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A theorem on the tail behaviour of probability distributions with an application to the stable family

P.C.B. PHILLIPS Yale University

Abstract. The theory of Fourier transforms of generalized functions is used to extract general formulae for the tail behaviour of a probability distribution from the behaviour of its characteristic function in the locality of the origin. The theory is applied to develop asymptotic formula for the tails of the stable distribution. The results of the present article yield immediate series representations of the densities of these distributions. In some cases (when the characteristic exponent $\alpha < 1$) the series are convergent; in others ($1 < \alpha < 2$) the series are asymptotic and thereby describe tail behaviour.

Un théorème sur le profil des extrémités des fonctions de probabilités et son application à une distribution stable. La théorie de la transformation de Fourier des fonctions généralisées est utilisée par l'auteur pour extraire, du comportement de sa fonction caractéristique dans le voisinage de l'origine, des formules générales définissant le profil des extrémités d'une fonction de probabilités. L'auteur applique cette théorie au développement de formules asymptotiques pour définir les profils des extrémités de distributions stables. Le travail développe des représentations en séries des densités de ces fonctions. Dans certains cas (quand l'exposant caractéristique $\alpha < 1$) les séries convergent; dans d'autres cas (quand $1 < \alpha < 2$), les séries sont asymptotiques et donc décrivent le profil des extrémités.

INTRODUCTION

The tail behaviour of a probability distribution is known to be closely related to the behaviour of the characteristic function of the distribution in the neighbourhood of the origin. In particular, it is well known that if the distribution has finite absolute moments to order M (where M is an integer) then the characteristic function is M times differentiable at the origin, and it admits a Taylor (asymptotic) expansion to this order in the locality of the origin. An extensive treatment of this aspect of the relationship has been given in the statistical literature, for example by Pitman (1960) and by Lukacs (1970). An excellent introduction to the subject is provided by Feller (1970).

Useful though these results are, they fall short of a full mathematical characterization of the relationship. Such a characterization is likely to be most interesting in those cases where the form of the characteristic function of a distribution is known and is

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easy to express in terms of elementary functions, whereas the density itself is either of unknown mathematical form or expressible only in terms of special functions of applied mathematics. Such cases arise frequently in problems of econometric distribution theory. They also arise more generally in economics when it becomes convenient to use explicit distributional laws. An obvious case in point is provided by the stable family of distributions. Under weak conditions this family describes the only possible limiting distributions of normal sums of stationarily dependent random variables.

In econometrics, disturbance terms are often taken to represent the combined effect of a large number of variables that are not explicit in the model itself. When the individual effects are sufficiently numerous, independent, identically distributed, and small enough, a central-limit theorem is often used to justify an assumption of normally distributed errors. If we wish to relax the assumptions of small individual errors (usually transmitted by a requirement of finite first and second moments) and independence, then the same argument leads naturally to the stable distributions as an alternative, wider family of distributions for disturbances in econometric models. Attention to the properties of econometric estimation and testing procedures under such wider distributional hypotheses has become an important concern of recent studies of robustness in econometrics, as exemplified by Bierens (1981), Koenker (1982), Andrews (1983), and the references therein. In the economics literature also, many different applications of stable distributions have been discovered, most frequently as a means of representing apparently non-normal populations that display thick tail-area behaviour. Both theoretical and empirical arguments have been put forward in support of these distributions, notably by Mandelbrot (1963a, 1963b, 1967) and Fama (1963, 1965) in connection with speculative commodity prices and stock market prices.

With the exception of a few special cases (the Cauchy, the normal, and the stable law with characteristic exponent parameter $\alpha = 1/2$) simple analytic formulae for the densities of the stable distributions are not available. This makes both analytic and empirical work within the family of stable distributions much more difficult than it is in other cases. For example, in the absence of analytic formulae, the formulation of maximum likelihood problems becomes very complex. On the other hand, a simple characteristic function formulae is known, which is general enough to include every member of the stable family (see, e.g., chap. 2 of Ibragimov and Linnik, 1971). It turns out, as will be seen in the third section of the paper, that very detailed information about the stable distributions may be extracted quite simply from this formula for the characteristic function. This information precisely describes the tail area behaviour of the distributions. In certain cases the information is even sufficient to allow for the numerical computation of the entire density. In others it will be sufficient to allow for the construction of good global approximants to the density by means of the Padé techniques explored in Phillips (1982a). Density formulae obtained in this way may then be used in statistical work at both descriptive and inferential levels. Tail area probability computations based on these formulae or the underlying series should also be directly useful in inferential work.

