}\int_{S>0}f(X-S)(\det S)^{\alpha-(n+1)/2}\,dS$$

when it exists. The integral is taken over the set of positive definite matrices $S>0$.

In (1) $\Gamma_n(\alpha)$ is the multivariate gamma function. The definition is motivated by the observation that $(\det\partial X)^{-\alpha}$ may be formally considered as the operator

$$(2)\qquad (\det\partial X)^{-\alpha}=\frac{1}{\Gamma_n(\alpha)}\int_{S>0}\operatorname{etr}(-\partial XS)(\det S)^{\alpha-(n+1)/2}\,dS,$$

$$\operatorname{etr}(\ )=\exp\{\operatorname{tr}(\ )\},$$

using the multivariate gamma integral [1]. We observe that, since f is analytic and S is symmetric, we have

$$(3)\qquad \operatorname{etr}(-\partial XS)f(X)=f(X-S)$$

which leads directly to (1).

The definition (1) is a matrix generalization of the Weyl fractional integral [4]. Indeed, when X is a scalar we deduce from (1) that, setting $D=d/dx$,

$$(4)\qquad D^{-\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_0^\infty f(x-s)s^{\alpha-1}\,ds=\frac{1}{\Gamma(\alpha)}\int_{-\infty}^x f(y)(x-y)^{\alpha-1}\,dy$$

which is one representation of the Weyl definition of a fractional integral. Here we require $\operatorname{Re}(\alpha)>0$. The Weyl integral (4), when it exists, satisfies the familiar law of exponents $D^{-\alpha}D^{-\beta}=D^{-(\alpha+\beta)}$ for all α and β. Fractional differentiation is defined in terms of fractional integration to an appropriate order of a traditional derivative to the nearest integral order: see [4] for a full development.

Equation (1) may be used to define a fractional matrix derivative $(\det\partial X)^{\mu}$. First we assume that μ is a complex quantity for which $\operatorname{Re}(\mu)\geq-(n-1)/2$ so that (1) does not apply as it stands. Next we introduce a positive integer m and complex number α which we define by:

$$m=[\operatorname{Re}(\mu)+(n-1)/2]+1,$$

$$\operatorname{Im}(\alpha)=-\operatorname{Im}(\mu),$$

$$\operatorname{Re}(\alpha)=m-\operatorname{Re}(\mu),$$

where [ ] denotes the integral part of its argument. Then $\mu = m-\alpha$ and $\mathrm{Re}(\alpha) > (n-1)/2$. Thus the operator $(\det \partial X)^{\mu}$ may be defined by:

$$(\det \partial X)^{\mu} = (\det \partial X)^{-\alpha}[(\det \partial X)^{m} f(X)]. \tag{5}$$

With these definitions we can verify that the fractional matrix operator satisfies the usual law of exponents. Writing $D_X = \det \partial X$ we have (for $\mathrm{Re}(\alpha)$, $\mathrm{Re}(\beta) > (n-1)/2$),

$$\begin{aligned}
D_X^{-\alpha}[D_X^{-\beta} f(X)] &= D_X^{-\alpha}\frac{1}{\Gamma_n(\beta)}\int_{S>0} f(X-S)(\det S)^{\beta-(n+1)/2}\,dS \\
&= \frac{1}{\Gamma_n(\alpha)\Gamma_n(\beta)}\int_{R>0}\int_{S>0} f(X-R-S)(\det S)^{\beta-(n+1)/2} \\
&\quad\cdot dS(\det R)^{\alpha-(n+1)/2}\,dR \\
&= \frac{1}{\Gamma_n(\alpha)\Gamma_n(\beta)}\int_{M>0} f(X-M) \\
&\quad\cdot \int_{Q=0}^{M}[\det(M-Q)]^{\alpha-(n+1)/2}(\det Q)^{\beta-(n+1)/2}\,dQ \\
&= \frac{1}{\Gamma_n(\alpha)\Gamma_n(\beta)}\int_{M>0} f(X-M)(\det M)^{\alpha+\beta-(n+1)/2} \\
&\quad\cdot \int_0^I [\det(I-T)]^{\alpha-(n+1)/2}(\det T)^{\beta-(n+1)/2}\,dT \\
&= \frac{1}{\Gamma_n(\alpha+\beta)}\int_{M>0} f(X-M)(\det M)^{\alpha+\beta-(n+1)/2}\,dM \\
&= D_X^{-\alpha-\beta} f(X).
\end{aligned}$$

