Cowles Foundation Paper 602

Journal of Econometrics 26 (1984) 387–398. North-Holland

THE EXACT DISTRIBUTION OF EXOGENOUS VARIABLE COEFFICIENT ESTIMATORS

P.C.B. PHILLIPS*

Cowles Foundation, and Yale University, New Haven, CT 06520, USA

Received July 1983, final version received March 1984

This paper derives the exact probability density function of the instrumental variable (IV) estimator of the exogenous variable coefficient vector in a structural equation containing $n+1$ endogenous variables and N degrees of overidentification. The derivations make use of an operator calculus which simplifies the algebra of invariant polynomials with multiple matrix arguments. A leading case of the general distribution that is more amenable to analysis and computation is also presented. Conventional classical assumptions of normally distributed errors and non-random exogenous variables are employed.

1. Introduction

Substantial progress has been made in recent years on the exact distribution theory of econometric estimators and test statistics in simultaneous equations models. The latest results cover general specifications of single-equation estimation which allow for the presence of any number of endogenous variables and an arbitrary degree of (either apparent or effective) equation overidentification. Thus, in earlier papers, the author (1980, 1983a, 1983b) has given the exact distributions of the instrumental variable (IV) and limited information maximum likelihood (LIML) estimators in this general setting; Rhodes (1981) extracted the exact density of the limited information identifiability test statistic; and Hillier, Kinal and Srivastava (1983) have provided exact moment formulae for the marginal distributions of the IV estimator. For a recent review of these and other developments in the field the reader is referred to Phillips (1983c).

The structural equation distribution theory cited above concentrates on the estimated coefficients of the endogenous variables. This is natural because these coefficients form the nucleus of the simultaneity problem and are therefore our primary concern. The coefficients of the exogenous variables are also an important, if subsidiary, component in the study of structural estimation. So far, our knowledge of the distribution of the estimated exogenous

*I am grateful to the referees for comments which helped to simplify the presentation of the results of this paper. My thanks also go to Glena Ames for her skill and effort in typing the manuscript and to the NSF for research support under grant number SES 8218792

variable coefficients comes from the moment formulae that can be deduced from the equations that define these estimators in terms of the estimated endogenous variable coefficients [see, for instance, Phillips (1983c)]. But the exogenous variable coefficients gain in significance in the transition from structure to reduced form. And an understanding of the distribution of these estimated coefficients provides an important stepping stone to the study of the estimated reduced forms.

The present paper derives the exact probability density function (p.d.f.) of a general IV estimator of the exogenous variable coefficient vector in the single-equation setting. Conventional assumptions of normally distributed errors and non-random exogenous variables are employed. A leading case is presented in section 3.

2. The model and notation

We work with the structural equation

$$y_1 = Y_2\beta + Z_1\gamma + u, \tag{1}$$

where y_1 $(T\times 1)$ and Y_2 $(T\times n)$ are an observation vector and observation matrix, respectively, of $n+1$ included endogenous variables; Z_1 is a $T\times K_1$ matrix of included exogenous variables; and u is a random disturbance vector. The reduced form of (1) is written

$$[y_1 \vdots Y_2] = [Z_1 \vdots Z_2]\begin{bmatrix}\pi_{11} & \Pi_{12}\\ \pi_{21} & \Pi_{22}\end{bmatrix} + [v_1 \vdots V_2] = Z\Pi + V, \tag{2}$$

where Z_2 is a $T\times K_2$ matrix of exogenous variables excluded from (1). The rows of the reduced form matrix V are assumed to be independent identically distributed normal random vectors. We assume that standardizing transformations have been carried out so that the covariance matrix of each row of V is the identity matrix and $T^{-1}Z'Z = I_K$ where $K = K_1 + K_2$. These transformations involve no loss of generality and their effect on the parameterization and resulting estimator distributions are fully discussed in Phillips (1983c). We assume that $K_2 \geq n$ and denote the degree of overidentification by $N = K_2 - n$. Finally we note that the relationship between (1) and (2) and the implied restrictions on (1) yield the equations

$$\pi_{11} - \Pi_{12}\beta = \gamma, \qquad \pi_{21} - \Pi_{22}\beta = 0. \tag{3}$$

