

THE EXACT DISTRIBUTION OF LIML: II*

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1. INTRODUCTION

In recent years there has been a renewed interest in the limited information maximum likelihood (LIML) method of estimation in the simultaneous equations model. Some of this interest has been directed towards modifications of LIML which yield estimators with certain improved features. The improvements have been measured by the criterion of second order asymptotic efficiency and by the extent to which the modifications thin out the tails of the distribution, thereby reducing the probability of extreme outliers in finite sample LIML estimation. Studies of such modifications have been made by Fuller [1977], Kunitomo [1981] and Morimune [1981]; and a review of this work may be found in Phillips [1983a]. A second direction of interest has involved extensive numerical tabulations of the exact distributions of competing estimators in the case of a single equation with two endogenous variables. These tabulations have led to a reassessment of the relative merits of LIML and two stage least squares (2SLS) as competing estimators. In particular, the distribution of LIML is shown to have a superior central location and a more rapid approach to its asymptotic distribution than the distribution of 2SLS. The differences working in favor of LIML are most striking when the degree of equation overidentification is large and when there is a high correlation between the endogenous regressor and the structural equation error. The reader is referred to Anderson [1982] for a detailed account of this work.

The present paper is concerned with the distribution of the LIML estimator in the general single equation case. As such, it is a sequel to an earlier paper by the author [1984a] (hereafter referred to as LIML: I) which dealt with a leading case of the general problem. The exact probability density function (p. d. f.) of the LIML estimator in an equation with two ($n+1=2$) endogenous variables, even degrees of freedom and an arbitrary degree of overidentification ($L \geq 1$) was found by Mariano and Sawa [1972].² In Basmann's [1974] notation, their result characterizes a subset of the following subclass of distributions corresponding to even degrees of freedom:

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² See Mariano and McDonald [1979] for a correction. But note that in their expression for the coefficients in their equation (5) there is a misprint in which $\left(\frac{\nu}{2}\right)$, reads, incorrectly, as an exponent.

$$\bigcup_{L=1}^{\infty} M_{1,L}$$

where $M_{n,L}$ denotes the joint distribution on R^n of the LIML estimator of the coefficients of the n right-hand side endogenous variables in an equation with L degrees of overidentification.

The results of the present paper characterize in the same notation the complete class of distributions

$$M = \bigcup_{n=1}^{\infty} \bigcup_{L=1}^{\infty} M_{n,L}$$

corresponding to a structural equation containing any number of endogenous variables, even or odd degrees of freedom and an arbitrary degree of overidentification. These results are made possible by the use of invariant polynomials of multiple matrix arguments and the deployment of a new operator calculus developed elsewhere by the author [1984b, 1984c]. Upon appropriate symbolic translation, our results also apply to the distribution of the maximum likelihood estimator in the multivariate linear functional relationship.

2. THE MODEL AND NOTATION

As in LIML: I, we work with the structural equation

$$(1) \quad y_1 = Y_2\beta + Z_1\gamma + u$$

where $y_1(T \times 1)$ and $Y_2(T \times n)$ are an observation vector and observation matrix, respectively, of $n+1$ included endogenous variables, Z_1 is a $T \times K_1$ matrix of included exogenous variables, and u is a random disturbance vector. The reduced form of (1) is written

$$(2) \quad [y_1 : Y_2] = [Z_1 : Z_2] \begin{bmatrix} \pi_{11} & \Pi_{12} \\ \pi_{21} & \Pi_{22} \end{bmatrix} + [v_1 : V_2] = Z\Pi + V,$$

where Z_2 is a $T \times K_2$ matrix of exogenous variables excluded from equation (1). The rows of the reduced form disturbance matrix V are assumed to be independent, identically distributed, normal random vectors. We assume that the standardizing transformations (see Phillips [1983a] for full details) have been carried out, so that the covariance matrix of each row of V is the identity matrix and $T^{-1}Z'Z = I_K$ where $K = K_1 + K_2$. We also assume that $K_2 \geq n+1$ so that the degree of overidentification is $L = K_2 - n \geq 1$. When the equation is just identified ($K_2 = n$) LIML reduces to indirect least squares and the exact distribution theory in Sargan [1976] and Phillips [1980] applies.

We write the LIML estimator of β in (1) as β_{LIML} and define the matrices $W = X'(P_Z - P_{Z_1})X$, $S = X'(I - P_Z)X$ where $X = [y_1 : Y_2]$ and $P_A = A(A'A)^{-1}A'$. β_{LIML} minimizes the ratio $\beta'_A W \beta_A / \beta'_A S \beta_A$, where $\beta'_A = (1, -\beta')$, and satisfies the system

$$(3) \quad (W - \lambda S)\beta_A = 0$$

where λ is the smallest latent root of the matrix $S^{-1/2}WS^{-1/2}$. β_A in (3) also satisfies

$$(4) \quad [S - f(W + S)]\beta_A = 0$$

where $f = (1 + \lambda)^{-1}$ is the largest latent root of $(W + S)^{-1/2}S(W + S)^{-1/2}$.

3. THE DISTRIBUTION OF LIML

The first steps of the derivation follow those of LIML: I. In particular, let the $m = n + 1$ roots of the equation

$$(5) \quad \det [S - f(W + S)] = 0$$

be ordered $f_1 > f_2 > \dots > f_m > 0$ and assembled into the matrix $F = \text{diag}(f_1, f_2, \dots, f_m)$. Further, let the corresponding vectors g_i satisfying $[S - f_i(W + S)]g_i = 0$ be normalized by $g_i'(W + S)g_i = 1$ and assembled into the matrix $G = [g_1, g_2, \dots, g_m]$. We set $E = G^{-1}$ and define a transformation $(S, W) \rightarrow (E, F)$ by the equations

$$(6) \quad S = E'FE, \quad W = E'(I - F)E.$$

This transformation is made one to one by the imposition of a sign requirement on a particular column of $E = (e_{ij})$. We choose the final column of E (as in LIML: I) and set $e_{i,n+1} \geq 0$ for all i .