To prove that the law holds for general indices as well as negative indices we use the argument that, if $\mu = n-\alpha$ and $\nu = m-\beta$ with n and m integer, and if the derivatives exist, then:

$$\begin{aligned}
D_X^{\mu}[D_X^{\nu} f] &= D_X^{-\alpha} D_X^{n}[D_X^{-\beta} D_X^{m} f] = D_X^{-\alpha}[D_X^{-\beta} D_X^{n+m} f] \\
&= D_X^{-\alpha-\beta}[D_X^{n+m} f] = D_X^{\mu+\nu} f.
\end{aligned}$$

The following examples illustrate the use of the fractional matrix operator D_X on elementary functions of matrix argument:

$$D_X^{\mu}\,\mathrm{etr}(AX) = \mathrm{etr}(AX)(\det A)^{\mu}, \quad \text{all } \mu, \tag{6}$$

$$D_X^{\mu}[\det(I-X)]^{-\alpha} = \frac{\Gamma_n(\alpha+\mu)}{\Gamma_n(\alpha)}[\det(I-X)]^{-\alpha-\mu}, \tag{7}$$

$$\mathrm{Re}(\alpha) > (n-1)/2, \quad \mathrm{Re}(\alpha+\mu) > (n-1)/2,$$

$$D^{\mu}_{X\,1}F_1(\alpha,\beta;X)=\frac{\Gamma_n(\beta)\Gamma_n(\alpha+\mu)}{\Gamma_n(\alpha)\Gamma_n(\beta+\mu)}\,{}_1F_1(\alpha+\mu,\beta+\mu;X),\tag{8}$$
$$\operatorname{Re}(\alpha)>(n-1)/2,\quad \operatorname{Re}(\beta)>(n-1)/2,$$
$$\operatorname{Re}(\alpha+\mu)>(n-1)/2,\quad \operatorname{Re}(\beta+\mu)>(n-1)/2.$$

It is frequently convenient to work with the adjoint of the matrix operator $\partial/\partial X$, which we will write in the form $\partial X_a=\operatorname{adj}(\partial/\partial X)$. The fractional calculus can be extended to this operator as in (2) and (5). We define

$$(\det\partial X_a)^{-\alpha}f(X)=\frac{1}{\Gamma_n(\alpha)}\int_{S>0}\{\operatorname{etr}(-\partial X_a S)f(X)\}(\det S)^{\alpha-(n+1)/2}\,dS,\tag{9}$$
$$\operatorname{Re}(\alpha)>(n-1)/2,$$

provided the integral exists; and, for complex μ with $\operatorname{Re}(\mu)\geqslant-(n-1)/2$, we define

$$(\det\partial X_a)^{\mu}f(X)=(\det\partial X_a)^{-\alpha}[(\det\partial X_a)^{m}f(X)],\tag{10}$$

where m and α are defined as in (5) above.

In our applications of this calculus $f(X)$ is often the elementary function $\operatorname{etr}(AX)$. Simple manipulations verify that for this function

$$(\det\partial X_a)^{\mu}\operatorname{etr}(AX)=\operatorname{etr}(AX)(\det A_a)^{\mu}\tag{11}$$

where $A_a=\operatorname{adj}(A)$.