We define $H = [Z_1 \vdots Z_3]$, where Z_3 $(T\times K_3)$ is a submatrix of Z_2 and $K_3 \geq n$. The IV estimators of β and γ in (1) obtained by using H as the matrix of instruments are

$$\beta_{IV} = \left(Y_2'Z_3Z_3'Y_2\right)^{-1}\left(Y_2'Z_3Z_3'y_1\right), \tag{4}$$

$$\gamma_{IV} = T^{-1}Z_1'y_1 - T^{-1}Z_1'Y_2\beta_{IV}. \tag{5}$$

The number of surplus instruments in this estimation is denoted by $L = K_3 - n$.

3. The leading case

We define the matrix variate

$$[b\,\vdots\,B] = \left[T^{-1}Z_1'y_1\,\vdots\,T^{-1}Z_1'Y_2\right] = T^{-1}Z_1'X, \tag{6}$$

which is normal with mean matrix $[\pi_{11}\,\vdots\,\Pi_{12}]$ and covariance matrix $I_{K_1(n+1)}$. From eq. (B3) in Phillips (1980) we know the p.d.f. of β_{IV} to be (using the notation $\beta_{IV} = r$)

$$\mathrm{pdf}(r) = \frac{\mathrm{etr}\left\{-\frac{T}{2}(I+\beta\beta')\overline{\Pi}_{22}'\overline{\Pi}_{22}\right\}\Gamma_n\left(\frac{L+n+1}{2}\right)}{\pi^{n/2}(1+r'r)^{(L+n+1)/2}}$$

$$\times \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{\varphi\in j\cdot\kappa} \frac{\left(\frac{L}{2}\right)_j\left(\frac{L+n+1}{2}\right)_\kappa \theta_\varphi^{j,\kappa}}{j!k!\Gamma_n\left(\frac{L+n}{2},\varphi\right)} C_\varphi^{j,\kappa}$$

$$\times\left(\frac{T}{2}\overline{\Pi}_{22}\beta\beta'\overline{\Pi}_{22}', \frac{T}{2}\overline{\Pi}_{22}(I+\beta r')(I+rr')^{-1}(I+r\beta')\overline{\Pi}_{22}'\right). \tag{7}$$

In this expression $C_\varphi^{j,\kappa}$ is an invariant polynomial in the elements of its two argument matrices. Such polynomials were introduced by Davis (1979, 1980) to extend the zonal polynomials and the reader is referred to his articles for a detailed presentation of their properties, together with a definition of the constants $\theta_\varphi^{j,\kappa}$ that appear in (7). φ is a partition of the integer $f = j + k$ into $\leq n$ parts, κ is a partition of k into $\leq n$ parts and the notation $\varphi \in (j, \kappa)$ which is defined by Davis (1979) relates the two sets of partitions in the summation. The matrix $\overline{\Pi}_{22}$ in (7) depends only on the submatrix Π_{22} of reduced form coefficients; it is defined in Phillips (1980).

We write γ_{IV}, as given by (5), in the form $s = b - Br$. The matrix variate (b, B) is independent of r since $Z_1'Z_3 = 0$. Thus, the conditional distribution of

s, given r, is $N(\pi_{11} - \Pi_{12}r, (1+r'r)I)$. The joint distribution of (s, r) may therefore be written as[1]