Our distribution theory begins with the joint p. d. f. of (W, S) . W and S are independent Wishart matrices. S is $W_m(T - K, I)$ (as in LIML: I) and W is noncentral Wishart $W_m(K_2, I, \bar{M})$. The noncentrality matrix \bar{M} is given by

$$(7) \quad \begin{aligned} \bar{M} &= MM' = E(T^{-1/2}X'Z_2)E(T^{-1/2}Z_2X) \\ &= T\Pi' \begin{bmatrix} 0 \\ \dots \\ I_{K_2} \end{bmatrix} [0 : I_{K_2}]\Pi \\ &= T \begin{bmatrix} \Pi'_{21} \\ \dots \\ \Pi'_{22} \end{bmatrix} [\Pi_{21} : \Pi_{22}] \\ &= T \begin{bmatrix} \beta' \\ \dots \\ I_n \end{bmatrix} \Pi'_{22}\Pi_{22}[\beta : I_n]. \end{aligned}$$

The joint density of (W, S) is

$$(8) \quad \text{pdf}(W, S) = \frac{\text{etr}\left(-\frac{1}{2}MM'\right)\text{etr}\left\{-\frac{1}{2}(W+S)\right\}}{2^{m(T-K_1)/2}\Gamma_m\left(\frac{K_2}{2}\right)\Gamma_m\left(\frac{T-K}{2}\right)} \cdot (\det W)^{(K_2-m-1)/2}(\det S)^{(T-K-m-1)/2} {}_0F_1\left(\frac{K_2}{2}; \frac{1}{4}MM'W\right).$$

Writing $\Pi'_{22}\Pi_{22}$ as $\bar{\Pi}'_{22}\bar{\Pi}_{22}$ where $\bar{\Pi}_{22}$ is an $n \times n$ matrix we find that

$$(9) \quad \text{etr}\left(-\frac{1}{2}MM'\right) = \text{etr}\left\{-\frac{T}{2}(I+\beta\beta')\bar{\Pi}'_{22}\bar{\Pi}_{22}\right\}$$

and

$$(10) \quad {}_0F_1\left(\frac{K_2}{2}; \frac{1}{4}MM'W\right) = {}_0F_1\left(\frac{K_2}{2}; \frac{T}{4}\bar{\Pi}_{22}[\beta : I]W\begin{bmatrix} \beta' \\ \vdots \\ I \end{bmatrix}\bar{\Pi}'_{22}\right).$$

The jacobian of the transformation $(S, W) \rightarrow (E, F)$ is

$$(11) \quad 2^m |\det E|^{m+2} \prod_{i < j} (f_i - f_j)$$

(as in equation (22) of LIML: I). We deduce from (8)–(11) that

$$(12) \quad \text{pdf}(E, F) = \frac{2^m \text{etr}\left\{-\frac{T}{2}(I+\beta\beta')\bar{\Pi}'_{22}\bar{\Pi}_{22}\right\}}{2^{m(T-K_1)/2} \Gamma_m\left(\frac{K_2}{2}\right) \Gamma_m\left(\frac{T-K}{2}\right)} \\ \cdot \text{etr}\left(-\frac{1}{2}E'E\right) [\det(E'E)]^{(T-K_1)/2-(m+1)} |\det E|^{m+2} \\ \cdot (\det F)^{(T-K-m-1)/2} [\det(I-F)]^{(K_2-m-1)/2} \prod_{i < j} (f_i - f_j) \\ \cdot {}_0F_1\left(\frac{K_2}{2}; \frac{T}{4}\bar{\Pi}_{22}[\beta : I]E'(I-F)E\begin{bmatrix} \beta' \\ \vdots \\ I \end{bmatrix}\bar{\Pi}'_{22}\right).$$

We now introduce the same partition of E that is used in LIML: I (see equation (32)) and again employ the notation $\beta_{\text{LIML}} = r$:

$$E = \begin{matrix} & 1 & & n \\ \begin{matrix} 1 \\ \vdots \\ n \end{matrix} & \begin{bmatrix} e_{11} & \vdots & e'_{12} \\ \dots & \dots & \dots \\ E_{22r} & \vdots & E_{22} \end{bmatrix} \end{matrix}.$$

We also partition F conformably as

$$F = \begin{bmatrix} f_1 & \vdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \vdots & F_2 \end{bmatrix}$$

where $F_2 = \text{diag}(f_2, f_3, \dots, f_m)$. With this notation the argument of the ${}_0F_1$ function in (12) becomes

$$(13) \quad (T/4)\bar{\Pi}_{22}[(1-f_1)(e_{11}\beta + e_{12})(e_{11}\beta' + e'_{12}) \\ + (I+\beta r')E'_{22}(I-F_2)E_{22}(I+r\beta')]\bar{\Pi}'_{22}.$$

Using the series representation of ${}_0F_1$ in zonal polynomials (Constantine [1963]) and the multinomial expansion of a zonal polynomial of a sum of matrices in terms of invariant polynomials of several matrix arguments (Davis [1980, 1981] and Chikuse [1980]) we deduce that

$$\begin{aligned}
(14) \quad {}_0F_1\left(\frac{K_2}{2}; \frac{T}{4} \bar{\Pi}_{22}[\beta : I]E'(I-F)E \begin{bmatrix} \beta' \\ I \end{bmatrix} \bar{\Pi}'_{22}\right) \\
= \sum_{f=0}^{\infty} \frac{1}{f!} \sum_{\varphi} \frac{1}{\left(\frac{K_2}{2}\right)_{\varphi}} \sum_{J[2]} \frac{f! \theta_{\varphi}^{J[2]}}{j_1! j_2!} C_{\varphi}^{J[2]}\left(\frac{T}{4} (1-f_1) \bar{\Pi}_{22}(e_{11}\beta + e_{12}) \right. \\
\left. \cdot (e_{11}\beta' + e'_{12}) \bar{\Pi}'_{22}, \frac{T}{4} \bar{\Pi}_{22}(I + \beta r')E'_{22}(I - F_2)E_{22}(I + r\beta') \bar{\Pi}'_{22}\right).
\end{aligned}$$

In this expression, φ represents an ordered partition of $f=j_1+j_2$ into at most n (hereafter $\leq n$) parts and $J[2]=(J_1, J_2)$ where J_i represents an ordered partition of the nonnegative integer j_i ($i=1, 2$) into $\leq n$ parts. The notation $\varphi \in J[2]$ relates the two sets of partitions and is explained in Davis [1980, 1981]. $C_{\varphi}^{J[2]}(X, Y)$ is a polynomial in the elements of the two matrices X and Y which is invariant under the simultaneous transformation $X \rightarrow H'XH$ and $Y \rightarrow H'YH$ for any orthogonal matrix H . These invariant polynomials in two matrix arguments are developed and tabulated to low orders by Davis [1980]. The constants $\theta_{\varphi}^{J[2]}$ that appear in (14) are given by

$$(15) \quad \theta_{\varphi}^{J[2]} = C_{\varphi}^{J[2]}(I, I)/C_{\varphi}(I)$$

(Davis [1980], equation (5.1)).