We also need to work with the operators $\det[S'(\partial X_a\otimes M)S]$ and $\det[S'(\partial X_a\Sigma\,\partial X_a\otimes M)S]$ where S is an $nm\times q$ matrix of rank q, Σ is $n\times n$ positive definite, and M is $m\times m$ positive definite. Extensions of (11) to these operators yield:

$$\begin{aligned}&\{\det[S'(\partial X_a\otimes M)S]\}^{\mu}\operatorname{etr}(AX)\\&=\operatorname{etr}(AX)\{\det[S'(A_a\otimes M)S]\}^{\mu}\\&=\operatorname{etr}(AX)\{\det[S'(A^{-1}\otimes M)S]\}^{\mu}(\det A)^{\mu q}\end{aligned}\tag{12}$$

$$\begin{aligned}&\{\det[S'(\partial X_a\Sigma\,\partial X_a\otimes M)S]\}^{\mu}\operatorname{etr}(AX)\\&=\operatorname{etr}(AX)\{\det[S'(A_a\Sigma A_a\otimes M)S]\}^{\mu}\\&=\operatorname{etr}(AX)\{\det[S'(A^{-1}\Sigma A^{-1}\otimes M)S]\}^{\mu}(\det A)^{2\mu q}.\end{aligned}\tag{13}$$

Another useful formula is:

$$\begin{aligned}&g'[S'(\partial X_a\otimes M)S][S'(\partial X_a\Sigma\,\partial X_a\otimes M)S]_a[S'(\partial X_a\otimes M)S]g\operatorname{etr}(AX)\\&\quad=\operatorname{etr}(AX)g'[S'(A_a\otimes M)S][S'(A_a\Sigma A_a\otimes M)S]_a[S'(A_a\otimes M)S]g\end{aligned}$$

where g is any $q\times 1$ vector; and repeated use of this operator yields:

$$\begin{aligned}&\{g'[S'(\partial X_a\otimes M)S][S'(\partial X_a\Sigma\,\partial X_a\otimes M)S]_a[S'(\partial X_a\otimes M)S]g\}^{\nu}\operatorname{etr}(AX)\\&\quad=\operatorname{etr}(AX)\{g'[S'(A_a\otimes M)S][S'(A_a\Sigma A_a\otimes M)S]_a[S'(A_a\otimes M)S]g\}^{\nu}.\end{aligned}\tag{14}$$

3 THE MULTIVARIATE LINEAR MODEL AND THE SUR SYSTEM

We write the multivariate linear model in the form:

$$y_t = Ax_t + u_t \qquad (t = 1, \ldots, T) \tag{15}$$

where y_t is an $n \times 1$ vector of endogenous variables, x_t is an $m \times 1$ vector of nonrandom exogenous variables, and the u_t $(t = 1, \ldots, T)$ are i.i.d. $N(0, \Sigma)$ with nonsingular covariance matrix Σ. We also write (15) as $Y' = AX' + U'$ where the data matrices are assembled in columns as in $Y' = [y_1, \ldots, y_T]$; and we assume that X has full rank m.

The coefficient matrix A in (15) is assumed to be parameterized in the form

$$\mathrm{vec}\,(A) = S\alpha - s \tag{16}$$

where vec ( ) denotes vectorization by rows, S is an $nm \times q$ matrix whose elements are known constants and whose rank is q, and s is a vector of known constants. In (16) α is taken as the $(q \times 1)$ vector of basic parameters.

The model given by (15) and (16) includes the SUR model as a special case as well as Malinvaud's general linear model [3] which allows for the same parameters to occur in more than one equation. Of course, in the SUR system S is a block diagonal selector matrix and $s = 0$. Additionally, the model (15) and explicit parameterization (16) are formally equivalent to the same model (15) with the coefficient matrix A subject to $p = mn - q$ general linear restrictions.

$$R\,\mathrm{vec}\,(A) = r \tag{17}$$

where R is a $p \times nm$ matrix of rank p and r is a $p \times 1$ vector. All of our results apply to the restricted regression model (15) and (17) upon appropriate symbolic translation. We will therefore confine our attention in what follows to the explicit parameterization (16).