$$\begin{aligned}
\mathrm{pdf}(s,r) &= \mathrm{pdf}(s|r)\mathrm{pdf}(r)\\
&= [2\pi(1+r'r)]^{-K_1/2}\\
&\quad \times \exp\{-\tfrac{1}{2}(s-\pi_{11}+\Pi_{12}r)'(s-\pi_{11}+\Pi_{12}r)\\
&\quad /(1+r'r)\}\mathrm{pdf}(r)\\
&= [2\pi(1+r'r)]^{-K_1/2}\exp\{-\tfrac{1}{2}(s-\pi_{11})'(s-\pi_{11})\}\\
&\quad \times \exp\{\tfrac{1}{2}[(s-\pi_{11})'(s-\pi_{11})r'r - 2r'\Pi_{12}'(s-\pi_{11})\\
&\quad - r'\Pi_{12}'\Pi_{12}r]/(1+r'r)\}\mathrm{pdf}(r).
\end{aligned} \tag{8}$$

We now examine the leading case that is characterized by the null hypothesis

$$\mathrm{H}_0:\quad \Pi_{12}=0,\quad \Pi_{22}=0. \tag{9}$$

Under H_0, the rank condition for identification of (1) fails, the parameter vector β is no longer identifiable and estimation by IV proceeds under conditions of only apparent overidentification. Additionally, (3) reduces to $\pi_{11} = \gamma$. Moreover, the density (7) takes on the simpler form

$$\mathrm{pdf}(r) = \frac{\Gamma\left(\dfrac{L+n+1}{2}\right)}{\pi^{n/2}\Gamma\left(\dfrac{L+1}{2}\right)(1+r'r)^{(L+n+1)/2}}, \tag{10}$$

[1] I am grateful to the referee for pointing out this derivation of (8) which is simpler than that presented in the first version of this paper.

and (8) becomes

$$\operatorname{pdf}(s,r)=\frac{\Gamma\left(\frac{L+n+1}{2}\right)\exp\{-\frac{1}{2}(s-\gamma)'(s-\gamma)\}\times\exp\{\frac{1}{2}(s-\gamma)'(s-\gamma)r'r/(1+r'r)\}}{(2\pi)^{K_1/2}\pi^{n/2}\Gamma\left(\frac{L+1}{2}\right)(1+r'r)^{(K_1+L+n+1)/2}} \tag{11}$$

$$=\frac{\Gamma\left(\frac{L+n+1}{2}\right)\exp\{-\frac{1}{2}(s-\gamma)'(s-\gamma)\}}{(2\pi)^{K_1/2}\pi^{n/2}\Gamma\left(\frac{L+1}{2}\right)(1+r'r)^{(K_1+L+n+1)/2}}\times\sum_{j=0}^{\infty}\frac{\{\frac{1}{2}(s-\gamma)'(s-\gamma)\}^j(r'r)^j}{j!(1+r'r)^j}. \tag{12}$$

We transform $r\to(m,h)$ according to the decomposition $r=(r'r)^{\frac{1}{2}}(r/(r'r)^{\frac{1}{2}})=m^{\frac{1}{2}}h$. The measure changes in accordance with the relation

$$\mathrm{d}r=2^{-1}m^{(n-2)/2}\,\mathrm{d}m(\mathrm{d}h), \tag{13}$$

where $(\mathrm{d}h)$ denotes the invariant measure over the Stiefel manifold $V_{1,n}$ [see James (1954, eq. (8.19))]. With this transformation we integrate out (m,h) and extract the density of s as follows:

$$\operatorname{pdf}(s)=\frac{\Gamma\left(\frac{L+n+1}{2}\right)\exp\{-\frac{1}{2}(s-\gamma)'(s-\gamma)\}}{(2\pi)^{K_1/2}\pi^{n/2}\Gamma\left(\frac{L+1}{2}\right)}\times\sum_{j=0}^{\infty}\frac{\{\frac{1}{2}(s-\gamma)'(s-\gamma)\}^j}{2j!}\times\int_0^{\infty}\frac{m^{n/2+j-1}\,\mathrm{d}m}{(1+m)^{(L+n+1+K_1)/2+j}}\int_{V_{1,n}}(\mathrm{d}h) \tag{14}$$