Noting that $\det E = (\det E_{22})(e_{11} - e'_{12}r)$ we transform $E \rightarrow (e_{11}, e_{12}, r, E_{22})$ and find the density:

$$\begin{aligned}
(16) \quad \text{pdf}(e_{11}, e_{12}, r, E_{22}, F) &= \frac{\text{etr} \left\{ -\frac{T}{2} (I + \beta\beta') \bar{\Pi}'_{22} \bar{\Pi}_{22} \right\} e^{-\frac{1}{2}(e_{11}^2 + e_{12}^2)}}{2^{m(T-K_1)/2-m} \Gamma_m\left(\frac{K_2}{2}\right) \Gamma_m\left(\frac{T-K}{2}\right)} \\
&\cdot \text{etr} \left\{ -\frac{1}{2} (I + rr') E'_{22} E_{22} \right\} [\det(E'_{22} E_{22})]^{(T-K_1-m-1)/2+1} |e_{11} - e'_{12}r|^{T-K_1-m} \\
&\cdot (\det F)^{(T-K-m-1)/2} [\det(I-F)]^{(K_2-m-1)/2} \prod_{i < j} (f_i - f_j) \\
&\cdot \sum_{f=0}^{\infty} \sum_{\varphi} \frac{1}{\left(\frac{K_2}{2}\right)_{\varphi}} \sum_{J[2]} \frac{\theta_{\varphi}^{J[2]}}{j_1! j_2!} (1-f_1)^{j_1} \\
&\cdot C_{\varphi}^{J[2]}\left(\frac{T}{4} \bar{\Pi}_{22}(e_{11}\beta + e_{12})(e_{11}\beta' + e'_{12}) \bar{\Pi}'_{22}, \frac{T}{4} \bar{\Pi}_{22}(I + \beta r')E'_{22} \right. \\
&\quad \left. \cdot (I - F_2)E_{22}(I + r\beta') \bar{\Pi}'_{22}\right).
\end{aligned}$$

The joint density of r is found by integrating out the surplus variates in (16). To facilitate the first step in this reduction we introduce a random orthogonal matrix H whose distribution is uniform over the orthogonal group $O(n)$. In the new joint distribution of the variates in (16) and H we then transform $E_{22} \rightarrow H'E_{22} = D_{22}$ and subsequently integrate out H over $O(n)$ where $O(n)$ is normalized so that

the measure over the whole group is unity. In performing this step we utilize the following integral, where (dH) denotes the normalized invariant measure on $O(n)$:

$$(17) \quad \int_{O(n)} C_{\varphi}^{\kappa, \lambda}(A'H'XHA, B)(dH) = C_{\varphi}^{\kappa, \lambda}(A'A, B)C_{\kappa}(X)/C_{\kappa}(I)$$

(Davis [1980], equation (5.13)). We find

$$(18) \quad \text{pdf}(e_{11}, e_{12}, r, D_{22}, F) \\ = \frac{\text{etr} \left\{ -\frac{T}{2}(I + \beta\beta')\bar{\Pi}'_{22}\bar{\Pi}_{22} \right\} e^{-\frac{1}{2}(e'_{11} + e'_{12}e_{12})}}{2^{m(T-K_1)/2-m} \Gamma_m\left(\frac{K_2}{2}\right) \Gamma_m\left(\frac{T-K}{2}\right)} \\ \cdot \text{etr} \left\{ -\frac{1}{2}(I + rr')D'_{22}D_{22} \right\} [\det(D'_{22}D_{22})]^{(T-K_1-m+1)/2} |e_{11} - e'_{12}r|^{T-K_1-m} \\ \cdot (\det F)^{(T-K-m-1)/2} [\det(I-F)]^{(K_2-m-1)/2} \prod_{i < j} (f_i - f_j) \\ \cdot \sum_{j=0}^{\infty} \sum_{\varphi} \frac{1}{\left(\frac{K_2}{2}\right)_{\varphi}} \sum_{\substack{J_1^{[2]} \\ (\varphi \in J_1^{[2]})}} \frac{\theta_{\varphi}^{J_1^{[2]}}}{j_1! j_2!} (1-f_1)^{j_1} \\ \cdot C_{\varphi}^{J_1^{[2]}} \left(\frac{T}{4} \bar{\Pi}_{22}(e_{11}\beta + e_{12})(e_{11}\beta' + e'_{12})\bar{\Pi}'_{22}, \frac{T}{4} \bar{\Pi}_{22}(I + \beta r')D'_{22}D_{22} \right. \\ \left. \cdot (I + r\beta')\bar{\Pi}' \right) C_{J_2}(I - F_2)/C_{J_2}(I).$$

The next step is to integrate F out of (18). The required integral is:

$$(19) \quad \int_F (1-f_1)^{j_1} (\det F)^{(T-K-m-1)/2} [\det(I-F)]^{(K_2-m-1)/2} C_{J_2}(I - F_2) \prod_{i < j} (f_i - f_j) dF.$$

where the region of integration is $0 < f_m < \dots < f_1 < 1$. First we transform $F_2 \rightarrow \bar{F}_2$ by defining $F_2 = f_1 \bar{F}_2$. The jacobian is f_1^m and (19) becomes

$$(20) \quad \int_0^1 f_1^{m(T-K)/2-1} (1-f_1)^{(K_2-m-1)/2+j_1} \int_{\bar{F}_2} \\ \cdot (\det \bar{F}_2)^{(T-K-m-1)/2} \det(I - \bar{F}_2) [\det(I - f_1 \bar{F}_2)]^{(K_2-m-1)/2} \\ \cdot C_{J_2}(I - f_1 \bar{F}_2) \prod_{2 \leq i < j} (\bar{f}_i - \bar{f}_j) d\bar{F}_2 df_1$$

where $\bar{F}_2 = \text{diag}(\bar{f}_2, \dots, \bar{f}_m)$ and the region of integration of \bar{F}_2 is $1 > \bar{f}_2 > \bar{f}_3 > \dots > \bar{f}_m > 0$. We write out the expansions:

$$(21) \quad [\det(I - f_1 \bar{F}_2)]^{(K_2-m-1)/2} = \sum_{J_3} \sum_{J_3} \frac{\binom{-K_2+m+1}{2}_{J_3}}{j_3!} C_{J_3}(f_1 \bar{F}_2)$$

where the summation over j_3 is finite if $K_2 - m - 1$ is even and infinite otherwise, and where J_3 is an ordered partition of j_3 with $\leq n$ parts; and

$$(22) \quad C_{J_2}(I - f_1 \bar{F}_2) = C_{J_2}(I) \sum_{j_4=0}^{j_2} \sum_{J_4} \binom{J_2}{J_4} C_{J_4}(-f_1 \bar{F}_2) / C_{J_4}(I)$$

where J_4 is a partition of j_4 with $\leq n$ parts and $\binom{J_2}{J_4}$ denotes the generalized binomial coefficient introduced by Constantine [1966]. The product formula

$$(23) \quad C_{J_3}(f_1 \bar{F}_2) C_{J_4}(-f_1 \bar{F}_2) = (-1)^{J_4} f_1^{j_3 + j_4} \sum_{(J_5 \in J_3 \cdot J_4)} (\theta_{J_5}^{j_3, J_4})^2 C_{J_5}(\bar{F}_2)$$