The GLS estimator of α is given by

$$\hat{\alpha} = \{S'(\Sigma^{-1} \otimes X'X)S\}^{-1}\{S'(\Sigma^{-1} \otimes X')\,\mathrm{vec}\,(Y') + S'(\Sigma^{-1} \otimes X'X)s\}. \tag{18}$$

The two-stage estimator of α is obtained by replacing Σ in (18) by an estimate that is typically based on the residuals of a preliminary least squares regression on (15). We take the estimate

$$\Sigma^* = (T - m)^{-1} Y'(I - P_X)Y, \quad P_X = X(X'X)^{-1}X', \tag{19}$$

from an unrestricted regression. The corresponding two-stage estimate of α we will denote by α^*. The error in this estimate satisfies:

$$\alpha^* - \alpha = \{S'(\Sigma^{*-1} \otimes X'X)S\}^{-1}\{S'(\Sigma^{*-1} \otimes X')\,\mathrm{vec}\,(U')\}. \tag{20}$$

It is also possible to select an estimator of Σ based on a restricted preliminary regression which takes (16) into account in the first stage. The treatment of the resulting restricted estimator of α is more complicated than it is for α^*, but our methods are still applicable. In what follows we will confine our attention to the estimator α^* given by (20) and leave the additional algebraic complications of the restricted estimator to later work.

4 THE EXACT DISTRIBUTION OF α^*

Define $M=T^{-1}X'X$, $p=\mathrm{vec}\,(U'X/T)$, and $D=Y'(I-P_X)Y$. We write the error in the estimator α^* given by (20) in the generic form

$$\alpha^*-\alpha=e(p,D). \tag{21}$$

Our approach is to work with the conditional distribution of e given D and then average over the distribution of D to achieve the marginal probability density function (pdf) of e.

Since p is $N(0,\Sigma\otimes M/T)$ the conditional pdf of e given D is

$$\mathrm{pdf}\,(e|D)=\frac{T^{q/2}\exp\{-(T/2)e'[B(\Sigma\otimes M)B']^{-1}e\}}{(2\pi)^{q/2}(\det[B(\Sigma\otimes M)B'])^{1/2}} \tag{22}$$

where

$$B=[S'(D^{-1}\otimes M)S]^{-1}[S'(D^{-1}\otimes I)]. \tag{23}$$

The matrix D is central Wishart with pdf given by:

$$\mathrm{pdf}\,(D)=\frac{\mathrm{etr}\,(-\frac{1}{2}\Sigma^{-1}D)(\det D)^{(T-m-n-1)/2}}{2^{n(T-m)/2}\Gamma_n((T-m)/2)(\det\Sigma)^{(T-m)/2}}. \tag{24}$$

It follows that the unconditional pdf of e is given by the integral:

$$\begin{aligned}\mathrm{pdf}\,(e)&=\frac{T^{q/2}}{(2\pi)^{q/2}2^{n(T-m)/2}\Gamma_n((T-m)/2)(\det\Sigma)^{(T-m)/2}}\\&\quad\cdot\int_{D>0}\frac{\mathrm{etr}\,(-\frac{1}{2}\Sigma^{-1}D)\exp\{-(T/2)e'[B(\Sigma\otimes M)B']^{-1}e\}(\det D)^{(T-m-n-1)/2}\,dD}{(\det[B(\Sigma\otimes M)B'])^{1/2}}\\&=\frac{(T/2\pi)^{q/2}}{2^{n(T-m)/2}\Gamma_n((T-m)/2)(\det\Sigma)^{(T-m)/2}}\sum_{j=0}^{\infty}\frac{(-T/2)^j}{j!}\\&\quad\cdot\int_{D>0}\frac{\mathrm{etr}\,(-\frac{1}{2}\Sigma^{-1}D)(\det D)^{(T-m-n-1)/2}(e'[B(\Sigma\otimes M)B]^{-1}e)^j\,dD}{(\det[B(\Sigma\otimes M)B'])^{1/2}}\end{aligned} \tag{25}$$

where term by term integration of the series is justified by uniform convergence.