$$=\frac{\Gamma\left(\frac{L+n+1}{2}\right)\exp\{-\frac{1}{2}(s-\gamma)'(s-\gamma)\}}{(2\pi)^{K_1/2}\pi^{n/2}\Gamma\left(\frac{L+1}{2}\right)}\times\sum_{j=0}^{\infty}\frac{\{\frac{1}{2}(s-\gamma)'(s-\gamma)\}^j}{2j!}$$

$$\times \frac{\Gamma\left(\frac{n}{2}+j\right)\Gamma\left(\frac{L+K_1+1}{2}\right)}{\Gamma\left(\frac{L+K_1+n+1}{2}+j\right)} \frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)}$$

$$= \frac{\Gamma\left(\frac{L+n+1}{2}\right)\Gamma\left(\frac{L+K_1+1}{2}\right)\exp\{-\frac{1}{2}(s-\gamma)'(s-\gamma)\}}{(2\pi)^{K_1/2}\Gamma\left(\frac{L+1}{2}\right)\Gamma\left(\frac{L+K_1+n+1}{2}\right)}$$

$$\times {}_1F_1\left(\frac{n}{2}, \frac{n+L+K_1+1}{2}; \tfrac{1}{2}(s-\gamma)'(s-\gamma)\right). \qquad (15)$$

It is simple to verify that the density given by (15) integrates to unity. In considering the order to which moments exist it is convenient to set $\gamma = 0$ and examine the convergence of the following series of positive terms:

$$\mathrm{E}\{(s's)^{d/2}\} = \frac{\Gamma\left(\frac{L+n+1}{2}\right)\Gamma\left(\frac{L+K_1+1}{2}\right)}{(2\pi)^{K_1/2}\Gamma\left(\frac{L+1}{2}\right)\Gamma\left(\frac{L+K_1+n+1}{2}\right)}$$

$$\times \sum_{j=0}^{\infty} \frac{\left(\frac{n}{2}\right)_j (\frac{1}{2})^j}{j!\left(\frac{n+L+K_1+1}{2}\right)_j} \int \mathrm{e}^{-s's/2}(s's)^{j+d/2}\,\mathrm{d}s$$

$$= \frac{\Gamma\left(\frac{L+n+1}{2}\right)\Gamma\left(\frac{L+K_1+1}{2}\right)}{(2\pi)^{K_1/2}\Gamma\left(\frac{L+1}{2}\right)\Gamma\left(\frac{L+K_1+n+1}{2}\right)}$$

$$\times \sum_{j=0}^{\infty} \frac{\left(\frac{n}{2}\right)_j (\frac{1}{2})^j}{j!\left(\frac{n+L+K_1+1}{2}\right)_j} \frac{\Gamma\left(\frac{d+K_1}{2}+j\right)2\pi^{K_1/2}}{2(\frac{1}{2})^{j+(d+K_1)/2}\Gamma\left(\frac{K_1}{2}\right)}$$

$$= \frac{\Gamma\left(\frac{L+n+1}{2}\right)\Gamma\left(\frac{L+K_1+1}{2}\right)\Gamma\left(\frac{d+K_1}{2}\right)2^{d/2}}{\Gamma\left(\frac{L+1}{2}\right)\Gamma\left(\frac{L+K_1+n+1}{2}\right)\Gamma\left(\frac{K_1}{2}\right)}$$

$$\times {}_2F_1\left(\frac{n}{2}, \frac{d+K_1}{2}, \frac{n+L+K_1+1}{2}; 1\right). \qquad (16)$$

The series converges absolutely provided $d \le L$ and diverges otherwise. Thus, integer moments of γ_{IV} are finite up to the number of surplus instruments $L = K_3 - n$ (or, in the case of 2SLS, to the degree of overidentification), as we know from earlier results on β_{IV} and from the form of (5).