(Davis [1980], equation (5.10)) enables us to write (20) in the form

$$(24) \quad C_{J_2}(I) \sum_{J_3} \sum_{J_3} \frac{\binom{-K_2 + m + 1}{2}}{j_3!} \sum_{j_4=0}^{j_2} \sum_{J_4} \binom{J_2}{J_4} (-1)^{J_4} \sum_{(J_5 \in J_3 \cdot J_4)} (\theta_{J_5}^{j_3, J_4})^2 \cdot \int_0^1 f_1^{m(T-K)/2 + j_3 + j_4 - 1} (1 - f_1)^{(K_2 - m - 1)/2 + j_1} df_1 \cdot \int_{\bar{F}_2} (\det \bar{F}_2)^{(T-K-m-1)/2} [\det(I - \bar{F}_2)] C_{J_5}(\bar{F}_2) \prod_{i < j} (\bar{f}_i - \bar{f}_j) d\bar{F}_2 / C_{J_4}(I).$$

To evaluate the integral over \bar{F}_2 we note that

$$(25) \quad \int_0^I (\det R)^{a-(n+1)/2} [\det(I - R)]^{b-(n+1)/2} C_J(R) dR = \frac{\Gamma_n(a, J) \Gamma_n(b)}{\Gamma_n(a+b, J)} C_J(I)$$

where the integration is over all positive definite R for which $0 < R < I$ (Constantine [1963], Theorem 3). In (25)

$$(26) \quad \Gamma_n(a, J) = \pi^{\frac{1}{4}n(n-1)} \prod_{i=1}^n \left(a + j_i - \frac{1}{2}(i-1) \right)$$

for the partition $J = (j_1, j_2, \dots, j_n)$. Following the approach used in LIML: I (equations (25)–(28)), we deduce from (25) a corresponding integral in terms of the latent roots ($1 > r_1 > \dots > r_n > 0$) of R :

$$(27) \quad \int_R (\prod_i r_i)^{a-(n+1)/2} (\prod_i (1 - r_i))^{b-(n+1)/2} C_J(R) \prod_{i < j} (r_i - r_j) \prod_i dr_i = \frac{\Gamma_n(a, J) \Gamma_n(b) C_J(I) \Gamma_n\left(\frac{n}{2}\right)}{\Gamma_n(a+b, J) \pi^{n^2/2}}.$$

This integral now may be used to reduce (24), leading in fact to

$$\begin{aligned}
(28) \quad & C_{J_2}(I) \sum_{i_3} \sum_{J_3} \frac{\binom{-K_2+m+1}{2}_{J_3}}{j_3!} \sum_{j_4=0}^{J_2} \sum_{J_4} \binom{J_2}{J_4} (-1)^{j_4} \sum_{(J_3 \in J_3, J_4)} (\theta_{J_3, J_4}^{j_3, j_4})^2 \\
& \cdot B\left(m \frac{T-K}{2} + j_3 + j_4, \frac{K_2}{2} - \frac{m+1}{2} + j_1 + 1\right) \\
& \cdot \frac{\Gamma_n\left(\frac{T-K}{2} - \frac{1}{2}, J_3\right) \Gamma_n\left(\frac{n+1}{2} + 1\right) \Gamma_n\left(\frac{n}{2}\right) C_{J_3}(I)}{\Gamma_n\left(\frac{T-K+n+2}{2}, J_3\right) \pi^{n^2/2} C_{J_4}(I)} \\
& = C_{J_2}(I) \omega_n(j_1, J_2), \text{ say.}
\end{aligned}$$

From this expression and (18) we deduce

$$\begin{aligned}
(29) \quad & \text{pdf}(e_{11}, e_{12}, r, D_{22}) \\
& = \frac{\text{etr}\left\{-\frac{T}{2}(I + \beta\beta') \bar{\Pi}'_{22} \bar{\Pi}_{22}\right\} e^{-(e'_{11} + e'_{12} e_{12})/2}}{2^{m(T-K_1)/2-m} \Gamma_m\left(\frac{K_2}{2}\right) \Gamma_m\left(\frac{T-K}{2}\right)} \\
& \cdot \text{etr}\left\{-\frac{1}{2}(I + rr') D'_{22} D_{22}\right\} [\det(D'_{22} D_{22})]^{(T-K_1-m+1)/2} |e_{11} - e'_{12} r|^{T-K_1-m} \\
& \cdot \sum_{j=0}^{\infty} \sum_{\phi} \frac{1}{\binom{K_2}{2}_{\phi}} \sum_{(j \in J[2])} \frac{\theta_{\phi}^{j[2]}}{j_1! j_2!} \omega_n(j_1, J_2) \\
& \cdot C_{\phi}^{j[2]}\left(\frac{T}{4} \bar{\Pi}_{22}(e_{11}\beta + e_{12}) (e_{11}\beta' + e'_{12}) \bar{\Pi}'_{22}, \frac{T}{4} \bar{\Pi}_{22}(I + \beta r') D'_{22}\right. \\
& \quad \left. \cdot D_{22}(I + r\beta') \bar{\Pi}_{22}\right).
\end{aligned}$$

We transform $D_{22} \rightarrow (H, D)$, where H is orthogonal and $D = D'_{22} D_{22}$ according to the unique decomposition $D_{22} = HD^{1/2}$. The measure changes according to (see equation (34) of LIML: I)

$$(30) \quad dD_{22} = 2^{-n} (\det D)^{-1/2} dD(dH)$$

where (dH) is the invariant measure on $O(n)$. Hence, by integrating over H and using (James [1954])

$$(31) \quad \text{vol}[O(n)] = \int_{O(n)} (dH) = \frac{2^n \pi^{n^2/2}}{\Gamma_n\left(\frac{n}{2}\right)}$$