We now decompose the matrix

$$\begin{aligned}B(\Sigma\otimes M)B'&=[S'(D^{-1}\otimes M)S]^{-1}[S'(D^{-1}\Sigma D^{-1}\otimes M)S]\\&\quad\cdot[S'(D^{-1}\otimes M)S]^{-1}\\&=[S'(D_a\otimes M)S]^{-1}[S'(D_a\Sigma D_a\otimes M)S][S'(D_a\otimes M)S]^{-1}\end{aligned} \tag{26}$$

where the suffix "a" is used to indicate the adjoint of the associated matrix. The integral we need to evaluate in (25) has the following form:

$$(27)\qquad \int_{D>0} \operatorname{etr}\left(-\tfrac{1}{2}\Sigma^{-1}D\right)(\det D)^{(T-m-n-1)/2} \cdot \{e'[S'(D_a\otimes M)S][S'(D_a\Sigma D_a\otimes M)S]_a[S'(D_a\otimes M)S]e\}^j \cdot \det[S'(D_a\otimes M)S]\{\det[S'(D_a\Sigma D_a\otimes M)S]\}^{-j-1/2}\,dD.$$

We introduce an $n\times n$ matrix W of auxiliary variables and using (12), (13), and (14) we deduce that (27) equals

$$(28)\qquad \Big[\{e'[S'(\partial W_a\otimes M)S][S'(\partial W_a\Sigma\partial W_a\otimes M)S]_a[S'(\partial W_a\otimes M)S]e\}^j \cdot \det[S'(\partial W_a\otimes M)S]\{\det[S'(\partial W_a\Sigma\partial W_a\otimes M)S]\}^{-j-1/2} \cdot \int_{D>0}\operatorname{etr}[-(\tfrac{1}{2}\Sigma^{-1}-W)D](\det D)^{(T-m-n-1)/2}\,dD\Big]_{W=0}.$$

In this expression ∂W_a denotes the adjoint of the matrix operator $\partial W=\partial/\partial W$ and the fractional matrix operator that appears in (28) is defined as in (9) and (10).

We evaluate the multivariate gamma integral in (28) and from (25) we then deduce:

$$(29)\qquad \operatorname{pdf}(e)=\frac{(T/2\pi)^{q/2}}{2^{n(T-m)/2}\Gamma_n((T-m)/2)(\det\Sigma)^{(T-m)/2}} \cdot \sum_{j=0}^{\infty}\frac{(-T/2)^j}{j!}\Big[\{e'[S'(\partial W_a\otimes M)S] \cdot [S'(\partial W_a\Sigma\partial W_a\otimes M)S]_a[S'(\partial W_a\otimes M)S]e\}^j \cdot \det[S'(\partial W_a\otimes M)S]\{\det[S'(\partial W_a\Sigma\partial W_a\otimes M)S]\}^{-j-1/2} \cdot \Gamma_n\left(\frac{T-m}{2}\right)[\det(\tfrac{1}{2}\Sigma^{-1}-W)]^{-(T-m)/2}\Big]_{W=0}$$

$$(30)\qquad =\left(\frac{T}{2\pi}\right)^{q/2}\sum_{j=0}^{\infty}\frac{(-T/2)^j}{j!}[\{e'[S'(\partial W_a\otimes M)S] \cdot [S'(\partial W_a\Sigma\partial W_a\otimes M)S]_a[S'(\partial W_a\otimes M)S]e\}^j \cdot \det[S'(\partial W_a\otimes M)S]\{\det[S'(\partial W_a\Sigma\partial W_a\otimes M)S]\}^{-j-1/2} \cdot [\det(I-2\Sigma W)]^{-(T-m)/2}]_{W=0}.$$

In generalized operator notation this expression for the pdf of $e=\alpha^*-\alpha$ may be written more simply as:

$$(31)\qquad \operatorname{pdf}(e)=\frac{T^{q/2}\exp\{-(T/2)e'G(\partial W_a)^{-1}e\}}{(2\pi)^{q/2}[\det G(\partial W_a)]^{1/2}}[\det(I-2\Sigma W)]^{-(T-m)/2}|_{W=0}$$

where

$$G(\partial W_a)=[S'(\partial W_a\otimes M)S]^{-1}[S'(\partial W_a\Sigma\partial W_a\otimes M)S][S'(\partial W_a\otimes M)S]^{-1}.$$