4. The general case

From (7) and (8) the joint p.d.f. of (s, r) under the alternative hypothesis $\Pi_{12} \neq 0$, $\Pi_{22} \neq 0$ is

$$\mathrm{pdf}(s,r) = \frac{\Gamma_n\left(\frac{L+n+1}{2}\right)\mathrm{etr}\left\{-\frac{T}{2}(I+\beta\beta')\bar{\Pi}_{22}'\bar{\Pi}_{22}\right\}}{(2\pi)^{K_1/2}\pi^{n/2}(1+r'r)^{(L+n+K_1+1)/2}}$$

$$\times \exp\{-(s-\pi_{11})'(s-\pi_{11})/2\}$$

$$\times \exp\left\{\frac{1}{2}\,\frac{(s-\pi_{11})'(s-\pi_{11})r'r - 2r'\Pi_{12}(s-\pi_{11}) - r'\Pi_{12}'\Pi_{12}r}{1+r'r}\right\}$$

$$\times \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{\varphi\in j\cdot\kappa} \frac{\left(\frac{L}{2}\right)_j\left(\frac{L+n+1}{2}\right)_\kappa \theta_\varphi^{j,\kappa}}{j!k!\Gamma_n\left(\frac{L+n}{2},\varphi\right)}$$

$$\times C_\varphi^{j,\kappa}\left(\frac{T}{2}\bar{\Pi}_{22}\beta\beta'\bar{\Pi}_{22}', \frac{T}{2}\bar{\Pi}_{22}(I+\beta r')(I+rr')^{-1}(I+r\beta')\bar{\Pi}_{22}'\right). \tag{17}$$

It will be convenient in what follows to use the identity

$$(I+\beta r')(I+rr')^{-1}(I+r\beta') = I + \beta\beta' - \frac{(r-\beta)(r-\beta)'}{1+r'r}. \tag{18}$$

Since the polynomial $C_\varphi^{j,\kappa}$ is an analytic function of its matrix arguments we employ the Taylor expansion

$$\mathrm{etr}\{-\partial Z(r-\beta)(r-\beta)'/(1+r'r)\}C_\varphi^{j,\kappa}$$

$$\times\left(\frac{T}{2}\bar{\Pi}_{22}\beta\beta'\bar{\Pi}_{22}', \frac{T}{2}\bar{\Pi}_{22}(1+\beta\beta'+Z)\bar{\Pi}_{22}'\right)\bigg|_{Z=0}$$

$$= C_\varphi^{j,\kappa}\left(\frac{T}{2}\bar{\Pi}_{22}\beta\beta'\bar{\Pi}_{22}', \frac{T}{2}\bar{\Pi}_{22}(I+\beta\beta')\bar{\Pi}_{22}'\right.$$

$$\left.-\frac{T}{2}\bar{\Pi}_{22}(r-\beta)(r-\beta)'\bar{\Pi}_{22}'/(1+r'r)\right), \tag{19}$$

which converges uniformly in r. The matrix Z in (19) is a matrix of auxiliary variables and ∂Z denotes the matrix operator $\partial/\partial Z$. The left side of (19) provides a simple algebraic representation of the multinomial expansion of the right-hand side polynomial, which involves a sum of matrix arguments. The latter no doubt admits an expansion in terms of polynomials with more matrix arguments; but the explicit form of this expansion has not yet been derived in the multivariate literature and (19) is a simpler alternative that is very convenient for our purpose.

