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The Distribution of Matrix Quotients

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Cramér's inversion formula for the distribution of a quotient is generalized to matrix variates and applied to give an alternative derivation of the matrix t-distribution. © 1985 Academic Press, Inc

1. INTRODUCTION

Useful inversion formulae which apply for scalar ratios of random variates and which proceed from the joint characteristic function of the component variates have been known for some time. In particular, if the scalar random variate $\eta \geqslant 0$ and has a finite mean, Cramér [1] and Geary [3] give the following formula for the density of the ratio $\zeta = \xi/\eta$:

$$f(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \left[\frac{\partial \phi(s_1, s_2)}{\partial s_2} \right]_{s_2 = -zs_1} ds_1 \tag{1}$$

where $\phi(s_1, s_2)$ is the joint characteristic function of (ξ, η). Gurland [4] generalized (1) by considering the multidimensional case of a vector of ratios and relaxed the requirements that η necessarily be positive or have a finite mean (by using principal values in the integrals that define the inversions).

Closely related statistics that take the form of matrix quotients arise frequently in multivariate analysis. A common situation (leading, for example, to the matrix t-distribution) is the following. Let A be a positive definite $n \times n$ matrix variate partitioned as

$$A = \overset{\begin{matrix} l & \quad k \end{matrix}}{\begin{bmatrix} A_{11} & A_{21}' \\ A_{21} & A_{22} \end{bmatrix}} \begin{matrix} l \\ k \end{matrix}, \qquad l + k = n \tag{2}$$

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and suppose interest centers on the distribution of the quotient $X = A_{22}^{-1}A_{21}$. For example, when A is central Wishart with degrees of freedom $T > k$, X is a regression coefficient matrix for multivariate normal samples and is known to have a matrix t-distribution [2, 6, 7].

In developing the analogue of the Cramér formula for matrix variates such as X, the following notation is convenient. Corresponding to the symmetric matrix $F = (f_{ij})_{n \times n}$ we define the matrix ${}_\eta F = (\eta_{ij} f_{ij})$ where $\eta_{ij} = 1$, $1/2$ for $i = j$, $i \neq j$ respectively. We denote the joint characteristic function of A by $\phi^*({}_\eta F) = E\{\operatorname{etr}(i_\eta FA)\}$. Partitioning F and ${}_\eta F$ conformably with A in (2), we define

$$\begin{aligned}\phi(F_{21}, {}_\eta F_{22}) &= E\{\operatorname{etr}(iF_{21}A_{21}' + i_\eta F_{22}A_{22})\} \\ &= E\{\operatorname{etr}(2i_\eta F_{21}A_{21}' + i_\eta F_{22}A_{22})\} \\ &= [\phi^*({}_\eta F)]_{{}_\eta F_{11}=0},\end{aligned} \tag{3}$$

which is the joint characteristic function of the distinct elements of (A_{21}, A_{22}). With this notation we develop an inversion formula for the joint density of the matrix quotient $X = A_{22}^{-1}A_{21}$ which generalizes (1) above.

THEOREM. *Suppose the joint density function $f(A_{21}, A_{22})$ of (A_{21}, A_{22}) exists everywhere and A_{22} is a positive definite matrix. Then, if $E(\det A_{22})^l$ exists, the density function of $X = A_{22}^{-1}A_{21}$ is given by*

$$f(X) = \left(\frac{1}{2\pi i}\right)^{kl} \int_{\mathbb{R}^{kl}} [D_{22}^l \phi(F_{21}, -(XF_{21}' + F_{21}X')/2)]\, dF_{21} \tag{4}$$

where D_{22} is the differential operator $\det(\partial/\partial F_{22}) = \det[(\partial/\partial f_{rs})]$ *where f_{rs} denotes the (r, s)th element of F_{22}.*

2. Proof of the Theorem

By direct transformation of $(A_{21}, A_{22}) \to (X, A_{22})$ we deduce that

$$f(X) = \int_{A_{22}>0} f(A_{22}X, A_{22})(\det A_{22})^l\, dA_{22}. \tag{5}$$

We observe that the joint density

$$f^*(A_{21}, A_{22}) = [E(\det A_{22})^l]^{-1}(\det A_{22})^l f(A_{21}, A_{22}) \tag{6}$$

defines a new distribution whose characteristic function $E\{\operatorname{etr}(iF_{21}A'_{21} + i_\eta F_{22}A_{22})\}$ is given by

$$[E(\det A_{22})^l]^{-1} \int \operatorname{etr}(iF_{21}A'_{21} + i_\eta F_{22}A_{22})(\det A_{22})^l f(A_{21}, A_{22})\, dA_{21}\, dA_{22}$$

$$= [E(\det A_{22})^l]^{-1} \int D_{22}^l \operatorname{etr}(iF_{21}A'_{21} + i_\eta F_{22}A_{22}) f(A_{21}, A_{22})\, dA_{21}\, dA_{22}/i^{kl}$$

$$= [E(\det A_{22})^l]^{-1} D_{22}^l \phi(F_{21}, {}_\eta F_{22})/i^{kl} \tag{7}$$

where the absolute convergence of the integral allows us to interchange the order of integration and differentiation.