5. MARGINAL DISTRIBUTIONS

Let F be an $f\times q$ matrix of known constants of full rank f ($<q$) and consider the marginal distribution of $g=Fe$. The joint marginal pdf of g is deduced by the same sequence of operations as those developed above for the full dimensional case. The final result corresponding to (30) is:

$$\text{(32)}\qquad \operatorname{pdf}(g)=\left(\frac{T}{2\pi}\right)^{f/2}\sum_{j=0}^{\infty}\frac{(-T/2)^j}{j!}$$
$$\cdot[\{g'[F\{S'(\partial W_a\otimes M)S\}_a\{S'(\partial W_a\Sigma\partial W_a\otimes M)S\}$$
$$\cdot\{S'(\partial W_a\otimes M)S\}_aF']_a g\}^j(\det[S'(\partial W_a\otimes M)S])^{f+2j}$$
$$\cdot(\det[F\{S'(\partial W_a\otimes M)S\}_a\{S'(\partial W_a\Sigma\partial W_a\otimes M)S\}$$
$$\cdot\{S'(\partial W_a\otimes M)S\}_aF'])^{-j-1/2}$$
$$\cdot[\det(I-2\Sigma W)]^{-(T-m)/2}]_{W=0};$$

or in generalized operator form:

$$\text{(33)}\qquad \operatorname{pdf}(g)=\frac{T^{f/2}\exp\{-(T/2)g'H(\partial W_a)^{-1}g\}}{(2\pi)^{f/2}[\det H(\partial W_a)]^{1/2}}[\det(I-2\Sigma W)]^{-(T-m)/2}|_{W=0}$$

where

$$H(\partial W_a)=F[S'(\partial W_a\otimes M)S]^{-1}[S'(\partial W_a\Sigma\partial W_a\otimes M)S]$$
$$\cdot[S'(\partial W_a\otimes M)S]^{-1}F'.$$

6. SPECIALIZATIONS

6.1. *The Unrestricted Model*

In this case $\alpha=\operatorname{vec}(A)$ and α^* is the unrestricted least squares estimator. To reduce the general expression (30) for the density in this case we note that $(\partial W_a)(\partial W_a)_a=I(\det\partial W)^{n-1}$ and thereby

$$(\partial W_a\otimes M)(\partial W_a\Sigma\partial W_a\otimes M)_a(\partial W_a\otimes M)$$
$$=(\partial W_a\otimes M)((\partial W_a)_a\Sigma_a(\partial W_a)_a\otimes M_a)(\partial W_a\otimes M)$$
$$\cdot[\det(\partial W_a\Sigma\partial W_a)]^{m-1}(\det M)^{n-1}$$
$$=(\Sigma_a\otimes MM_aM)(\det\partial W)^{2n-2}(\det\partial W_a)^{2m-2}(\det\Sigma)^{m-1}(\det M)^{n-1}$$
$$=(\Sigma^{-1}\otimes M)(\det\Sigma)^m(\det M)^n(\det\partial W)^{2m(n-1)}.$$

Additionally,

$$\det(\partial W_a \otimes M)\{\det(\partial W_a \Sigma \partial W_a \otimes M)\}^{-j-1/2}$$
$$= (\det M)^n \{\det(\Sigma \otimes M)\}^{-j-1/2} (\det \partial W_a)^{m-2m(j+1/2)}$$
$$= (\det M)^n \{\det(\Sigma \otimes M)\}^{-j-1/2} (\det \partial W)^{m(n-1)-2m(n-1)(j+1/2)}.$$

Using these reductions in (30) we deduce that:

$$\mathrm{pdf}(e) = \left(\frac{T}{2\pi}\right)^{q/2} \sum_{j=0}^{\infty} \frac{(-T/2)^j}{j!} \{e'(\Sigma^{-1} \otimes M)e\}^j$$
$$\cdot [(\det \Sigma)^{mj} (\det M)^{nj} (\det \partial W)^{2m(n-1)j} (\det M)^n$$
$$\cdot \{\det(\Sigma \otimes M)\}^{-j-1/2} (\det \partial W)^{m(n-1)-2m(n-1)(j+1/2)}$$
$$\cdot \{\det(I - 2\Sigma W)\}^{-(T-m)/2}]_{W=0}$$
$$= \left(\frac{T}{2\pi}\right)^{q/2} \sum_{j=0}^{\infty} \frac{(-T/2)^j}{j!} \{e'(\Sigma^{-1} \otimes M)e\}^j (\det \Sigma)^{-m/2} (\det M)^{n/2}$$

$$= \frac{T^{q/2} \exp\{-T/2e'(\Sigma^{-1} \otimes M)e\}}{(2\pi)^{q/2} [\det(\Sigma \otimes M^{-1})]^{1/2}}, \quad q = nm. \tag{34}$$

Thus, the joint density of $e = \alpha^* - \alpha$ reduces to the well known multivariate $N(0, \Sigma \otimes (X'X)^{-1})$.