In the earlier article (1982a) referenced above I proved a result (theorem 4 of 1982a) which can be used to extract general formulae for the tail behaviour of a probability distribution from the behaviour of its characteristic function in the locality of the origin. The present paper reports a generalization of this result that covers a more extensive class of distributions. In particular, the results of the present paper apply to the stable family of distributions and yield immediate series representations of the densities of this family. In some cases (when the stable family exponent parameter $\alpha < 1$) the series are convergent; in others (when $1 < \alpha < 2$) the series are asymptotic (as the argument of the density $x \to \infty$) and thereby describe tail behaviour. The approach adopted here uses the theory of Fourier transforms of generalized functions. The advantage of this approach is that the formulae obtained are very general indeed and describe tail behaviour directly. In the case of the stable family this avoids the use of the lengthy derivations by which the formulae have previously been obtained, involving contour integrations that are specially tailored for individual cases.

GENERAL FORMULAE FOR THE TAILS OF A DISTRIBUTION

We let $\text{CF}(s)$ be the characteristic function of a real valued random variable. The behaviour of $\text{CF}(s)$ as $s \to 0$ is assumed to be given by the following asymptotic series:

$$\text{CF}(s) \sim e^{i\eta s}\left\{\sum_{m=0}^{M-1} p_m(is)^m + |s|^{\mu} \sum_{j=0}^{\infty} \sum_{k=0}^{K(j)} \sum_{l=0}^{L(j)} q_{jkl}|s|^{\nu j}[i \operatorname{sgn}(s))^k(\ln|s|]^l\right\},$$

where η, μ, ν, p_m, q_{jkl} are real constants and $\operatorname{sgn}(s) = 1, 0, -1$ for $s > 0, = 0, < 0$. In general, we shall find in most applications that $\mu \geq M$, $\nu > 0$, $K(j) = 0$, and $L(j) = 0$ or 1 for all j.

The representation (1) is sufficiently general to include a very wide class of distributions and should cover most distributions of practical interest in statistics. The first component in braces on the right side of (1) is analytic and ensures, when $\mu \geq M$, that integral moments of the distribution will exist to order $M - 1$ if this is an even integer and to order $M - 2$ if $M - 1$ is odd (see, e.g., Lukacs, 1970). In cases where M is finite and the distribution does not possess all its moments, the second component of (1) is important in the local behaviour of $\text{CF}(s)$ in the locality of the origin and is instrumental in determining the form of the tails of the distribution.

A simple example of (1) is given by the Cauchy distribution whose characteristic function is $\text{CF}(s) = e^{-|s|}$ with the obvious (convergent) series representation

$$\text{CF}(s) = 1 + |s| \sum_{j=1}^{\infty} [(-1)^{j+1}/(j + 1)!]|s|^j$$

so that $\eta = 0, M = 1, p_0 = 1, K(j) = L(j) = 0$ for all j, $\nu = 1$ and $q_{j00} = (-1)^{j+1}/(j + 1)!$ in (1).

A more complicated example is provided by the F distribution whose characteristic function admits an expansion of the form (1) involving powers of $\ln |s|$ (so that $L(j) \neq 0$); details of this example have been given by the author in (1982b). As a third example we shall consider in the following section the entire family of stable distributions.

The following result is an extension of theorem 4 in an earlier article by the author (1982a):

THEOREM. *If the characteristic function* CF(s) *is absolutely integrable and can be decomposed into the form*

$$CF(s) = CF_1(s) + CF_2(s) + CF_3(s),$$

where

$$CF_1(s) = e^{i\eta s} \sum_{m=0}^{M-1} p_m (is)^m$$

$$CF_2(s) = e^{i\eta s}|s|^{\mu} \sum_{j=0}^{J} \sum_{k=0}^{K(j)} \sum_{l=0}^{L(j)} q_{jkl}|s|^{\nu j}[i \text{ sgn } (s)]^k (ln|s|)^l, \ \mu \geq M, \ \nu > 0$$