Next consider the distribution of the matrix variate $W = A_{21} - A_{22}X$ given X where the joint distribution of (A_{21}, A_{22}) is defined by (6). The density of W is

$$f(W) = \int_{A_{22}>0} f^*(W + A_{22}X, A_{22})\, dA_{22}. \tag{8}$$

From (5) and (6) we see that $f(W)$ reduces to $[E(\det A_{22})^l]^{-1} f(X)$ when $W = 0$. We further note that the characteristic function of W is obtained by setting ${}_\eta F_{22} = -\frac{1}{2}(XF'_{21} + F_{21}X')$ in (7), that is

$$[E(\det A_{22})^l]^{-1} D_{22}^l \phi(F_{21}, {}_\eta F_{22}) \big|_{{}_\eta F_{22} = -\frac{1}{2}(XF'_{21} + F_{21}X')}. \tag{9}$$

The required formula (4) for the density function $f(X)$ now follows from inversion of the characteristic function (9) and from taking its value at $W = 0$.

3. Application to the Matrix t-Distribution

Consider the canonical case of a central Wishart matrix A with degrees of freedom T and covariance matrix I_n. The joint characteristic function of A is

$$\phi^*({}_\eta F) = [\det(I - 2i_\eta F)]^{-T/2}$$

and simple manipulations yield

$$\phi(F_{21}, {}_\eta F_{22}) = [\det(I - 2i_\eta F_{22} + F_{21}F'_{21})]^{-T/2}. \tag{10}$$

Moreover,

$$D_{22}^{l}\phi(F_{21}, {}_{\eta}F_{22})$$

$$= D_{22}^{l}\int_{S>0} \operatorname{etr}\{-S(I - 2i_{\eta}F_{22} + F_{21}F_{21}')\}$$

$$\times (\det S)^{T/2-(k+1)/2}\, dS/\Gamma_{n}(T/2)$$

$$= (2i)^{kl}[\Gamma_{k}(T/2)]^{-1}\int_{S>0} \operatorname{etr}\{-S(I - 2i_{\eta}F_{22} + F_{21}F_{21}')\}$$

$$\times (\det S)^{T/2+l-(k+1)/2}\, dS$$

$$= (2i)^{kl}(\Gamma_{k}(T/2+l)/\Gamma_{k}(T/2))[\det(I - 2i_{\eta}F_{22} + F_{21}F_{21}')]^{-T/2-l}. \tag{11}$$

Substitution of (11) in (4) leads to the following expression for the joint density of $X = A_{22}^{-1}A_{21}$:

$$f(X) = \frac{\Gamma_{k}(T/2+l)}{\pi^{kl}\Gamma_{k}(T/2)}\int_{\mathbb{R}^{kl}} [\det\{I - i(XF_{21}' + F_{21}X') + F_{21}F_{21}'\}]^{-T/2-l}\, dF_{21}$$

$$= \frac{\Gamma_{k}(T/2+l)}{\pi^{kl}\Gamma_{k}(T/2)}[\det(I + XX')]^{-T/2-l}$$

$$\cdot \int_{\mathbb{R}^{kl}} [\det\{I - (I + XX')^{-1/2}(X + iF_{21})(X + iF_{21})'$$

$$\cdot (I + XX')^{-1/2}\}]^{-T/2-l}\, dF_{21}. \tag{12}$$

We transform $F_{21} \to Z = (I + XX')^{-1/2}(F_{21} - iX)$ in the integral in (12). This transformation has Jacobian $[\det(I + XX')]^{l/2}$ and we deduce that

$$f(X) = \frac{\Gamma_{k}(T/2+l)}{\pi^{kl}\Gamma_{k}(T/2)}[\det(I + XX')]^{-(T+l)/2}\int_{\mathbb{R}^{kl}} [\det(I + ZZ')]^{-T/2-l}\, dZ. \tag{13}$$

The domain of integration in (13) can be taken to be $\mathbb{R}^{kl}$ as before, since the integrand is analytic in a strip of $\mathbb{C}^{kl}$ that contains $\mathbb{R}^{kl} - i(I + XX')^{-1/2}X$. In view of the integral

$$\int_{\mathbb{R}^{kl}} [\det(I + ZZ')]^{-T/2-l}\, dZ = \frac{\pi^{kl/2}\Gamma_{k}(T/2 + l/2)}{\Gamma_{k}(T/2+l)}$$

(for example, [2, p. 512]) it follows that the joint density of $X = A_{22}^{-1}A_{21}$ is

$$f(X) = \frac{\Gamma_{k}(T/2 + l/2)}{\pi^{kl/2}\Gamma_{k}(T/2)}[\det(I + XX')]^{-(T+l)/2}.$$

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