6.2. *The Zellner Model with Pairwise Orthogonal Regressors*

We specialize the multivariate system (15) and (16) to the Zellner model

$$y_k = X_k \beta_k + u_k \qquad (k = 1, \ldots, n) \tag{35}$$

where y_k is the observation vector on the kth dependent variable, X_k is a $T \times L_k$ observation matrix on L_k regressors, β_k is a vector of coefficients, and u_k is a vector of normally distributed serially independent errors. The covariance matrix of $u' = (u_1', \ldots, u_n')$ is $\Sigma \otimes I$.

When we stack the model (35) and set $\alpha' = (\beta_1', \ldots, \beta_n')$ the Zellner SUR estimator of α is α^* and its exact distribution is given by (30). The marginal distribution of β_k^*, the SUR estimator of the subvector of the coefficients in the kth equation can be deduced directly from (32).

To relate our results to the existing literature we now assume pairwise orthogonal regressors across the equations of (35), so that $X_i'X_j = 0$ for $i \neq j$. We may concentrate on the first equation of the system without loss of generality and set $k = 1$, $e_{11} = \beta_1^* - \beta_1$, $m = \sum_{k=1}^n L_k$, and $M_{11} = T^{-1}X_1'X_1$. From the general expression (32) the joint marginal density of e_{11} is found to be:

$$\mathrm{pdf}(e_{11}) = \left(\frac{T}{2\pi}\right)^{L_1/2} (\det M_{11})^{1/2} \sum_{j=0}^{\infty} \frac{(-T/2)^j}{j!} (e_{11}' M_{11} e_{11})^j$$
$$\cdot \left[\frac{(\partial W_a)_{11}^{2j+L_1}}{(\partial W_a \Sigma\, \partial W_a)_{11}^{j+L_1/2}} [\det(I - 2\Sigma W)]^{-(T-m)/2}\right]_{W=0}. \tag{36}$$

Simple manipulations now verify that

$$(\partial W_a)_{11}^{2j+L_1}(\partial W_a\Sigma\,\partial W_a)_{11}^{-j-L_1/2}[\det(I-2\Sigma W)]^{-(T-m)/2}|_{W=0}$$

$$=\frac{1}{\Gamma_n((T-m)/2)}\int_{S>0}\operatorname{etr}(-S)(\det S)^{(T-m)/2-(n+1)/2}$$

$$\cdot(c'S^{-1}c)^{2j+L_1}(c'S^{-2}c)^{-j-L_1/2}\,dS$$

$$=\frac{(c'c)^{j+L_1/2}}{\Gamma_n((T-m)/2)}\int_{S>0}\operatorname{etr}(-S)(\det S)^{(T-m)/2-(n+1)/2}$$

$$\cdot(S^{-1})_{11}^{2j+L_1}(S^{-2})_{11}^{-j-L_1/2}\,dS$$

$$=\frac{(c'c)^{j+L_1/2}\Gamma\left(\frac{T-m}{2}+\frac{1}{2}\right)\Gamma\left(\frac{T-m}{2}+\frac{L_1}{2}-\frac{n}{2}+j+1\right)}{\Gamma\left(\frac{T-m}{2}-\frac{n}{2}+1\right)\Gamma\left(\frac{T-m}{2}+\frac{L_1}{2}+\frac{1}{2}+j\right)} \tag{37}$$

where c' denotes the first row of $\Sigma^{-1/2}$. The final step in the argument leading to (23) is an interesting exercise in integration for the reader. If we write Σ in partitioned form as