From (17) and (19) we obtain

$$\begin{aligned}
\mathrm{pdf}(s,r) = {} & \frac{\Gamma_n\left(\frac{L+n+1}{2}\right)\operatorname{etr}\left\{-\frac{T}{2}(I+\beta\beta')\overline{\Pi}_{22}'\overline{\Pi}_{22}\right\}}{(2\pi)^{K_1/2}\pi^{n/2}(1+r'r)^{(L+n+K_1+1)/2}} \\
& \times \exp\{-(s-\pi_{11})'(s-\pi_{11})/2\} \\
& \times \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{\varphi\in j\cdot\kappa} \frac{\left(\frac{L}{2}\right)_j\left(\frac{L+n+1}{2}\right)_\kappa \theta_\varphi^{j,\kappa}}{j!k!\Gamma_n\left(\frac{L+n}{2},\varphi\right)} \\
& \times \sum_l \frac{\left\{\frac{1}{2}(s-\pi_{11})'(s-\pi_{11})\right\}^{l_1}\left(-\frac{1}{2}\right)^{l_2}(-1)^{l_3+l_4}}{l_1!l_2!l_3!l_4!} \\
& \times \left(\frac{r'r}{1+r'r}\right)^{l_1}\left(\frac{r'\Pi_{12}'\Pi_{12}r}{1+r'r}\right)^{l_2}\left(\frac{r'\Pi_{12}(s-\pi_{11})}{1+r'r}\right)^{l_3} \\
& \times \left(\frac{(r-\beta)'\partial Z(r-\beta)}{1+r'r}\right)^{l_4} \\
& \times C_\varphi^{j,\kappa}\left(\frac{T}{2}\overline{\Pi}_{22}'\beta\beta'\overline{\Pi}_{22}', \frac{T}{2}\overline{\Pi}_{22}(I+\beta\beta'+Z)\overline{\Pi}_{22}'\right)\Bigg|_{Z=0},
\end{aligned} \tag{20}$$

where $\sum_l$ denotes $\sum_{l_1,l_2,l_3,l_4}$. Since the series converges uniformly in r we may integrate termize to remove r. The typical term is then

$$\int \frac{(r'\Pi_{12}'\Pi_{12}r)^{l_2}\{r'\Pi_{12}(s-\pi_{11})\}^{l_3}\{(r-\beta)'\partial Z(r-\beta)\}^{l_4}(r'r)^{l_1}\,\mathrm{d}r}{(1+r'r)^{(L+n+K_1+1)/2+l}}$$

$$= (-\partial x'\Pi_{12}'\Pi_{12}\partial x)^{l_2}\{-\mathrm{i}\,\partial x'\Pi_{12}(s-\pi_{11})\}^{l_3}$$

$$\times\{(\mathrm{i}\,\partial x+\beta)'\partial Z(\mathrm{i}\,\partial x+\beta)\}^{l_4}$$

$$\times\int \frac{\mathrm{e}^{\mathrm{i}x'r}(r'r)^{l_1}\,\mathrm{d}r}{(1+r'r)^{(L+n+K_1+1)/2+l}}\Bigg|_{x=0}, \tag{21}$$

where $l=l_1+l_2+l_3+l_4$ and ∂x denotes the operator $\partial/\partial x$ taken with respect to a vector of auxiliary variables x.

We transform $r\to Hr=p$ for H orthogonal and integrate over the orthogonal group O(n), normalized so that the measure over the whole group is unity. The latter measure will be denoted by $(\underline{\mathrm{d}H})$. (21) becomes

$$\int_p\Bigg[(-\partial x'\Pi_{12}'\Pi_{12}\partial x)^{l_2}\{-\mathrm{i}\,\partial x'\Pi_{12}(s-\pi_{11})\}^{l_3}$$

$$\times\{(\mathrm{i}\,\partial x+\beta)'\partial Z(\mathrm{i}\,\partial x+\beta)\}^{l_4}\int_{\mathrm{O}(n)}\operatorname{etr}(\mathrm{i}Hxp')(\underline{\mathrm{d}H})\Bigg]_{x=0}$$

$$\times\frac{(p'p)^{l_1}\mathrm{d}p}{(1+p'p)^{(L+n+K_1+1)/2+l}}$$

$$=\int_p\Bigg[(-\partial x'\Pi_{12}'\Pi_{12}\partial x)^{l_2}\{-\mathrm{i}\,\partial x'\Pi_{12}(s-\pi_{11})\}^{l_3}$$