we obtain

$$\begin{aligned}
(32) \quad & \text{pdf}(e_{11}, e_{12}, r, D) \\
&= \frac{2^n \pi^{n^2/2} \text{etr} \left\{ -\frac{T}{2} (I + \beta \beta') \bar{\Pi}'_{22} \bar{\Pi}_{22} \right\} e^{-(e_{11}^2 + e'_{12} e_{12})/2}}{2^{m(T-K_1)/2-m+2n} \Gamma_m \left(\frac{K_2}{2} \right) \Gamma_m \left(\frac{T-K}{2} \right) \Gamma_n \left(\frac{n}{2} \right)} \\
&\cdot \text{etr} \left[-\frac{1}{2} (I + rr') D \right] (\det D)^{(T-K_1-m)/2} |e_{11} - e'_{12} r|^{T-K_1-m} \\
&\cdot \sum_{f=0}^{\infty} \sum_{\varphi} \frac{1}{\left(\frac{K_2}{2} \right)_{\varphi}} \sum_{\substack{J_1^{[2]} \\ (\varphi \in J_1^{[2]})}} \frac{\theta_{\varphi}^{J_1^{[2]}}}{J_1! J_2!} \omega_n(j_1, J_2) \\
&\cdot C_{\varphi}^{J_1^{[2]}} \left(\frac{T}{4} \bar{\Pi}_{22} (e_{11} \beta + e_{12}) (e_{11} \beta' + e'_{12}) \bar{\Pi}'_{22}, \frac{T}{4} \bar{\Pi}_{22} (I + \beta r') D (I + r \beta') \bar{\Pi}'_{22} \right)
\end{aligned}$$

where the additional factor $(1/2^n)$ arises because of the original sign restriction on the final column of E_{22} which is now relaxed. To integrate out D we use the following result (where x is an $n \times 1$ vector and X is an $n \times n$ matrix):

$$\begin{aligned}
(33) \quad & \int_{D>0} \text{etr} \left\{ -\frac{1}{2} (I + rr') D \right\} (\det D)^{(T-K_1-m)/2} C_{\varphi}^{J_1, J_2} (xx', XDX') dD \\
&= \Gamma_n \left(\frac{T-K_1}{2}, J_2 \right) \left[\det \left\{ \frac{1}{2} (I + rr') \right\} \right]^{-(T-K_1)/2} C_{\varphi}^{J_1, J_2} (xx', 2X(I + rr')^{-1} X')
\end{aligned}$$

which can be deduced from Davis [1980], equation (5.14). In (33) the explicit notation J_1, J_2 replaces $J[2]$ in (32). Additionally, since xx' has only one nonzero latent root, the right side of (33) may be written in the form

$$\begin{aligned}
(34) \quad & 2^{n(T-K_1)/2} \Gamma_n \left(\frac{T-K_1}{2}, J_2 \right) [\det (I + rr')]^{-(T-K_1)/2} C_{\varphi}^{[j_1], J_2} \\
&\quad \cdot (xx', 2X(I + rr')^{-1} X')
\end{aligned}$$

where $[j_1]$ denotes the partition of j_1 with leading part j_1 i. e. $\{j_1, 0, \dots, 0\}$. We now deduce

$$\begin{aligned}
(35) \quad & \text{pdf}(e_{11}, e_{12}, r) \\
&= \frac{\pi^{n^2/2} \text{etr} \left\{ -\frac{T}{2} (I + \beta \beta') \bar{\Pi}'_{22} \bar{\Pi}_{22} \right\} e^{-(e_{11}^2 + e'_{12} e_{12})/2}}{2^{(T-K_1)/2-1} \Gamma_n \left(\frac{n}{2} \right) \Gamma_m \left(\frac{K_2}{2} \right) \Gamma_m \left(\frac{T-K}{2} \right) [\det (I + rr')]^{(T-K_1)/2}} \\
&\cdot |e_{11} - e'_{12} r|^{T-K_1-m} \sum_{f=0}^{\infty} \sum_{\varphi} \frac{1}{\left(\frac{K_2}{2} \right)_{\varphi}} \sum_{\substack{J_1, J_2 \\ (\varphi \in J_1, J_2)}} \frac{\theta_{\varphi}^{j_1, J_2}}{J_1! J_2!} \omega_n(j_1, J_2) \Gamma_n \left(\frac{T-K_1}{2}, J_2 \right) \\
&\cdot C_{\varphi}^{[j_1], J_2} \left(\frac{T}{4} \bar{\Pi}_{22} (e_{11} \beta + e_{12}) (e_{11} \beta' + e'_{12}) \bar{\Pi}'_{22}, \frac{T}{2} \bar{\Pi}_{22} (I + \beta r') (I + rr')^{-1} \right. \\
&\quad \left. \cdot (I + r \beta') \bar{\Pi}'_{22} \right).
\end{aligned}$$

In what follows we define $q' = (e_{11}, e_{22})$, $a' = (1, r')$ and $g = T - K_1 - m$. Then a typical term of (35) is a constant multiple of the following product:

$$(36) \quad e^{-q'q/2} |a'q|^g C_{\phi}^{[j_1, j_2]} \left(\frac{T}{4} \bar{\Pi}_{22} [\beta : I] q q' \begin{bmatrix} \beta' \\ I \end{bmatrix} \bar{\Pi}'_{22}, \frac{T}{2} \bar{\Pi}_{22} (I + \beta r') (I + r r')^{-1} \cdot (I + r \beta') \bar{\Pi}'_{22} \right).$$

We now define a vector $d = a/(a'a)^{1/2}$ and construct an orthogonal matrix $D = [d, d_2, \dots, d_m]$. Under the transformation $q \rightarrow D'q = p$ the density pdf(q, r) becomes:

$$(37) \quad \text{pdf}(p, r) = \frac{\pi^{n^2/2} \text{etr} \left\{ -\frac{T}{2} (I + \beta \beta') \bar{\Pi}'_{22} \bar{\Pi}_{22} \right\}}{2^{(T-K_1)/2-1} \Gamma_n \left(\frac{n}{2} \right) \Gamma_m \left(\frac{K_2}{2} \right) \Gamma_m \left(\frac{T-K}{2} \right) (1+r'r)^{(T-K_1)/2}} \\ \cdot e^{-p'p/2} (a'a)^{g/2} |p_1|^g \sum_{j=0}^{\infty} \sum_{\phi} \frac{1}{\left(\frac{K_2}{2} \right)_{\phi}} \sum_{(j_1, j_2)_{\phi}} \frac{\theta_{\phi}^{[j_1, j_2]}}{j_1! j_2!} \omega_n(j_1, j_2) \Gamma_n \left(\frac{T-K_1}{2}, j_2 \right) \\ \cdot C_{\phi}^{[j_1, j_2]} \left(\frac{T}{4} \bar{\Pi}_{22} B D(r) p p' D(r)' B' \bar{\Pi}'_{22}, \frac{T}{2} \bar{\Pi}_{22} (I + \beta r') (I + r r')^{-1} (I + r \beta') \bar{\Pi}'_{22} \right)$$

where $B = [\beta : I_n]$ and we write $D = D(r)$ to emphasize the dependence of D on r .