$$\Sigma=\left[\begin{array}{c|c}\sigma_{11} & \sigma_{12}\\ \hline \sigma_{21} & \Sigma_{22}\end{array}\right]$$

and set $\sigma_{11\cdot 2}=\operatorname{var}(u_{1t}|u_{2t}\cdots u_{nt})=\sigma_{11}-\sigma_{12}\Sigma_{22}^{-1}\sigma_{21}$ we find that $c'c=\sigma^{11}=(\sigma_{11}-\sigma_{12}\Sigma_{22}^{-1}\sigma_{21})^{-1}=\sigma_{11\cdot 2}^{-1}$. We may now deduce from (36) and (37) the following expression for the joint marginal density of e_{11}:

$$\operatorname{pdf}(e_{11})=\left(\frac{T}{2\pi}\right)^{L_1/2}[\det(\sigma_{11\cdot 2}M_{11}^{-1})]^{-1/2} \tag{38}$$

$$\cdot\frac{\Gamma\left(\frac{T-m}{2}+\frac{1}{2}\right)}{\Gamma\left(\frac{T-m}{2}-\frac{n}{2}+1\right)}\sum_{j=0}^{\infty}\frac{\Gamma\left(\frac{T-m}{2}+\frac{L_1}{2}-\frac{n}{2}+j+1\right)}{\Gamma\left(\frac{T-m}{2}+\frac{L_1}{2}+\frac{1}{2}+j\right)j!}$$

$$\cdot\left(-\frac{Te_{11}'M_{11}e_{11}}{2\sigma_{11\cdot 2}}\right)^j$$

$$=(T/2\pi)^{L_1/2}[\det(\sigma_{11\cdot 2}M_{11}^{-1})]^{-1/2}$$

$$\cdot\frac{\Gamma\left(\frac{T-m+1}{2}\right)\Gamma\left(\frac{T-m+L_1-n}{2}+1\right)}{\Gamma\left(\frac{T-m-n}{2}+1\right)\Gamma\left(\frac{T-m+L_1+1}{2}\right)}$$

$$\cdot{}_1F_1\left(\frac{T-m+L_1-n}{2}+1,\frac{T-m+L_1+1}{2};-\frac{Te_{11}'M_{11}e_{11}}{2\sigma_{11\cdot 2}}\right)$$

$$= (T/2\pi)^{L_1/2}[\det(\sigma_{11\,2}M_{11}^{-1})]^{-1/2}\exp(-Te_{11}'M_{11}e_{11}/2\sigma_{11\,2})$$

$$\cdot \frac{\Gamma\left(\frac{T-m+1}{2}\right)\Gamma\left(\frac{T-m+L_1-n}{2}+1\right)}{\Gamma\left(\frac{T-m-n}{2}+1\right)\Gamma\left(\frac{T-m+L_1-1}{2}\right)}$$

$$\cdot\, {}_1F_1\left(\frac{n-1}{2}, \frac{T-m+L_1+1}{2}; \frac{Te_{11}'M_{11}e_{11}}{2\sigma_{11\,2}}\right).$$

Upon translation of notation (38) is the expression found in **[2]** by direct methods.

7. FINAL REMARKS

The general formulae (30) and (32) may be used to deduce the corresponding asymptotic distributions in a simple way. We replace $[\det(I-2\Sigma W)]^{-(T-m)/2}$ in (30) or (32) by an asymptotic approximation as $W\to 0$. It is simplest to use $[\det(I-2\Sigma W)]^{-(T-m)/2} \sim \operatorname{etr}[(T-m)\Sigma W]$. Upon evaluation, we see that (30) now yields the asymptotic $N(0, T^{-1}[S'(\Sigma^{-1}\otimes M)S]^{-1})$ approximation directly. Higher order asymptotics may be obtained in a similar way although the algebra is more complicated.

The operator calculus developed in Section 2 may be applied to a variety of other unsolved problems in econometric distribution theory, including systems estimators in simultaneous equations, Stein-like estimators, and other commonly used two-stage GLS estimators. Research on some of these problems has commenced and is reported in **[5** and **7]**.

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Manuscript received November, 1983; revision received August, 1984.

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