$CF_3^{(j)}(s)$ *is absolutely integrable over every finite interval for* $j = 0, 1, \ldots, N$ *where* N *is the smallest integer* $\geq \mu + J\nu + 1$, *and* $CF^{(N)}(s)$ *is well-behaved at infinity (Lighthill, 1958, 49), then the corresponding probability density function* PDF(x) *has the following asymptotic expansion as* $|x| \to$:

$$PDF(x) = \frac{1}{\pi|x - \eta|^{\mu+1}} \sum_{j=0}^{J} \sum_{k=0}^{K(j)} \left[\sum_{l=0}^{L(j)} (q_{jkl}\partial^l/\partial z^l)\Gamma(z + \mu + 1)|y|^{-z} \right.$$

$$\left. \cdot \frac{1}{2} \left\{ i^k e^{-\frac{1}{2}i\pi sgn(y)(z+\mu+1)} + (-1)^k e^{\frac{1}{2}i\pi sgn(y)(z+\mu+1)} \right\} \right]_{\substack{z=j\nu \\ y=x-\eta}} + O(|x|^{-N}). \quad (2)$$

Proof. The derivation of (2) follows in an identical way the lines laid out in detail in the proof of theorem 4 in the earlier article. This proof draws on the theory of asymptotic expansions of Fourier transforms of generalized functions as developed by Lighthill (1958) and Jones (1966). In fact, Lighthill's theorem 19 on page 52 of his book provides the basis for this result. The only point of difference with the earlier proof arises in the treatment of the terms in $CF_2(s)$ involving $[i \text{ sgn } (s)]^k (\ln |s|)^l$, $(l > 1)$ which did not occur in the (1982a) paper. We shall now show how these terms may be analysed; the remainder of the argument follows as before.

We use the notation $FT_i)x)$ to denote the inverse Fourier transform of $CF_i(s)$. Since $CF_2(s)$ is not absolutely integrable, we proceed by defining it as a generalized function, and we interpret $FT_2(x)$ as the inverse transform of the generalized function $CF_2(s)$.

We now write

$$CF_2(s) = e^{i\eta s}|s|^{\mu} \sum_{j=0}^{J} \sum_{k=0}^{K(j)} \sum_{l=0}^{L(j)} q_{jkl}|s|^{\nu j}(i \text{ sgn } (s))^k (\ln |s|)^l$$

$$= e^{i\eta s} \sum_{j=0}^{J} \sum_{k=0}^{K(j)} \left[\sum_{l=0}^{L(j)} (q_{jkl}\partial^l/\partial z^l)|s|^{z+\mu}(i \text{ sgn } (s))^k \right] z = j\nu$$

$$= \lim_{t \to 0+} e^{i\eta s} \sum_{j=0}^{J} \sum_{k=0}^{K(j)} \left[\sum_{l=0}^{L(j)} (q_{jkl}\partial^l/\partial z^l)|s|^{z+\mu}(i \text{ sgn } (s))^k e^{-|s|t} \right] z = j\nu.$$

By definition we have the inverse transform:

$$\mathrm{FT}_2(x) = \lim_{t\to 0+}\left\{\frac{1}{2\pi}\sum_{j=0}^{J}\sum_{k=0}^{K(j)}\left[\sum_{l=0}^{L(j)}(q_{jkl}\partial^l/\partial z^l)\int_{-\infty}^{\infty} e^{-isx+i\eta s-|s|t}|s|^{z+\mu}(i\,\mathrm{sgn}\,(s))^k ds\right]_{z=j\nu}\right\}$$

$$= \lim_{t\to 0+}\left\{\frac{1}{2\pi}\sum_{j=0}^{J}\sum_{k=0}^{K(j)}\left[\sum_{l=0}^{L(j)}(q_{jkl}\partial^l/\partial z^l)\left(i^k\int_0^{\infty} e^{-(iy+t)s}s^{z+\mu}ds + (-i)^k\int_0^{\infty} e^{-(-iy+t)s}s^{z+\mu}ds\right)\right]_{\substack{z=j\nu\\ y=x-\eta}}\right\}$$

$$= \frac{1}{2\pi}\sum_{j=0}^{J}\sum_{k=0}^{K(j)}\left(\sum_{l=0}^{L(j)}(q_{jkl}\partial^l/\partial z^l)\left\{\lim_{t\to 0+}\Gamma(z+\mu+1)[i^k(t+iy)^{-z-\mu-1}] + (-i)^k(t-iy)^{-z-\mu-1}\right\}\right)_{\substack{z=j\nu\\ y=x-\eta}}$$

$$= \frac{1}{2\pi}\sum_{j=0}^{J}\sum_{k=0}^{K(j)}\left\{\sum_{l=0}^{L(j)}(q_{jkl}\partial l/\partial z^l)\Gamma(z+\mu+1)|y|^{-z-\mu-1}[i^k e^{-\frac{1}{2}i\pi\,\mathrm{sgn}\,(y)(z+\mu+1)} + (-i)^k e^{\frac{1}{2}i\pi\,\mathrm{sgn}\,(y)(z+\mu+1)}\right\}_{\substack{z=j\nu\\ y=x-\eta}}$$