We note that since $C_{\phi}^{[j_1, j_2]}$ is a polynomial in the elements of its argument matrices we have the equivalence

$$(38) \quad C_{\phi}^{[j_1, j_2]}(X p p' X', Y) = C_{\phi}^{[j_1, j_2]}(X \partial x \partial x' X', Y) \exp(p'x)|_{x=0}$$

where ∂x denotes the vector operator $\partial/\partial x$. To reduce (37), we may work first with the integral

$$(39) \quad 2^{-1} \int e^{-p'p/2 + p'x} |p_1|^g dp$$

since the order of differentiation and integration may be interchanged in view of the uniform convergence of the integral. The domain of integration in (39) is taken to be unrestricted (i.e. $-\infty < p_i < \infty$ all i) while that of q in (36) satisfies the restriction $0 < q_m < \infty$ and this explains the presence of the factor $1/2$ in (39). We partition p and x conformably as $p' = (p_1, p_2)$, $x' = (x_1, x_2)$ and integrate p_2 out of (39) giving:

$$(40) \quad 2^{-1} (2\pi)^{n/2} e^{x_2' x_2 / 2} \int_{-\infty}^{\infty} e^{-p_1^2 + p_1 x_1} |p_1|^g dp_1 \\ = 2^{-1} (2\pi)^{n/2} e^{x_2' x_2 / 2} \sum_{k=0}^{\infty} \frac{x_1^k}{k!} \int_{-\infty}^{\infty} e^{-p_1^2 / 2} |p_1|^g p_1^k dp_1.$$

Term by term integration is permissible here because the series is uniformly convergent. In fact, only even terms in the series are nonvanishing and we obtain:

$$\begin{aligned}
(41) \quad & (2\pi)^{n/2} e^{x_2' x_2 / 2} \sum_{l=0}^{\infty} \frac{x_1^{2l}}{(2l)!} \int_0^{\infty} e^{-p_1^2 / 2} p_1^{g+2l} dp_1 \\
& = (2\pi)^{n/2} e^{x_2' x_2 / 2} \sum_{l=0}^{\infty} \frac{x_1^{2l}}{(2l)!} \frac{\Gamma\left(\frac{g+2l+1}{2}\right)}{\left(\frac{1}{2}\right)^{(g+2l-1)/2}} \\
& = (2\pi)^{n/2} e^{x_2' x_2 / 2} 2^{(g-1)/2} \Gamma\left(\frac{g+1}{2}\right) \sum_{l=0}^{\infty} \frac{(2x_1^2)^l \left(\frac{g+1}{2}\right)_l}{(2l)!}.
\end{aligned}$$

Since $(2l)! = \pi^{-1/2} 2^{2l} \Gamma(l+1/2) \Gamma(l+1)$ by the duplication formula we may write (41) in the form

$$\begin{aligned}
(42) \quad & (2\pi)^{n/2} e^{x_2' x_2 / 2} 2^{(g-1)/2} \Gamma\left(\frac{g+1}{2}\right) \sum_{l=0}^{\infty} \frac{(2x_1^2)^l \left(\frac{g+1}{2}\right)_l}{l! 2^{2l} \left(\frac{1}{2}\right)_l} \\
& = (2\pi)^{n/2} e^{x_2' x_2 / 2} 2^{(g-1)/2} \Gamma\left(\frac{g+1}{2}\right) {}_1F_1\left(\frac{g+1}{2}, \frac{1}{2}; \frac{x_1^2}{2}\right).
\end{aligned}$$

Noting that $(a'a)^{g/2} = (1+r'r)^{(T-K_1-n-1)/2}$ we now deduce from (37), (38), (39) and (42) that

$$\begin{aligned}
(43) \quad \text{pdf}(r) & = \frac{\pi^{nm/2} \text{etr}\left\{-\frac{T}{2}(I+\beta\beta')\bar{\Pi}'_{22}\bar{\Pi}_{22}\right\} \Gamma\left(\frac{T-K_1-n}{2}\right)}{\Gamma_n\left(\frac{n}{2}\right) \Gamma_m\left(\frac{K_2}{2}\right) \Gamma_m\left(\frac{T-K}{2}\right) (1+r'r)^{(n+1)/2}} \\
& \cdot \sum_{f=0}^{\infty} \sum_{\phi} \frac{1}{\left(\frac{K_2}{2}\right)_{\phi}} \sum_{\substack{j_1, j_2 \\ (\phi \in j_1, j_2)}} \frac{\theta_{\phi}^{j_1, j_2}}{j_1! j_2!} \omega_n(j_1, j_2) \Gamma_n\left(\frac{T-K_1}{2}, j_2\right) \\
& \cdot \left[C_{\phi}^{j_1, j_2} \left(\frac{T}{4} \bar{\Pi}_{22} B D(r) \partial x \partial x' D(r)' B' \bar{\Pi}'_{22}\right) \right. \\
& \left. \frac{T}{2} \bar{\Pi}_{22} (I+\beta r') (I+r'r)^{-1} (I+r\beta') \bar{\Pi}'_{22} \right] e^{x_2' x_2 / 2} {}_1F_1\left(\frac{T-K_1-n}{2}, \frac{1}{2}; \frac{x_1^2}{2}\right) \Big|_{x=0}.
\end{aligned}$$

4. THE LEADING CASE

When $\bar{\Pi}_{22}=0$ each term in the series (43) is zero with the exception of the leading term, for which $f=j_1=j_2=0$. The general expression for the joint density reduces in this case to:

$$\begin{aligned}
(44) \quad \text{pdf}(r) & = \frac{\pi^{nm/2} \Gamma_n\left(\frac{T-K_1}{2}\right) \Gamma\left(\frac{T-K_1-n}{2}\right) \omega_n(0, 0)}{\Gamma_n\left(\frac{n}{2}\right) \Gamma_m\left(\frac{K_2}{2}\right) \Gamma_m\left(\frac{T-K}{2}\right) (1+r'r)^{(n+1)/2}} \\
& \propto \frac{1}{(1+r'r)^{(n+1)/2}}.
\end{aligned}$$

Thus, β_{LIML} has a multivariate Cauchy distribution in this leading case as shown by direct methods in LIML: I.