$$\times\{(\mathrm{i}\,\partial x+\beta)'\partial Z(\mathrm{i}\,\partial x+\beta)\}^{l_4}{}_0F_1\left(\frac{n}{2};-\tfrac{1}{4}x'xp'p\right)\Bigg]_{x=0}$$

$$\times\frac{(p'p)^{l_1}\mathrm{d}p}{(1+p'p)^{(L+n+K_1+1)/2+l}}$$

$$= \sum_{\tau} \Big[\left(-\partial x' \Pi_{12}' \Pi_{12} \partial x \right)^{l_2} \{ -\mathrm{i} \partial x' \Pi_{12} (s - \pi_{11}) \}^{l_3}$$

$$\times \{ (\mathrm{i} \partial x + \beta)' \partial Z (\mathrm{i} \partial x + \beta) \}^{l_4} \left(-\tfrac{1}{4} x' x \right)^{t} \Big]_{x=0}$$

$$\times \frac{1}{t! \left(\frac{n}{2} \right)_t} \int_p \frac{(p'p)^{l_1 + t} \, \mathrm{d} p}{(1 + p'p)^{(L+n+K_1+1)/2 + l}}, \tag{22}$$

where the summation $\sum_\tau$ is over all values of t for which the quantity in square brackets is non-zero. Since the latter quantity is zero whenever $t > l_2 + l_3 + l_4$ the integral in (22) is convergent within this summation. Upon evaluation of this integral [as in (14) above] we obtain

$$\frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)} \sum_{\tau} \Big[\left(-\partial x' \Pi_{12}' \Pi_{12} \partial x \right)^{l_2} \{ -\mathrm{i} \partial x' \Pi_{12} (s - \pi_{11}) \}^{l_3}$$

$$\times \{ (\mathrm{i} \partial x + \beta)' \partial Z (\mathrm{i} \partial x + \beta) \}^{l_4} \left(-\tfrac{1}{4} x' x \right)^{t} \Big]_{x=0} \frac{1}{t! \left(\frac{n}{2} \right)_t}$$

$$\times \frac{\Gamma\left(l_1 + \frac{n}{2} + t \right) \Gamma\left(\frac{L + K_1 + 1}{2} + l_2 + l_3 + l_4 - t \right)}{\Gamma\left(\frac{L + n + K_1 + 1}{2} + l_1 + l_2 + l_3 + l_4 \right)}. \tag{23}$$

From (20) and (23) we deduce the following general expression for the p.d.f. of γ_{IV}:

$$\mathrm{pdf}(s) = \frac{\Gamma_n\left(\frac{L+n+1}{2} \right) \mathrm{etr}\left\{ -\frac{T}{2} (I + \beta\beta') \overline{\Pi}_{22}' \overline{\Pi}_{22} \right\}}{(2\pi)^{K_1/2} \Gamma\left(\frac{n}{2} \right)}$$

$$\times \exp\{ -(s - \pi_{11})'(s - \pi_{11})/2 \}$$

$$\times \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{\varphi \in j \cdot \kappa} \frac{\left(\frac{L}{2} \right)_J \left(\frac{L+n+1}{2} \right)_\kappa \theta_\varphi^{J,\kappa}}{j! k! \Gamma_n\left(\frac{L+n}{2}, \varphi \right)}$$

$$\times \sum_{l} \frac{\{\frac{1}{2}(s-\pi_{11})'(s-\pi_{11})\}^{l_1}(-\frac{1}{2})^{l_2}(-1)^{l_3+l_4}}{l_1!l_2!l_3!l_4!}$$