$$= \frac{1}{\pi|x-\eta|^{\mu+1}}\sum_{j=0}^{J}\sum_{k=0}^{K(j)}\left\{\sum_{l=0}^{L(j)}(q_{jkl}\partial^l/\partial z^l)\Gamma(z+\mu+1)|y|^{-z}\cdot\frac{1}{2}\left[i^k e^{-\frac{1}{2}i\pi\,\mathrm{sgn}\,(y)(z+\mu+1)} + (-i)^k e^{\frac{1}{2}i\pi\,\mathrm{sgn}\,(y)(z+\mu+1)}\right]\right\}_{\substack{z=j\nu.\\ y=x-\eta}} \tag{3}$$

As in the (1982a) paper we now deduce that (3) provides the dominating terms in the asymptotic expansion which yields the tail behaviour of PDF(x). Specifically, we have as $x \to \infty$:

$$\mathrm{PDF}(x) = \frac{1}{\pi|x-\eta|^{\mu+1}}\sum_{j=0}^{J}\sum_{k=0}^{K(j)}\left\{\sum_{l=0}^{L(j)}(q_{jkl}\partial^l/\partial z^l)\Gamma(z+\mu+1)|y|^{-z}\cdot\frac{1}{2}\left[i^k e^{-\frac{1}{2}i\pi\,\mathrm{sgn}\,(y)(z+\mu+1)} + (-i)^k e^{\frac{1}{2}i\pi\,\mathrm{sgn}\,(y)(z+\mu+1)}\right]\right\}_{\substack{z=j\nu\\ y=x-\eta}} + 0(|x|^{-N}) \qquad \square$$

AN APPLICATION TO THE FAMILY OF STABLE DISTRIBUTIONS

If a distribution is stable, then its characteristic function must be expressible in one or other of the following two forms (see, e.g., Ibragimov and Linnik, 1971, chap. 2):

$$\mathrm{CF}(s) = \exp\{i\gamma s - c|s|^{\alpha}[\exp(-\tfrac{1}{2}i\pi K(\alpha)\beta s/|s|)]\}, \qquad \alpha \neq 1 \tag{4}$$

or

$$\text{CF}(s) = \exp\{i\gamma s - c|s|[1 - i\beta(2/\pi)(s/|s|)\ln|s|]\}, \qquad \alpha = 1 \tag{5}$$

where α, β, γ, and c are constants ($c \geq 0, 0 < \alpha \leq 2, |\beta| \leq 1$), and $K(\alpha) = 1 - |1 - \alpha|$.

Both (4) and (5) fall within the class of characteristic functions with local expansions at the origin of the form given by (1). Without loss of generality in the consideration of these distributions, we may restrict ourselves to the case in which $\gamma = 0$, $c = 1$, $x \geq 0$ (once again, see Ibragimov and Linnik, 1971, chap. 2). The behaviour of (4) is then governed by the series

$$\text{CF}(s) = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^j|s|^{j\alpha}}{j!k!}\left(-\frac{1}{2}j\pi K(\alpha)\beta\right)^k (i\,\text{sgn}\,(s))^k.$$

Setting $\mu = 0$, $\eta = 0$, $\nu = \alpha$ in (1), we then obtain directly from (2) the following asymptotic series for the probability density as $x \to \infty$:

$$\begin{aligned}\text{PDF}(x) &\sim \frac{1}{\pi x}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^j(-\frac{1}{2}j\pi K(\alpha)\beta)^k}{j!k!}\Bigg\{\Gamma(z+1)y^{-z} \\ &\qquad\cdot\frac{1}{2}\left[i^k e^{-\frac{1}{2}i\pi\,\text{sgn}\,(y)(z+1)} + (-i)^k e^{\frac{1}{2}i\pi\,\text{sgn}\,(y)(z+1)}\right]\Bigg\}_{\substack{z=j\alpha\\ y=x}} \\ &= \frac{1}{\pi x}\sum_{j=0}^{\infty}\frac{(-1)^j}{j!}\Gamma(j\alpha+1)x^{-j\alpha}\frac{1}{2}\Big(e^{-\frac{1}{2}ij\pi K(\alpha)\beta}e^{-\frac{1}{2}i\pi\,\text{sgn}\,(x)(j\alpha+1)} \\ &\qquad + e^{\frac{1}{2}ij\pi K(\alpha)\beta}e^{\frac{1}{2}i\pi\,\text{sgn}\,(x)(j\alpha+1)}\Big) \\ &= \frac{1}{\pi x}\sum_{j=0}^{\infty}\frac{(-1)^j}{j!}\Gamma(j\alpha+1)x^{-j\alpha}\left[\frac{i}{2}(e^{iz} - e^{-iz})\right]_{z=\frac{1}{2}j\pi K(\alpha)\beta+\frac{1}{2}\pi j\alpha} \\ &= \frac{1}{\pi x}\sum_{j=0}^{\infty}\frac{(-1)^{j+1}}{j!}\Gamma(j\alpha+1)\sin\left\{\frac{1}{2}\pi j(\alpha + K(\alpha)\beta)\right\}x^{-j\alpha}. \end{aligned}\tag{6}$$

This formula has been obtained by contour integration in the separate cases $\alpha < 1$ and $\alpha > 1$ by other authors (see Ibragimov and Linnik, 1971, 54–6 for a discussion). When $\alpha < 1$, the series is convergent; when $\alpha > 1$, the series is asymptotic.

When $\alpha = 1$ we have from (5) (setting $\gamma = 0$, $c = 1$):

$$\begin{aligned}\text{CF}(s) &= \exp\{-|s|[1 - \beta(2/\pi)\ln|s|(i\,\text{sgn}\,(s))]\} \\ &= \sum_{j=0}^{\infty}\sum_{k=0}^{j}\frac{(-1)^{j+k}\binom{j}{k}\left(\frac{2\beta}{\pi}\right)^k}{j!}|s|^j(\ln|s|)^k(i\,\text{sgn}\,(s))^k,\end{aligned}$$

which is also of the general form (1). We set $K(j) = j$, $l = k$, $\eta = 0$, $\mu = 0$ and $\nu = 1$ in (2), and we find the corresponding asymptotic series for the probability density as $x \to \infty$

$$\text{PDF}(x) \sim \frac{1}{\pi x}\sum_{j=0}^{\infty}\left\{\sum_{k=0}^{j}\frac{(-1)^j(-1)^k\binom{j}{k}\left(\frac{2\beta}{\pi}\right)^k}{j!}\,\partial^k/\partial z^k\Gamma(z+1)y^{-z}\right.$$
$$\left.\cdot\frac{1}{2}\left[i^k e^{-\frac{1}{2}i\pi\,\text{sgn}\,(y)(z+1)} + (-i)^k e^{\frac{1}{2}i\pi\,\text{sgn}\,(y)(z+1)}\right]\right\}_{\substack{z=j\\y=x}}$$
$$= \frac{1}{\pi x}\sum_{j=0}^{\infty}\frac{(-1)^j}{j!}\left\{\frac{1}{2}\left[\left(1-\frac{2i\beta}{\pi}\frac{\partial}{\partial z}\right)^j\Gamma(z+1)y^{-z}e^{-\frac{1}{2}i\pi(z+1)}\right.\right.$$
$$\left.\left.+\left(1+\frac{2i\beta}{\pi}\frac{\partial}{\partial z}\right)^j\Gamma(z+1)y^{-z}e^{\frac{1}{2}i\pi(z+1)}\right]\right\}_{\substack{z=j\\y=x}} \qquad (7)$$

This series representation of the density when $\alpha = 1$ appears to be new.

We note that when $\beta = 0$, (7) becomes

$$pdf(x) = \frac{1}{\pi x}\sum_{j=0}^{\infty}(-1)^j x^{-j}\,\frac{1}{2}\left[e^{-i\pi(j+1)/2} + e^{i\pi(j+1)/2}\right]$$
$$= \frac{1}{\pi x}\sum_{j=0}^{\infty}(-1)^j x^{-j}\cos(\pi(j+1)/2)$$
$$= \frac{1}{\pi x^2}\sum_{n=0}^{\infty}(-1)^n x^{-2n}$$
$$= (1/\pi x^2)(1+x^{-2})^{-1}, \qquad (8)$$

which we identify as the tail expansion for the Cauchy density $pdf(x) = [\pi(1 + x^2)]^{-1}$. In this case, therefore, the asymptotic series (7) is convergent for $x^2 > 1$; and the formula for the summed series given by (8) actually represents the density over the whole real axis, as we would expect by analytic continuation.

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