The reduction of the constant coefficient in (44) presents some difficulties when $n > 1$ in view of the multiple series expression for $\omega_n(0, 0)$, viz from (28):

$$(45) \quad \omega_n(0, 0) = \pi^{-n^2/2} \Gamma_n\left(\frac{m}{2} + 1\right) \Gamma_n\left(\frac{n}{2}\right) \sum_{j_3} \sum_{j_3} \frac{\left(\frac{-K_2+m+1}{2}\right)_{j_3}}{j_3!} \\ \cdot B\left(m \frac{T-K}{2} + j_3, \frac{K_2}{2} - \frac{n}{2}\right) \cdot \frac{\Gamma_n\left(\frac{T-K-1}{2}; J_3\right)}{\Gamma_n\left(\frac{T-K+n}{2} + 1; J_3\right)} C_{J_3}(I).$$

When $n=1$, direct evaluation of the series can be achieved as follows:

$$(46) \quad \omega_1(0,0) = \pi^{-1/2} \frac{\Gamma(2) \Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{T-K}{2} - \frac{1}{2}\right)}{\Gamma\left(\frac{T-K}{2} + \frac{3}{2}\right)} \sum_{j_3} \frac{\left(\frac{-K_2+3}{2}\right)_{j_3}}{j_3!} \\ \cdot \frac{\left(\frac{T-K}{2} - \frac{1}{2}\right)_{j_3}}{\left(\frac{T-K}{2} + \frac{3}{2}\right)_{j_3}} B\left(T-K+j_3, \frac{K_2}{2} - \frac{1}{2}\right) \\ = \frac{\Gamma\left(\frac{T-K}{2} - \frac{1}{2}\right) \Gamma\left(\frac{K_2}{2} - \frac{1}{2}\right) \Gamma(T-K)}{\Gamma\left(\frac{T-K}{2} + \frac{3}{2}\right) \Gamma\left(T-K + \frac{K_2}{2} - \frac{1}{2}\right)} \sum_{j_3} \frac{\left(\frac{-K_2+3}{2}\right)_{j_3}}{j_3!} \\ \cdot \frac{(T-K)_{j_3} \left(\frac{T-K}{2} - \frac{1}{2}\right)_{j_3}}{\left(T-K + \frac{K_2}{2} - \frac{1}{2}\right)_{j_3} \left(\frac{T-K}{2} + \frac{3}{2}\right)_{j_3}} \\ = \frac{\Gamma\left(\frac{T-K}{2} - \frac{1}{2}\right) \Gamma\left(\frac{K_2}{2} - \frac{1}{2}\right) \Gamma(T-K)}{\Gamma\left(\frac{T-K}{2} + \frac{3}{2}\right) \Gamma\left(T-K + \frac{K_2}{2} - \frac{1}{2}\right)} {}_3F_2 \\ \cdot \left(T-K, \frac{T-K}{2} - \frac{1}{2}, \frac{-K_2+3}{2}; \frac{T-K+3}{2}, T-K + \frac{K_2}{2} - \frac{1}{2}; 1\right).$$

The ${}_3F_2$ series is well poised (see, for example, Rainville [1963] page 92) and therefore, sums to:

$$(47) \quad \frac{\Gamma\left(\frac{T-K}{2} + 1\right) \Gamma\left(\frac{T-K}{2} + \frac{3}{2}\right) \Gamma\left(T-K + \frac{K_2}{2} - \frac{1}{2}\right) \Gamma\left(\frac{K_2}{2}\right)}{\Gamma(T-K+1) \Gamma\left(\frac{3}{2}\right) \Gamma\left(\frac{T-K_1}{2} - \frac{1}{2}\right) \Gamma\left(\frac{T-K_1}{2}\right)}.$$

We deduce that

Thus, β_{LIML} has a multivariate Cauchy distribution in this leading case as shown by direct methods in LIML: I.

The reduction of the constant coefficient in (44) presents some difficulties when $n > 1$ in view of the multiple series expression for $\omega_n(0, 0)$, viz from (28):

$$(45) \quad \omega_n(0, 0) = \pi^{-n^2/2} \Gamma_n\left(\frac{m}{2} + 1\right) \Gamma_n\left(\frac{n}{2}\right) \sum_{j_3} \sum_{j_3} \frac{\left(\frac{-K_2+m+1}{2}\right)_{j_3}}{j_3!} \\ \cdot B\left(m \frac{T-K}{2} + j_3, \frac{K_2}{2} - \frac{n}{2}\right) \cdot \frac{\Gamma_n\left(\frac{T-K-1}{2}; J_3\right)}{\Gamma_n\left(\frac{T-K+n}{2} + 1; J_3\right)} C_{J_3}(I).$$

When $n = 1$, direct evaluation of the series can be achieved as follows:

$$(46) \quad \omega_1(0, 0) = \pi^{-1/2} \frac{\Gamma(2) \Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{T-K}{2} - \frac{1}{2}\right)}{\Gamma\left(\frac{T-K}{2} + \frac{3}{2}\right)} \sum_{j_3} \frac{\left(\frac{-K_2+3}{2}\right)_{j_3}}{j_3!} \\ \cdot \frac{\left(\frac{T-K}{2} - \frac{1}{2}\right)_{j_3}}{\left(\frac{T-K}{2} + \frac{3}{2}\right)_{j_3}} B\left(T-K+j_3, \frac{K_2}{2} - \frac{1}{2}\right) \\ = \frac{\Gamma\left(\frac{T-K}{2} - \frac{1}{2}\right) \Gamma\left(\frac{K_2}{2} - \frac{1}{2}\right) \Gamma(T-K)}{\Gamma\left(\frac{T-K}{2} + \frac{3}{2}\right) \Gamma\left(T-K + \frac{K_2}{2} - \frac{1}{2}\right)} \sum_{j_3} \frac{\left(\frac{-K_2+3}{2}\right)_{j_3}}{j_3!} \\ \cdot \frac{(T-K)_{j_3} \left(\frac{T-K}{2} - \frac{1}{2}\right)_{j_3}}{\left(T-K + \frac{K_2}{2} - \frac{1}{2}\right)_{j_3} \left(\frac{T-K}{2} + \frac{3}{2}\right)_{j_3}} \\ = \frac{\Gamma\left(\frac{T-K}{2} - \frac{1}{2}\right) \Gamma\left(\frac{K_2}{2} - \frac{1}{2}\right) \Gamma(T-K)}{\Gamma\left(\frac{T-K}{2} + \frac{3}{2}\right) \Gamma\left(T-K + \frac{K_2}{2} - \frac{1}{2}\right)} {}_3F_2 \\ \cdot \left(T-K, \frac{T-K}{2} - \frac{1}{2}, \frac{-K_2+3}{2}; \frac{T-K+3}{2}, T-K + \frac{K_2}{2} - \frac{1}{2}; 1\right).$$

The ${}_3F_2$ series is well poised (see, for example, Rainville [1963] page 92) and therefore, sums to:

$$(47) \quad \frac{\Gamma\left(\frac{T-K}{2} + 1\right) \Gamma\left(\frac{T-K}{2} + \frac{3}{2}\right) \Gamma\left(T-K + \frac{K_2}{2} - \frac{1}{2}\right) \Gamma\left(\frac{K_2}{2}\right)}{\Gamma(T-K+1) \Gamma\left(\frac{3}{2}\right) \Gamma\left(\frac{T-K_1}{2} - \frac{1}{2}\right) \Gamma\left(\frac{T-K_1}{2}\right)}.$$