$$\times \sum_{\tau} \frac{\Gamma\left(l_1+\frac{n}{2}+t\right)\Gamma\left(\frac{L+K_1+1}{2}+l_2+l_3+l_4-t\right)}{t!\left(\frac{n}{2}\right)_t \Gamma\left(\frac{L+n+K_1+1\cdot}{2}+l_1+l_2+l_3+l_4\right)}$$

$$\times \left[\left[(-\partial x'\Pi_{12}'\Pi_{12}\partial x)^{l_2}\{-\mathrm{i}\,\partial x'\Pi_{12}(s-\pi_{11})\}^{l_3}\right.\right.$$

$$\left.\times\{(\mathrm{i}\,\partial x+\beta)'\partial Z(\mathrm{i}\,\partial x+\beta)\}^{l_4}(-\tfrac{1}{4}x'x)^{t}\right]_{x=0}$$

$$\left.\times C_{\varphi}^{j,\kappa}\left(\frac{T}{2}\overline{\Pi}_{22}\beta\beta'\overline{\Pi}_{22}', \frac{T}{2}\overline{\Pi}_{22}(I+\beta\beta'+Z)\overline{\Pi}_{22}'\right)\right]_{Z=0}. \tag{24}$$

The leading case that occurs when $\Pi_{12}=0$, $\Pi_{22}=0$ may be deduced from (24) by noting that non-zero terms in the various summations arise only when $j=k=l_2=l_3=l_4=t=0$. Moreover, $\Gamma_n((L+n)/2;\ \varphi)=\Gamma_n((L+n)/2)$ when φ is a partition of zero. We find in this case

$$\mathrm{pdf}(s)=\frac{\Gamma_n\left(\frac{L+n+1}{2}\right)\exp\{-(s-\pi_{11})'(s-\pi_{11})/2\}}{(2\pi)^{K_1/2}\Gamma\left(\frac{n}{2}\right)\Gamma_n\left(\frac{L+n}{2}\right)}$$

$$\times \sum_{l_1=0}^{\infty} \frac{\left\{\frac{1}{2}(s-\pi_{11})'(s-\pi_{11})\right\}^{l_1}}{l_1!} \frac{\Gamma\left(l_1+\frac{n}{2}\right)\Gamma\left(\frac{L+K_1+1}{2}\right)}{\Gamma\left(\frac{L+n+K_1+1}{2}+l_1\right)}$$

$$=\frac{\Gamma_n\left(\frac{L+n+1}{2}\right)\Gamma\left(\frac{L+K_1+1}{2}\right)\exp\{-(s-\pi_{11})'(s-\pi_{11})/2\}}{(2\pi)^{K_1/2}\Gamma_n\left(\frac{L+n}{2}\right)\Gamma\left(\frac{L+n+K_1+1}{2}\right)}$$

$$\times {}_1F_1\left(\frac{n}{2}, \frac{L+n+K_1+1}{2}; \tfrac{1}{2}(s-\pi_{11})'(s-\pi_{11})\right)$$

$$= \frac{\Gamma\left(\frac{L+n+1}{2}\right)\Gamma\left(\frac{L+K_1+1}{2}\right)\exp\{-(s-\pi_{11})'(s-\pi_{11})/2\}}{(2\pi)^{K_1/2}\Gamma\left(\frac{L+1}{2}\right)\Gamma\left(\frac{L+n+K_1+1}{2}\right)}$$

$$\times {}_1F_1\left(\frac{n}{2}, \frac{n+L+K_1+1}{2}; \tfrac{1}{2}(s-\pi_{11})'(s-\pi_{11})\right),$$

which is the same as (15) derived earlier by direct methods.

5. Remarks

The exact densities (15) and (24) relate to the standardized model. The corresponding densities for the non-standardized model may be obtained from these results by transformation using the formulae in Phillips (1983c).

Accurate approximations to these densities that will permit wide-ranging numerical computations and the analysis of marginal distributions are the next step in studying these distributions. Methods used by the author (1983d) elsewhere seem promising in this respect.

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