We deduce that

$$(48) \quad \omega_1(0, 0) = \frac{\Gamma\left(\frac{T-K}{2} - \frac{1}{2}\right)\Gamma\left(\frac{K_2}{2} - \frac{1}{2}\right)\Gamma(T-K)\Gamma\left(\frac{T-K}{2} + 1\right)}{\Gamma\left(\frac{T-K}{2} + \frac{3}{2}\right)\Gamma(T-K+1)\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{T-K_1}{2} - \frac{1}{2}\right)} \cdot \frac{\Gamma\left(\frac{T-K}{2} + \frac{3}{2}\right)\Gamma\left(\frac{K_2}{2}\right)}{\Gamma\left(\frac{T-K_1}{2}\right)}.$$

It now follows from (44) and (48) that when $n=1$

$$(49) \quad \text{pdf}(r) = \frac{\pi\Gamma\left(\frac{T-K}{2} - \frac{1}{2}\right)\Gamma\left(\frac{K_2}{2} - \frac{1}{2}\right)\Gamma(T-K)\Gamma\left(\frac{T-K}{2} + 1\right)\Gamma\left(\frac{K_2}{2}\right)}{(1/2)\pi\Gamma_2\left(\frac{K_2}{2}\right)\Gamma_2\left(\frac{T-K}{2}\right)\Gamma(T-K+1)(1+r^2)}.$$

Note that

$$\Gamma_2\left(\frac{K_2}{2}\right) = \pi^{1/2}\Gamma\left(\frac{K_2}{2}\right)\Gamma\left(\frac{K_2}{2} - \frac{1}{2}\right)$$

$$\Gamma_2\left(\frac{T-K}{2}\right) = \pi^{1/2}\Gamma\left(\frac{T-K}{2}\right)\Gamma\left(\frac{T-K}{2} - \frac{1}{2}\right)$$

so that

$$\text{pdf}(r) = \frac{2\Gamma(T-K)\Gamma\left(\frac{T-K}{2} + 1\right)}{\pi\Gamma(T-K+1)\Gamma\left(\frac{T-K}{2}\right)(1+r^2)}.$$

Finally, using the duplication formula for $\Gamma(z)$ we deduce that

$$\frac{\Gamma(T-K)}{\Gamma(T-K+1)} = \frac{1}{2} \frac{\Gamma\left(\frac{T-K}{2}\right)}{\Gamma\left(\frac{T-K}{2} + 1\right)}.$$

Hence,

$$(50) \quad \text{pdf}(r) = \frac{1}{\pi(1+r^2)},$$

the univariate Cauchy distribution which we obtained by other means in LIML: I.

5. FINAL REMARKS

The exact theory for the LIML estimator developed in this paper has application beyond the realm of the simultaneous equations model. There is a well-known formal mathematical equivalence between the model studied here and the linear

functional relationship in mathematical statistics. This equivalence has been explored in detail by Anderson [1976], who applied the distribution theory for the LIML estimator (in the two endogenous variable case) within the setting of a linear functional relationship involving only two variables. Upon appropriate symbolic translation, our distribution theory for the LIML estimator in the general case may be applied directly to the maximum likelihood estimator in the multivariate linear functional relationship. Thus, our results generalize the presently known exact theory in the latter setting as well as the simultaneous equations model.

The numerical implementation of the general expression for the joint density given by (43) is hampered by computational difficulties at present. First, multiple series involving matrix argument polynomials like (43) are often very slow to converge, particularly when the argument matrices have some large latent roots. Second, available tabulations of the polynomials and constants that appear in (43) are currently limited to low orders (involving only single digits in most cases); and algorithms for their generation are only in the incipient stages of development. (A recent discussion of these issues has been given by the author elsewhere [1983b].) Computational work with the series (43) in its general form, therefore, must await improvements in technology which enhance computational speed and the development of general purpose software for the algorithmic generation of the required polynomials and constants. In the meantime, these practical shortcomings of the general exact theory increase the present value of the leading case analyses developed in LIML: I.

The exact distribution theory given here and in my earlier paper LIML: I provide results which apply not only in finite samples but also go beyond conventional asymptotic theory in the sense that they provide a limiting distribution theory in cases where conventional asymptotic methods break down. Presently known limiting distribution theory for LIML and other simultaneous equations estimators proceeds from the assumption that the model is identified; this assumption is critical to the mechanics of traditional asymptotic methods. But, as is well known (and contrary to some textbook analyses), it is not necessary that a structural equation be identified for that equation to be estimated by simultaneous equations methods. Moreover, strong arguments have recently been advanced which suggest that the restrictions in simultaneous equations models are often only *apparently* identifying or overidentifying the structures of these models in practice. The analysis of underidentified models is, therefore, of some considerable interest in itself and possibly of greater relevance than has earlier been thought. When an equation is completely underidentified ($\Pi_{22}=0$), our analysis shows that simultaneous equations methods like LIML have distributions which continue, even for infinitely large samples, to represent the uncertainty about the coefficients that is implicit in their lack of identification. In particular, the distribution of LIML is multivariate Cauchy and this distribution is invariant to the size of the sample. Thus, the limiting distribution of LIML is in this case multivariate Cauchy also. In this sense, therefore, our results provide a limiting distribution theory for LIML (and other estimators in LIML: I) which correctly applies to

underidentified models, unlike existing asymptotic theory. More general cases in which Π_{22} has less than full rank may also be considered and yield a greater variety of finite sample and asymptotic results.

Finally, it may be worth mentioning the effect of departures from the assumption of nonrandom exogenous variables. Observations on these variables affect the distribution (43) through the parameter matrix $\bar{\Pi}_{22}$. When the exogenous variables are random but statistically independent of the reduced form errors (43) provides the conditional distribution given Z . Note that when the reduced form coefficient submatrix $\Pi_{22}=0$, (43) reduces to a multivariate Cauchy distribution as we have seen. Since this distribution is independent of Z , it is also the unconditional distribution. Thus, our leading case analyses here and in LIML: I also to models with independent random exogenous variables. The standardizing transformations which transform $TZ'Z \rightarrow I$ do not affect these results since the LIML (and IV) estimators of the endogenous variable coefficients are invariant to these transformations (Phillips [1983a]). We may also note that in the leading case both LIML and instrumental variable estimators of β are invariant to the scale parameter of the reduced form. It is easy to deduce that the leading case densities we have obtained for these estimators remain valid for error distributions within the spherically symmetric class of compound normal distributions.

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